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Krein-Milman's Extreme Point Theorem and Weak Topology on Hyperspace

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Abstract. Let WCC(X) be the collection of all non-empty, weakly compact, convex subsets of a Banach space X endowed with the Hausdorff metric h. Weak topology $\mathcal{T}_w$ will be defined on WCC(X). We shall prove that every weakly compact ($\mathcal{T}_w$-compact) convex subset $\mathcal{K} \subset (\mathrm{WCC}(X), \mathcal{T}_w)$ has an extreme point. We also show that there exists strongly bounded (h-bounded), closed (h-closed) convex subsets which are not weakly closed (i.e., not $\mathcal{T}_w$-closed).

1. Introduction

The study of convex sets has always been an important and useful research area in various branches of mathematics. Suppose X is a Banach space and BCC(X) is the collection of all non-empty bounded, closed, convex subsets of X endowed with the Hausdorff metric, h. Then $(\mathrm{BCC}(X), h)$ is a complete metric space and is known as the hyperspace over X. Let WCC(X) be the collection of all non-empty, weakly compact, convex subsets of X, and CC(X) be the collection of all non-empty compact, convex subsets of X. For general X, we have $\mathrm{CC}(X) \subsetneqq \mathrm{WCC}(X) \subsetneqq \mathrm{BCC}(X)$.

Blaschke [2] proved that a bounded sequence A_n of the hyperspace $(\mathrm{CC}(\mathbb{R}^k), h)$ over $\mathbb{R}^k$ has a subsequence A_{n_j} such that A_{n_j} converges to some $A \in \mathrm{CC}(\mathbb{R}^k)$ (i.e., $A_{n_j} \xrightarrow{h} A$, or $h(A_{n_j}, A) \to 0$). Many mathematicians have studied convergence of convex sets on different spaces [1, 6, 10, 11]. De Blasi and Myjak [4] introduced the concept of weak sequential convergence in BCC(X) and proved an infinite dimensional version of Blaschke's theorem and some other interesting results. Hu and company [3, 7–9] first introduced the notation of weak topology on CC(X) and later to WCC(X). They showed that the classical Browder-Kirk and Brodskii-Milman theorems can be extended to the hyperspace WCC(X) [7, 9]. If X is assumed to be separable, they proved that Alaoglu's theorem [3] can be extended, too. The main purpose of this paper is to investigate further on the hyperspace WCC(X). We obtain a hyperspace version of Krein-Milman's extreme point theorem and other results. We also give examples to illustrate results that are valid on the underlying space X that cannot be extended to the hyperspace WCC(X).

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2. Notations and preliminaries

Let X be a Banach space, X^* its topological dual and $\mathrm{BCC}(X)$ the collection of all non-empty, bounded closed convex subsets of X. For $A, B \in \mathrm{BCC}(X)$, define $N(A;\varepsilon) = \{x \in X : d(x,a) = \|x-a\| < \varepsilon \text{ for some } a \in A\}$, and $h(A,B) = \inf\{\varepsilon > 0 : A \subset N(B,\varepsilon)$ and $B \subset N(A,\varepsilon)\}$, equivalently $h(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\}$. Then h is known as the Hausdorff metric and $(\mathrm{BCC}(X), h)$ is the hyperspace over X and which is known to be a complete metric space. If $\dim(X) < \infty$, we have $\mathrm{CC}(X) = \mathrm{WCC}(X) = \mathrm{BCC}(X)$. If X is reflexive, we have $\mathrm{WCC}(X) = \mathrm{BCC}(X)$. We shall consider only $\mathrm{CC}(X)$ and $\mathrm{WCC}(X)$ in this paper.

First, we shall summarize some elementary properties of the Hausdorff metric h, and its relationship with $x^* \in X^*$ in the following lemmas.

Lemma 2.1. *Let $A, B, C, D \in \mathrm{WCC}(X)$, and $\alpha \in \mathbb{C}$. Then we have*

(i) $h(A, \{0\}) = \sup\{\|a\| : a \in A\}$,

(ii) $h(A+B, C+D) \le h(A,C) + h(B,D)$,

(iii) $A \subset B \subset C$ *implies* $h(A,C) \ge h(B,C)$,

(iv) $h(\alpha A, \alpha B) = |\alpha|\, h(A,B)$,

(v) $h([\alpha_1,\alpha_2],[\beta_1,\beta_2]) = \max(|\beta_1-\alpha_1|, |\beta_2-\alpha_2|)$, *for* $[\alpha_1,\alpha_2],[\beta_1,\beta_2] \in (\mathrm{CC}(\mathbb{R}^1), h)$.

Lemma 2.2. *Let $A, B \in \mathrm{WCC}(X)$, and $x^* \in X^*$. Then we have $x^*(A)$, $x^*(B) \in (\mathrm{CC}(\mathbb{C}), h)$, and*

(i) $A = B$ *if and only if* $x^*(A) = x^*(B)$ *for each* $x^* \in \mathbb{X}^*$,

(ii) $h(x^*(A), x^*(B)) \le \|x^*\|\, h(A,B)$,

(iii) $h(x^*(A), y^*(A)) \le \|x^* - y^*\|\, h(A, \{0\})$,

(iv) $h(A,B) = \sup\{h(x^*(A), x^*(B)) : \|x^*\| \le 1\}$.

