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Stability of Traveling Wave Fronts for Nonlocal Diffusion Equation with Delayed Nonlocal Response

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Abstract. In this paper, we consider with the stability of traveling wave fronts for the nonlocal diffusion equation with delay and global response. We first establish the existence and comparison theorem of solutions for the nonlocal reaction-diffusion equation by appealing to the theory of abstract functional differential equation. Then we further show that the traveling wave fronts are asymptotical stability with phase shift. Our main technique is the super and subsolution method coupled with the comparison principle and squeezing method.

1. Introduction

The reaction diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)(1-u(x,t)), \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

has been investigated by Fisher [16] to model the spatial spreading of a mutant in an infinite one-dimensional habitat. Since then, the traveling waves for reaction-diffusion systems have been widely studied in biology, chemistry, epidemiology and physics, see [2, 4, 14, 15, 35]. There are many methods to deal with the existence and stability of traveling waves, for example, the phase space analysis [33] or the Conley index [36] for the proof of the existence, and spectral analysis and energy estimates [18, 21, 25, 32, 41] or squeezing technique based on comparison principle as well as super and subsolution [7, 34] for the study of the stability.

As mentioned by Murray [26], since some biological processes are free and random, the Laplacian operator as a local operator is not accurately describe the phenomenon of spatial diffusion. To overcome these problems with the Laplacian operator, some researchers

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introduce an integral operator describing the spatial diffusion. A typical mathematical model is the following form

$$\frac{\partial u(x,t)}{\partial t} = d\left(\int_{\mathbb{R}} J(x-y)u(y,t)\,dy - u(x,t)\right) + f(u(x,t)), \quad x \in \mathbb{R},\ t > 0. \tag{1.2}$$

Meanwhile, many researchers give more attention on the study of traveling waves of nonlocal reaction diffusion equation (1.2). The existence, nonexistence, uniqueness, propagation speed and stability of traveling waves for (1.2) are widely studied, see [6, 9–12, 39] and the references cited therein. Obviously, the model (1.2) is closely related to local reaction-diffusion model. If we set the diffusion kernel $J(x) = \delta(x) + \delta''(x)$, where $\delta(\cdot)$ is the Dirac delta function, then (1.2) can be reduced to the local equation $\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^2 u(x,t)}{\partial x^2} + f(u(x,t))$, (see Medlock [24]).

Due to the practical background, delays and nonlocal delays are incorporated into reaction-diffusion equations, see [27, 28, 34, 38] and the references therein. For example, authors in [29] showed the existence, symptotic behavior, uniqueness and stability of the traveling wave fronts for (1.2) with $f(u(x,t), u(x,t-\tau)) = -du(x,t) + b(u(x,t-\tau))$, where $d > 0$, b is a continuous function. In [37], Wang et al. considered the existence and stability of traveling wave fronts in reaction-advection-diffusion with nonlocal delay and the effect of the advection term for the wave speed.

Notice that the drift of some individuals depends on their present positions from all possible positions at previous time, Britton in [5] first considered the model for the reaction term with delayed nonlocal response to address this phenomena. For instance, the model with local diffusion and nonlocal response is as follows

$$\frac{\partial u(x,t)}{\partial t} = d\frac{\partial^2 u(x,t)}{\partial x^2} + f\left(u(x,t), \int_{-\infty}^{\infty} k(y)u(x-y,t-\tau)\,dy\right), \quad x \in \mathbb{R},\ t > 0, \tag{1.3}$$

where $d > 0$, $\tau > 0$ and k is nonnegative kernel function. Since traveling waves play an important role in understanding the dynamics of (1.3), the model (1.3) has attracted significant attention. Under some monostable assumptions, Wang et al. [38] studied the existence, uniqueness and global asymptotical stability of traveling wave fronts for (1.3). In particular, equation (1.3) with $f(u,v) = -au + b(1-u)v$ is proposed by Ruan and Xiao [31]. When $f(u,v) = bv\exp\{-\gamma\tau\} - \delta u^2$, and $k(y) = \frac{1}{4\pi\alpha\tau}\exp\left\{\frac{-y^2}{4\alpha\tau}\right\}$, equation (1.3) is the age-structured reaction-diffusion model of a single species proposed by Al-Omari and Gourley [1]. In recent years, Guo and Zimmer studied a spatially discrete version of reaction-diffusion equations with delayed nonlocal response