Proof. (i), (ii), and (iii) are obvious. To prove (iv), note that, in [4] and [6], they assume X is a real Banach space, $\sigma_A(x^*) = \sup\{x^*(a) : a \in A\}$ is the support functional and they prove that $h(A,B) = \sup\{|\sigma_A(x^*) - \sigma_B(x^*)| : \|x^*\| \le 1\}$. In our case, X is a complex Banach space, and A, B are weakly compact, convex subsets of X. Let $\alpha_2 = \sigma_A(\mathrm{Re}\, x^*) = \sup\{\mathrm{Re}\, x^*(a) : a \in A\} = \max\{\mathrm{Re}\, x^*(a) : a \in A\}$, $\beta_2 = \sigma_B(\mathrm{Re}\, x^*) = \sup\{\mathrm{Re}\, x^*(b) : b \in B\} = \max\{\mathrm{Re}\, x^*(b) : b \in B\}$, $\alpha_1 = \min\{\mathrm{Re}\, x^*(a) : a \in A\}$ and $\beta_1 = \min\{\mathrm{Re}\, x^*(b) : b \in B\}$. We have then $\mathrm{Re}\, x^*(A) = [\alpha_1,\alpha_2]$, $\mathrm{Re}\, x^*(B) = [\beta_1,\beta_2] \in (\mathrm{CC}(\mathbb{R}^1), h)$. Thus $|\sigma_A(\mathrm{Re}\, x^*) - \sigma_B(\mathrm{Re}\, x^*)| = |\alpha_2 - \beta_2| \le \max\{|\alpha_2-\beta_2|, |\alpha_1-\beta_1|\} =$

$h(\mathrm{Re}\, x^*(A), \mathrm{Re}\, x^*(B)) \le h(x^*(A), x^*(B))$. Consequently, $h(A, B) = \sup\{|\sigma_A(\mathrm{Re}\, x^*) - \sigma_B(\mathrm{Re}\, x^*)| : \|x^*\| \le 1\} \le \sup\{h(x^*(A), x^*(B)) : \|x^*\| \le 1\}$. Reverse inequality follows from $h(x^*(A), x^*(B)) \le \|x^*\|\, h(A, B) \le h(A, B)$ since $\|x^*\| \le 1$. □

Now, we let $\mathbb{C}$ denote the complex plane and $\mathrm{CC}(\mathbb{C})$ the collection of all non-empty compact, convex subsets of $\mathbb{C}$. It follows from the weak continuity and linearity of x^* that for each $A \in \mathrm{WCC}(X)$ (i.e., A is a weakly compact convex subset of X), we have $x^*(A) \in \mathrm{CC}(\mathbb{C})$ (i.e., $x^*(A)$ is a compact, convex subset of the complex plane $\mathbb{C}$). Hence x^* maps the space $\mathrm{WCC}(X)$ into $\mathrm{CC}(\mathbb{C})$, or $x^* \colon (\mathrm{WCC}(X), h)) \to (\mathrm{CC}(\mathbb{C}), h)$. Note that both the domain and range of x^* are now hyperspaces endowed with corresponding Hausdorff metric h. By Lemma 2.2(ii), $x^* \colon (\mathrm{WCC}(X), h) \to (\mathrm{CC}(\mathbb{C}), h)$ is continuous. Recall that the weak topology τ_w on X is defined to be the weakest topology which makes each $x^* \colon (X, \tau_w) \to (\mathbb{C}, |\cdot|)$ continuous. Thus, we may define $\mathcal{T}_w$ to be the weakest topology on $\mathrm{WCC}(X)$ such that each $x^* \colon (\mathrm{WCC}(X), \mathcal{T}_w) \to (\mathrm{CC}(\mathbb{C}), h)$ is continuous. A typical $\mathcal{T}_w$ neighborhood of $A \in \mathrm{WCC}(X)$ is denoted by $\mathcal{W}(A; x_1^*, \ldots, x_n^*; \varepsilon) = \{B \in \mathrm{WCC}(X) : h(x_i^*(B), x_i^*(A)) < \varepsilon \text{ for } i = 1, 2, \ldots, n\}$.

To avoid confusion, we shall use small letters $a, b, c, \ldots, x, y, z$ to denote elements of the underlying space X, capital letters $A, B, \ldots, Z$ to denote elements of the hyperspace $\mathrm{WCC}(X)$ as well as subsets of X, e.g., $A, B \subset X$; $A, B \in \mathrm{WCC}(X)$. We shall use script letters to denote subsets of the hyperspace $\mathrm{WCC}(X)$, e.g., $\mathcal{K} \subset \mathrm{WCC}(X)$, $\mathcal{E} \subset \mathrm{WCC}(X)$. The strong topology on X is the norm topology and will be denoted by $\|\cdot\|$-topology and the weak topology on X will be denoted by τ_w-topology. The strong topology on the hyperspace $\mathrm{WCC}(X)$ is the metric topology and will be denoted by h-topology; the weak topology on $\mathrm{WCC}(X)$ will be denoted by $\mathcal{T}_w$-topology.