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= d\left[u(x+1,t) - 2u(x,t) + u(x-1,t)\right] \\ &\quad + f\left(u(x,t), \int_{-\infty}^{\infty} k(y)u(x-y,t-\tau)\,dy\right).\end{aligned} \tag{1.4}$$

They have studied the existence and uniqueness of traveling waves for (1.4) in [19] and the stability of traveling waves for (1.4) in [20]. At the same time, Yu and Yuan in [40] also incorporated the nonlocal delayed response into nonlocal diffusion model (1.2), that is

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= d\left(\int_{\mathbb{R}} J(x-y)u(y,t)\,dy - u(x,t)\right) \\ &\quad + f\left(u(x,t), \int_{-\infty}^{\infty} k(y)u(x-y,t-\tau)\,dy\right),\end{aligned} \tag{1.5}$$

where $d > 0$, $\tau > 0$ and J, k are nonnegative kernel functions. They have investigated the existence, asymptotic behavior and uniqueness of traveling wave fronts for equation (1.5).

Motivated by these, in this paper, we further study the stability of the traveling wave fronts by using the squeezing technique based on comparison principle as well as super and subsolution. Authors in [3, 8, 36] obtained the stability of traveling wave fronts for several evolution equations by considering the long time behavior of corresponding Cauchy type problem if its initial value is a spatial disturbance of a traveling wave front. Due to these, our results can imply that the traveling wave fronts of nonlocal diffusion model with delayed nonlocal response are useful to apprehend the long term dynamical behavior of the corresponding initial value problem.

In this paper, for the functions J, k, $f(u,v)$, we impose the following conditions.

(J) $J(x) = J(-x) \geq 0$, $\int_{\mathbb{R}} J(x)\,dx = 1$, $\int_{\mathbb{R}} J(x)e^{-\lambda x}\,dx < \infty$, $\forall\,\lambda \geq 0$.

(K) $k(x) = k(-x) \geq 0$, $\int_{\mathbb{R}} k(x)\,dx = 1$, $\int_{\mathbb{R}} k(x)e^{-\lambda x}\,dx < \infty$, $\forall\,\lambda \geq 0$.

(A1) $f \in C^1(\mathbb{R}^2, \mathbb{R})$, $f(0,0) = f(K,K) = 0$, $f(u,u) > 0$ for $u \in (0,K)$, and $\partial_2 f(u,v) \geq 0$ for $(u,v) \in [0,K]^2$, where K is a positive constant.

(A2) $\partial_1 f(0,0)u + \partial_2 f(0,0)v \geq f(u,v)$ for any $(u,v) \in [0,K]^2$.

(A3) There exist numbers $L, \kappa > 0$ and $\sigma_1, \sigma_2 \in (0,1]$ such that

$$|f(u,v) - \partial_1 f(0,0)u - \partial_2 f(0,0)v| \leq L(u^{1+\sigma_1} + v^{1+\sigma_2})$$

for any $(u,v) \in [0,\kappa]^2$.

(A4) There exists a constant $\delta > 0$ such that

$$\partial_1 f(u_1,v_1) \geq \partial_1 f(u_2,v_1), \quad \partial_2 f(u_1,v_1) > \partial_2 f(u_2,v_2)$$

for any $(u_i,v_i) \in [0,(1+2\delta)K]^2$ satisfying $u_1 < u_2$, $v_1 < v_2$, $(i = 1,2)$.

We are interested in traveling waves that connect the two equilibria 0 and K. Without loss of generality, throughout this paper, a traveling wave solution of (1.5) always refers to a solution of special form of $u(x,t) = \phi(x+ct)$, $x \in \mathbb{R}$, $t > 0$, with the speed $c > 0$. Namely, the wave profile ϕ satisfies the following functional differential equation

$$(1.6) \quad c\phi'(\xi) = d\left(\int_{\mathbb{R}} J(\xi - y)\phi(y)\,dy - \phi(\xi)\right) + f\left(\phi(\xi), \int_{-\infty}^{\infty} k(y)\phi(\xi - c\tau - y)\,dy\right)$$

and the asymptotic boundary conditions

$$(1.7) \quad \lim_{\xi\to\infty} \phi(\xi) = K, \quad \lim_{\xi\to-\infty} \phi(\xi) = 0,$$

where $\xi = x + ct$. We say that a traveling wave solution $\phi(x+ct)$ is called the traveling wave front if ϕ is monotone, that is, $\phi(\cdot)\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing function.