Definition 2.3. (i) A sequence $\{A_n\} \subset \mathrm{WCC}(X)$ is said to be weakly Cauchy (or $\mathcal{T}_w$-Cauchy) if and only if for each $x^* \in X^*$, the sequence $\{x^*(A_n) \subset \mathrm{CC}(\mathbb{C})\}$ is Cauchy.

(ii) $A_n, A \in \mathrm{WCC}(X)$. $\{A_n\}$ is said to converge strongly to A (denoted by $A_n \xrightarrow{s} A$ or $A_n \xrightarrow{h} A$) if and only if $\lim_{n\to\infty} h(A_n, A) = 0$; $\{A_n\}$ is said to converge weakly (denoted by $A_n \xrightarrow{w} A$ or $A_n \xrightarrow{\mathcal{T}_w} A$) if and only if $\lim_{n\to\infty} h(x^*(A_n), x^*(A)) = 0$ for each $x^* \in X^*$.

(iii) $\mathcal{K} \subset \mathrm{WCC}(X)$ is said to be weakly sequentially closed if and only if every sequence $\{A_n\} \subset \mathcal{K}$ with $A_n \xrightarrow{\mathcal{T}_w} A$, we have $A \in \mathcal{K}$.

(iv) $\mathcal{K}$ is said to be weakly sequentially complete if and only if every weak Cauchy sequence $\{A_n\} \subset \mathcal{K}$ converges weakly to some $A \in \mathcal{K}$.

(v) $\mathcal{K}$ is said to be weakly sequentially compact if and only if every sequence $\{A_n\} \subset \mathcal{K}$ contains a subsequence $\{A_{n_k}\}$ that converges weakly to some $A \in \mathcal{K}$.

(vi) $\mathcal{K}$ is said to be weakly bounded if and only if for each $x^* \in X^*$, the set $\{x^*(A) : A \in \mathcal{K}\}$ is bounded in $(\mathrm{CC}(\mathbb{C}), h)$.

3. Main results

We now list some of the basic properties of the weak topology $\mathcal{T}_w$ in the following.

Lemma 3.1. (i) *The weak topology $\mathcal{T}_w$ is Hausdorff.*

(ii) *Addition and scalar multiplications are continuous operations on the space $(\mathrm{WCC}(X), \mathcal{T}_w)$.*

(iii) *Suppose $\mathcal{K} \subset \mathrm{WCC}(X)$ is weakly bounded ($\mathcal{T}_w$-bounded). Then $\mathcal{K}$ is strongly bounded (h-bounded).*

Proof. The verification of (i) and (ii) is routine. To prove (iii), let $\mathcal{K}$ be weakly bounded. Thus for each $x^* \in X^*$, $\{x^*(A) : A \in \mathcal{K}\}$ is a bounded set of $(\mathrm{CC}(\mathbb{C}), h)$ and we have some $M > 0$ such that $\sup\{h(x^*(A), x^*(\{0\})) : A \in \mathcal{K}\} \leq M < \infty$. Note that $h(x^*(A), x^*(\{0\})) = \sup\{\|x^*(a)\| : a \in A\}$. Thus, if we set $K = \bigcup_{A\in\mathcal{K}} A = \bigcup_{A\in\mathcal{K}} \{a : a \in A\} \subset X$, we have $\sup\{h(x^*(A), x^*(\{0\})) : A \in \mathcal{K}\} = \sup_{A\in\mathcal{K}}\{\sup_{a\in A}(\|x^*(a)\|)\} = \sup_{a\in K}\{\|x^*(a)\|\} \leq M$.

It follows now from the uniform boundedness principle that K is a bounded subset of X, i.e., $\sup\{\|a\| : a \in K\} \leq N < \infty$ for some N. Since $A \subset K$, we have $h(A, \{0\}) = \sup\{\|a\| : a \in A\} \leq \sup\{\|a\| : a \in K\} \leq N$ for each $A \in \mathcal{K}$ proving that $\mathcal{K}$ is bounded. □

Theorem 3.2. *Let X be a Banach space with $\dim(X) < \infty$, and $\mathrm{CC}(X) = \{A \subseteq X :$ A is a non-empty compact, convex subset of $X\}$ is the corresponding hyperspace. Suppose $\mathcal{K} \subset \mathrm{CC}(X)$. Then the following are equivalent:*

(a) *$\mathcal{K}$ is weakly compact ($\mathcal{T}_w$-compact).*

(b) *$\mathcal{K}$ is weakly sequentially compact ($\mathcal{T}_w$-sequentially compact).*

(c) *$\mathcal{K}$ is h-sequentially compact.*

(d) *$\mathcal{K}$ is compact (h-compact).*

Proof. Assume $\mathcal{K}$ is weakly compact. Then $\mathcal{K}$ is weakly bounded. And it follows from Lemma 3.1 that $\mathcal{K}$ is bounded. Hence, there exists some $M > 0$ such that

$$\sup\{h(A, \{0\}) : A \in \mathcal{K}\} \leq M < \infty.$$

Next, $B_1^* = \{x^* \in X^* : \|x^*\| \le 1\}$ is compact, and $B_1^* \subset \bigcup_{x^* \in B_1^*} B^*(x^*, \varepsilon/(3M))$ implies that $B_1^* \subset \bigcup_{k=1}^n B^*(x_k^*, \varepsilon/(3M))$ with $x_k^* \in B_1^*$ for $k = 1, 2, \ldots, n$. Suppose now $\{A_\alpha\}$ is a given infinite sequence of $\mathcal{K}$. $\mathcal{K}$ is weakly compact ($\mathcal{T}_w$-compact), and x_1^* is $\mathcal{T}_w$-continuous implies that $x_1^*(\mathcal{K})$ is compact in $(\mathrm{CC}(\mathbb{C}), h)$. Thus $\{x_1^*(A_\alpha)\} \subset x_1^*(\mathcal{K})$ has a convergent subsequence $\{x_1^*(A_{1j})\}_{j=1}^\infty$. Similarly, $x_2^*(\mathcal{K})$ is compact, and $\{x_2^*(A_{1j})\}_{j=1}^\infty \subset x_2^*(\mathcal{K})$ implies that $\{A_{1j}\}$ has a subsequence $\{A_{2j}\}$ such that $\{x_2^*(A_{2j})\}$ converges. Note that $\{x_1^*(A_{2j})\} \subset \{x_1^*(A_{1j})\}$ also converges. Inductively, we obtain a subsequence $\{A_{nj}\}_{j=1}^\infty$ such that $\{x_k^*(A_{nj})\}_{j=1}^\infty$ converges for $k = 1, 2, \ldots, n$. To simplify the notation we let $\{A_j\}_{j=1}^\infty = \{A_{nj}\}_{j=1}^\infty$. Hence, $\{A_j\}_{j=1}^\infty$ is a sequence such that $\{x_k^*(A_j)\}_{j=1}^\infty$ converges for $k = 1, 2, \ldots, n$. Consequently, for $\varepsilon/3 > 0$, there exists N_k such that $i, j \ge N_k$ implies $h(x_k^*(A_i), x_k^*(A_j)) < \varepsilon/3$. Let $N = \max\{N_1, N_2, \ldots, N_n\}$, we have $i, j \ge N$ implies $h(x_k^*(A_i), x_k^*(A_j)) < \varepsilon/3$ for $k = 1, 2, \ldots, n$. Claim that $i, j \ge N$ implies

$$h(x^*(A_i), x^*(A_j)) < \varepsilon \quad \text{for each } x^* \in B_1^*. \tag{3.1}$$

Indeed for each $x^* \in B_1^* \subset \bigcup_{k=1}^n B^*(x_k^*, \varepsilon/(3M))$, there exists some k such that $x^* \in B^*(x_k^*, \varepsilon/(3M))$ and hence $\|x^* - x_k^*\| < \varepsilon/(3M)$. If $i, j \ge N$, we have

$$\begin{aligned} h(x^*(A_i), x^*(A_j)) &\le h(x^*(A_i), x_k^*(A_i)) + h(x_k^*(A_i), x_k^*(A_j)) + h(x_k^*(A_j), x^*(A_j)) \\ &< \|x^* - x_k^*\|\, h(A_i, \{0\}) + \frac{\varepsilon}{3} + \|x_k^* - x^*\|\, h(A_i, \{0\}) \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and (3.1) is proved. For each $x^* \in B_1^*$, $\{x^*(A_j)\}$ is Cauchy implies that the sequence $\{A_j\}$ is weakly Cauchy ($\mathcal{T}_w$-Cauchy).

$\mathcal{K}$ is weakly compact implies there exists some $A_0 \in \mathcal{K}$ such that every weak neighborhood $\mathcal{W}(A_0)$ contains infinitely many terms of the sequence $\{A_j\}$. We now claim that $\{A_j\}$ converges weakly to A_0. Indeed, it follows from (3.1) that for each $x^* \in B_1^*$, $\{x^*(A_j)\}$ is Cauchy in $(\mathrm{CC}(\mathbb{C}), h)$ and hence there exists $D_{x^*} \in \mathrm{CC}(\mathbb{C})$ such that

$$\lim_{j \to \infty} h(x^*(A_j), D_{x^*}) = 0. \tag{3.2}$$

And consequently for given $\varepsilon > 0$, there exists k such that $j \ge k$ implies $h(x^*(A_j), D_{x^*}) < \varepsilon/2$. On the other hand, since $\mathcal{W}(A_0, x^*, \varepsilon/2)$ contains infinitely many A_j's, there exists some $j_0 \ge k$ such that $A_{j_0} \in \mathcal{W}(A_0, x^*, \varepsilon/2)$, or $h(x^*(A_{j_0}), x^*(A_0)) < \varepsilon/2$. Consequently,

$$\begin{aligned} h(x^*(A_0), D_{x^*}) &\le h(x^*(A_0), x^*(A_{j_0})) + h(x^*(A_{j_0}), D_{x^*}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $x^*(A_0) = D_{x^*}$ and hence $\lim_{j \to \infty} h(x^*(A_j), x^*(A_0)) = \lim_{j \to \infty} h(x^*(A_j), D_{x^*}) = 0$ by (3.2). Hence A_j converges weakly to A_0 showing that $\mathcal{K}$ is weakly sequentially compact. Thus $\mathcal{K}$ is weakly sequentially compact. Also, it follows from (3.1) that for $\varepsilon > 0$ there exists N such that $j \geq N$ implies $h(x^*(A_j), x^*(A_0)) \leq \varepsilon$ for $x^* \in B_1^*$. Hence by Lemma 2.2(iv), we have $h(A_j, A_0) = \sup\{h(x^*(A_j), x^*(A_0)) : \|x^*\| \leq 1\} \leq \varepsilon$. Thus $\{A_j\}$ also h-converges to A_0 showing that $\mathcal{K}$ is h-sequentially compact. Finally, $(\mathcal{K}, h)$ is a metric space implies that $\mathcal{K}$ is also h-compact since compact and sequentially compact are equivalent in metric spaces. □

Since $A, B \in \mathrm{WCC}(X)$, $\alpha \in \mathbb{C}$, we have $A + B \in \mathrm{WCC}(X)$, $\alpha A \in \mathrm{WCC}(X)$. We may define algebraic line segments and convex sets analogous to their counterparts on the underlying Banach Space X. We may also define extreme points and extremal sets. However, on the underlying Banach space X with $\|\cdot\|$-topology and weak topology τ_w, we usually define extreme points on norm-closed convex sets which are of course applicable to τ_w-closed convex sets since closed convex sets are also weakly closed. On the hyperspace $\mathrm{WCC}(X)$, carrying the h-topology and the weak topology $\mathcal{T}_w$ we see from the Example 3.7 below that h-closed convex subsets are not necessarily $\mathcal{T}_w$-closed. Thus we define extreme points and extremal sets on $\mathcal{T}_w$-closed, convex sets so that the definition is applicable to h-closed convex sets too.

Definition 3.3. (i) $[A, B] = \{\alpha A + (1 - \alpha)B : 0 \leq \alpha \leq 1, A, B \in \mathrm{WCC}(X)\}$ is called the closed line segment joining A and B.

(ii) A subset $\mathcal{K} \subset \mathrm{WCC}(X)$ is said to be convex if and only if $A_1, A_2, \ldots, A_n \in \mathcal{K}$, $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^n \alpha_i = 1$ implies $\sum_{i=1}^n \alpha_i A_i \in \mathcal{K}$.

(iii) A mapping $T\colon \mathcal{K}_1 \to \mathcal{K}_2$ is said to be affine if and only if $T(\alpha A + (1 - \alpha)B) = \alpha T(A) + (1 - \alpha)T(B)$ for $A, B \in \mathcal{K}_1$, $0 \leq \alpha \leq 1$ and $\mathcal{K}_1$, $\mathcal{K}_2$ are convex subsets of $\mathrm{WCC}(X)$.

(iv) If $\mathcal{K}_1 \subset \mathcal{K}_2$, $\mathcal{K}_1, \mathcal{K}_2$ are $\mathcal{T}_w$-closed, convex subsets of $\mathrm{WCC}(X)$, then $\mathcal{K}_1$ is said to be an extremal subset of $\mathcal{K}_2$ if $A, B \in \mathcal{K}_2$, $\alpha A + (1 - \alpha)B \in \mathcal{K}_1$ for some $0 < \alpha < 1$ implies that $A, B \in \mathcal{K}_1$.

(v) Suppose $P \in \mathcal{K}$ where $\mathcal{K}$ is $\mathcal{T}_w$-closed convex, then P is said to be an extreme point of $\mathcal{K}$ if and only if $A, B \in \mathcal{K}$, $0 < \alpha < 1$, $\alpha A + (1 - \alpha)B = P$ implies $A = B = P$.

We state the following lemma whose proofs are similar as in the underlying space X.

Lemma 3.4. *Let $\mathcal{K}$ be a $\mathcal{T}_w$-closed, convex subset of the hyperspace* $\mathrm{WCC}(X)$.

(i) *If $P \in \mathcal{K}$, then $\{P\}$ is an extremal subset of $\mathcal{K}$ if and only if P is an extreme point of $\mathcal{K}$.*

(ii) $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_3$ *are* $\mathcal{T}_w$*-closed, convex sets and if* $\mathcal{K}_2$ *is an extremal subset of* $\mathcal{K}_3$ *and* $\mathcal{K}_1$ *is an extremal subset of* $\mathcal{K}_2$*, then* $\mathcal{K}_1$ *is an extremal subset of* $\mathcal{K}_3$*.*

(iii) *If* $\mathcal{K}$ *is* $\mathcal{T}_w$*-compact, convex, and* $T\colon (\mathcal{K}, \mathcal{T}_w) \to (\mathbb{R}^1, |\cdot|)$ *is a continuous affine mapping and* $\alpha_1 = \inf_{A\in\mathcal{K}} \{T(A)\} = \min_{A\in\mathcal{K}} \{T(A)\}$*,* $\alpha_2 = \sup_{A\in\mathcal{K}} \{T(A)\} = \max_{A\in\mathcal{K}} \{T(A)\}$*, then* $T^{-1}(\alpha_1), T^{-1}(\alpha_2)$ *are both extremal subsets of* $\mathcal{K}$*.*

(iv) *Suppose* $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in (\mathrm{CC}(\mathbb{R}^1), h)$*. Then* $h([\alpha_1, \alpha_2], [\beta_1, \beta_2]) = \max(|\beta_1 - \alpha_1|, |\beta_2 - \alpha_2|)$*. Then the mapping* $T\colon (\mathrm{CC}(\mathbb{R}^1), h) \to (\mathbb{R}^1, |\cdot|)$ *defined by* $T[\alpha_1, \alpha_2] = \alpha_2$ *is a continuous (in fact, nonexpansive) affine mapping.*

Theorem 3.5. *Let* X *be a Banach sapce and* $\mathrm{WCC}(X)$ *the corresponding hyperspace. Suppose* $\mathcal{K} \subset \mathrm{WCC}(X)$ *is a weakly compact (*$\mathcal{T}_w$*-compact), convex subset. Then* $\mathcal{K}$ *has an extreme point.*

Proof. Let Ω denote the collection of all $\mathcal{T}_w$-closed, convex extremal subsets of $\mathcal{K}$. $\Omega \neq \Phi$ since $\mathcal{K} \subset \Omega$. Define a partial order in Ω by $\mathcal{K}_2 \leq \mathcal{K}_1$ if and only if $\mathcal{K}_1 \subset \mathcal{K}_2$. If $\{\mathcal{K}_i\}_{i\in I} \subset \Omega$ is a totally ordered subset, we shall show that $\mathcal{K}_0 = \bigcap_{i\in I} \mathcal{K}_i$ is an upper bound of $\{\mathcal{K}_\alpha\}$. Each $\mathcal{K}_\alpha$ is $\mathcal{T}_w$-compact, convex and $\{\mathcal{K}_\alpha\}$ has finite intersection property implies that $\mathcal{K}_0$ is a non-empty, $\mathcal{T}_w$-compact, convex set. Suppose now we have $A, B \in \mathcal{K}$, $0 < \alpha < 1$ and $\alpha A + (1-\alpha)B \in \mathcal{K}_0$. $\mathcal{K}_0 \subset \mathcal{K}_i$ for each $i \in I$ implies that $\alpha A + (1-\alpha)B \in \mathcal{K}_i$ which in turn implies $A, B \in \mathcal{K}_i$, since $\mathcal{K}_i$ is an extremal subset of $\mathcal{K}$. Thus $A, B, \in \mathcal{K}_0$ showing that $\mathcal{K}_0$ is an extremal subset of $\mathcal{K}$ and consequently $\mathcal{K}_0$ is an upper bound of $\{\mathcal{K}_i\}_{i\in I}$. It follows now from Zorn's Lemma that Ω has a maximal element, denoted by $\mathcal{K}_\infty$. Finally, we claim that $\mathcal{K}_\infty$ is a singleton. Otherwise, there exist $A_0, B_0 \in \mathcal{K}_\infty$ with $A_0 \neq B_0$ w.l.o.g., assume there exists some $b_0 \in B_0$ such that $b_0 \notin A_0$. By Hahn-Banach theorem, there exists some $x^* \in X^*$ such that $\sup\{\mathrm{Re}\, x^*(a) : a \in A_0\} < \mathrm{Re}\, x^*(b_0)$. Let $\mathrm{Re}\, x^*(A_0) = [\alpha_1, \alpha_2]$, $\mathrm{Re}\, x^*(B_0) = [\beta_1, \beta_2] \in \mathrm{CC}(\mathbb{R})$. We have

$$\alpha_2 = \sup\{\mathrm{Re}\, x^*(a) : a \in A_0\} < \mathrm{Re}\, x^*(b_0) \leq \beta_2. \tag{3.3}$$

Define $G\colon (\mathrm{CC}(\mathbb{R}), h) \to (\mathbb{R}, |\cdot|))$ by $G([\alpha_1, \alpha_2]) = \alpha_2$. It follows from Lemma that G is a nonexpnasive (hence continuous) affine mapping. Now let $F\colon (\mathrm{WCC}(X), \mathcal{T}_w) \to (\mathbb{R}, |\cdot|)$ be defined by $F(A) = G(\mathrm{Re}\, x^*(A))$. $F\colon (\mathcal{K}_\infty, \mathcal{T}_w) \to (\mathbb{R}, |\cdot|)$ is continuous implies F attains its maximum on $\mathcal{K}_\infty$, i.e., there exists $\beta_\infty \in \mathbb{R}$ ($\beta_\infty \geq \beta_2 > \alpha_2$) and $A_\infty \in \mathcal{K}_\infty$ such that $F(A_\infty) = \beta_\infty = \sup\{F(A) : A \in \mathcal{K}_\infty\}$. It follows from Lemma 3.4(iii) that $F^{-1}(\beta_\infty)$ is a non-empty, extremal subset of $\mathcal{K}_\infty$. It follows from (3.3) that $A_0 \notin F^{-1}(\beta_\infty)$. Hence, $F^{-1}(\beta_\infty) \geq \mathcal{K}_\infty$ and $F^{-1}(\beta_\infty) \neq \mathcal{K}_\infty$. That is a contradiction to the maximality of $\mathcal{K}_\infty$. Hence $\mathcal{K}_\infty$ is a singleton, say $\mathcal{K}_\infty = \{D\}$, and D is an extremal point of $\mathcal{K}$. □

Since (X, τ) is a l.c.t.v.s, it is known from Hahn-Banach theorem that the dual space X^* contains enough linear functionals x^* that separate two disjoint weakly compact (τ_w-compact) convex sets in the sense that if K_1, K_2 are weakly compact, convex subsets of X and $K_1 \cap K_2 = \Phi$, then there exists some $x^* \in X^*$ such that $\operatorname{dist}(K_1, K_2) = \inf\{\|k_1 - k_2\| = d(k_1, k_2) : k_1 \in K_1, k_2 \in K_2\} = \delta > 0$. We shall show in the following example that X^* does not have enough x^* to separate two disjoint compact, convex sets on the hyperspace WCC(X).

Example 3.6. Let x_1, x_2, x_3 be vertices of an equilateral triangle of $(\mathbb{R}^2, \|\cdot\|)$. Let $A_1 = [x_1, x_2]$, $A_2 = [x_2, x_3]$, $A_3 = [x_3, x_1]$ be the closed line segments of unit length, and $D = \operatorname{conv}(x_1, x_2, x_3)$. Then $A_1, A_2, A_3, D \in (\mathrm{CC}(\mathbb{R}^2), h)$. Suppose $\mathcal{K}_1 = \operatorname{conv}(A_1, A_2, A_3)$, $\mathcal{K}_2 = \{D\} \subset \mathrm{CC}(\mathbb{R}^2)$. Then $\mathcal{K}_1$ and $\mathcal{K}_2$ are disjoint compact subsets of the hyperspace $\mathrm{CC}(\mathbb{R}^2)$. In fact, it can be shown that

$$\begin{aligned}\operatorname{dist}(\mathcal{K}_1, \mathcal{K}_2) &= \inf\{h(K_1, K_2) : K_1 \in \mathcal{K}_1, K_2 \in \mathcal{K}_2\} \\ &= \inf\left\{h\left(\sum_{i=1}^{3} \alpha_i A_i, D\right) : \alpha_i \geq 0, \sum_{i=1}^{3} \alpha_i = 1\right\} \\ &= h\left(\frac{A_1}{3} + \frac{A_2}{3} + \frac{A_3}{3}, D\right) = \frac{\sqrt{3}}{6},\end{aligned}$$

where $A_1/3 + A_2/3 + A_3/3 = \operatorname{conv}(y_1, y_2, y_3, y_4, y_5, y_6)$, where $y_1 = 2x_1/3 + x_2/3$, $y_2 = x_1/3 + 2x_2/3$, and thus the line segment $[y_1, y_2]$ is the middle one-third of $[x_1, x_2] = A_1$, etc. However, since each $x^* \colon (D, \|\cdot\|_2) \to (\mathbb{R}^1, |\cdot|)$ is continuous linear, x^* attains its maximum and minimum on extreme points of D. Assume $x^*(x_1) = \min\{x^*(x) : x \in D\} = a$ and $x^*(x_2) = \max\{x^*(x) : x \in D\} = b$. We have then $x^* \colon (\mathrm{CC}(\mathbb{R}^2), h) \to (\mathrm{CC}(\mathbb{R}^1), h)$ is such that $x^*(A_1) = [a, b] \in \mathrm{CC}(\mathbb{R}^1)$ and $x^*(D) = [a, b] \in \mathrm{CC}(\mathbb{R}^1)$ and consequently

$$\begin{aligned}\operatorname{dist}(x^*(\mathcal{K}_1), x^*(\mathcal{K}_2)) &= \min\{h(x^*(K_1), x^*(K_2)) : K_1 \in \mathcal{K}_1, K_2 \in \mathcal{K}_2\} \\ &= h(x^*(A_1), x^*(D)) = h([a, b], [a, b]) = 0.\end{aligned}$$

It is well-known that in the underlying Banach space X, every norm-closed convex set K is weakly closed ($\mathcal{T}_w$-closed) and hence if a sequence $\{x_n\}$ converges weakly to x ($x_n \to x$), then there exists a sequence $\{y_n\}$ which are convex combinations of the x_n's such that $y_n \to x$ strongly which is Mazur's theorem. We show in the following example that the above result cannot be extended to the hyperspace WCC(X).

Example 3.7. Suppose $\ell_2 = \left\{a = (a_1, a_2, \ldots) : \sum |a_i|^2 < \infty\right\}$ is the Hilbert space with norm $\|a\| = \sum_{i=1}^{\infty} \left|a_i^2\right|^{1/2}$, $B_1 = \left\{a \in \ell^2 : \|a\| \leq 1\right\}$, and $A_n = \{a = (a_1, a_2, \ldots, a_n, 0, 0, \ldots) : \|a\| \leq 1\}$. Then $A_n \xrightarrow{\mathcal{T}_w} B_1$, $A_n \overset{h}{\not\to} B_1$. Moreover $\mathcal{K} = \operatorname{conv}\{A_1, A_2, \ldots, A_n, \ldots\} = \left\{D \in \mathrm{WCC}(\ell_2) : D = \sum_{i=j}^{k} \alpha_j A_{ij}, \sum_{j=1}^{k} \alpha_j = 1, 0 \leq \alpha_j \leq 1\right\} \subset \mathrm{WCC}(\ell_2)$, $\overline{\mathcal{K}}$ is an h-closed convex subset of the hyperspace WCC(ℓ_2) but $\overline{\mathcal{K}}$ is not $\mathcal{T}_w$-closed.

Proof. Let $e_n = \{0, 0, \ldots, 0, 1, 0, 0, \ldots\} \in B_1$ with the n^{th} co-ordinate equal to 1 and 0 elsewhere. Then, we have $\operatorname{dist}(e_{n+1}, A_n) = 1$ which implies that $h(B_1, A_n) = 1$ for $n = 1, 2, 3, \ldots$, which in turn implies that $A_n \overset{h}{\nrightarrow} B_1$. Also, if $D \in \operatorname{conv}(A_1, A_2, \ldots)$, we have $D = \alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \cdots + \alpha_k A_{i_k}$ (assume $i_1 < i_2 < \cdots < i_k$ w.l.o.g), $\alpha_j \geq 0$, $\sum_{j=1}^{k} \alpha_j = 1$. Thus $D \subset A_{i_k} \subset B_1$ and thus $h(D, B_1) \geq h(A_{i_k}, B_1) = 1$. Consequently $\operatorname{dist}(B_1, \operatorname{conv}(A_1, A_2, \ldots)) \geq 1$ which in turn implies that $\operatorname{dist}(B_1, \overline{\operatorname{conv}(A_1, A_2, \ldots)} = \overline{\mathcal{K}}) = 1$.

On the other hand, let $x^* = (x_1, x_2, \ldots) \in \ell_2^*$, $y_n^* = (0, 0, \ldots, 0, x_n, x_{n+1}, \ldots) \in \ell_2^*$. Then $\|x^*\| < \infty$, and $\|y_n^*\| \to 0$, and we have $|y_n^*(b)| \leq \|y_n^*\| \, \|b\| \leq \|y_n^*\| \to 0$ for each $b \in B_1$. Thus for each $b = (b_1, b_2, \ldots) \in B_1$, if we let $b^{(n)} = (b_1, b_2, \ldots, b_n, 0, 0, \ldots)$, we have $b^{(n)} \in A_n$ and $\left|x^*(b) - x^*(b^{(n)})\right| = |\sum_{i=1}^{\infty} b_i x_i - \sum_{i=1}^{n} b_i x_i| = \left|\sum_{i=n+1}^{\infty} b_i x_i\right| = \left|y_{n+1}^*(b)\right| \leq \left\|y_{n+1}^*\right\| \to 0$ as $n \to \infty$. Hence for any given $\varepsilon > 0$, given $x^*(b)$, choose n_0 such that $n \geq n_0$ implies $\|y_n^*\| < \varepsilon/2$. Consequently, for each $x^*(b) \in x^*(B_1)$, and $n \geq n_0$, there exists $b^{(n)} \in A_n$ such that $d(x^*(b), x^*(b^{(n)})) = |x^*(b) - x^*(b^n)| < \varepsilon$ and we have $x^*(B) \subseteq N(x^*(A_n), \varepsilon)$. Also $A_n \subset B$ implies $x^*(A_n) \subset N(x^*(B), \varepsilon)$. It follows from the definition of Hausdorff metric that $h(x^*(B_1), x^*(A_n)) < \varepsilon$. Now we have $x^*(A_n) \xrightarrow{h} x^*(B_1)$ for each $x^* \in \ell_2^*$. That is, $A_n \xrightarrow{\mathcal{T}_w} B_1$ (or $A_n \to B_1$ weakly in the hyperspace $(\operatorname{WCC}(\ell_2), \mathcal{T}_w)$. Hence $B_1 \in \overline{\mathcal{K}}^{\mathcal{T}_w}$, but $B_1 \notin \overline{\mathcal{K}}$. Thus Mazur's Theorem is not valid on hyperspaces. □

Remark 3.8. Let $\overline{X} = \{\overline{x} = \{x\} : x \in X\}$. Then $(\overline{X}, h) \subset (\operatorname{WCC}(X), h)$ and $(\overline{X}, h)$ is isomorphic to $(X, \|\cdot\|)$; $(\overline{X}, \mathcal{T}_w)$ is isomorphic to (X, τ_w). Thus every theorem proved on the hyperspace is a natural extension of its counterpart on the original space as illustrated by Blaschke's Theorem and Bolzano-Weierstrass Theorem. However, from the two given examples of this paper, we see there are many results that are valid on the original space but cannot be extended analogously to the hyperspace. In fact, we do not know whether the classical Eberlein-Smulian Theorem can be extended to the hyperspace or not. Also Krein-Milman's extreme point theorem has many important applications. De Branges [5] used it to prove the famous Stone-Weierstrass Theorem. We hope further investigation in this area will lead to more results and useful applications on the hyperspace.

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