The rest of this paper is organized as follows. In Section 2, we show the existence and uniqueness of the traveling wave fronts obtained in [40]. In Section 3, we consider the corresponding initial value problem of (1.5) by appealing to the theory of abstract functional differential equations and operator semigroup. For the Section 4, we get the stability of traveling wave fronts by applying the squeezing technique. In the end, we show the applications to another version of the classical logistic model and the Nicholoson's blowflies model with delayed nonlocal response.

2. Existence and uniqueness of traveling wave fronts

In this section, we will show the existence and uniqueness of traveling wave fronts of equation (1.5).

Theorem 2.1. (Existence, [40, Theorem 1.1]) *Assume that* (J), (K) *and* (A1)–(A3) *hold. Then there exists a positive constant c_* such that for each $c \ge c_*$, the equation* (1.5) *admits a nondecreasing positive traveling wave front $u(x,t) = \phi(x+ct)$ with $\phi(-\infty) = 0$ and $\phi(+\infty) = K$. Moreover, if $c > c_*$, then*

$$(2.1) \quad \lim_{\xi\to-\infty} \phi(\xi)e^{-\lambda_1\xi} = 1, \quad \lim_{\xi\to-\infty} \phi'(\xi)e^{-\lambda_1\xi} = \lambda_1,$$

where $\xi = x + ct$, $\lambda_1 > 0$ is the smallest root of the equation

$$(2.2) \quad \Delta(c,\lambda) = c\lambda - d\left[\int_{\mathbb{R}} J(y)e^{-\lambda y}\,dy - 1\right] - \partial_1 f(0,0) - \partial_2 f(0,0)\int_{\mathbb{R}} k(y)e^{-\lambda(y+c\tau)}\,dy = 0.$$

Theorem 2.2. (Uniqueness, [40, Theorem 1.3]) *Assume that* (J), (K) *and* (A1)–(A3) *hold. For $c \ge c_*$, let ϕ, ψ be two traveling wave fronts of* (1.5) *with wave speed c. Then ϕ is a translation of ψ; more precisely, there exists $\overline{\xi} \in \mathbb{R}$ such that $\phi(\xi) = \psi(\xi + \overline{\xi})$.*

Here we give a pair of super and subsolutions which have been obtained in the proof of the existence of the traveling wave fronts for (1.5) in [40].

Define continuous functions $\phi^+(\xi)$ and $\phi^-(\xi)$ as follows

$$\phi^+(\xi) = \min\left\{e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi}, K\right\}, \quad \phi^-(\xi) = \max\left\{e^{\lambda_1\xi} - qe^{\eta\lambda_1\xi}, 0\right\}, \tag{2.3}$$

where $\eta \in \left(1, \min\left\{1+\sigma_1, 1+\sigma_2, \frac{\lambda_2}{\lambda_1}\right\}\right)$, $q > 1$, λ_1, λ_2 are the roots of the equation (2.2) satisfying $\lambda_2 > \lambda_1$. It is clear that for sufficiently large q, we have that $\phi^+(\xi)$ and $\phi^-(\xi)$ satisfy $0 \le \phi^-(\xi) \le \phi^+(\xi) \le K$, $\sup_{\xi\in\mathbb{R}} \phi^-(\xi) > 0$ and $\inf_{\xi\in\mathbb{R}} \phi^+(\xi) < K$.

Under the assumption conditions (J), (K) and (A1)–(A3), it is easy to show that $\phi^+(\xi)$ and $\phi^-(\xi)$ are a pair of super and subsolution of (1.6), respectively. We can find the proof in [40]. Here we omit it.

3. Existence and comparison of solutions for the initial value problem

In this section, we consider the following initial value problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\displaystyle\int_{\mathbb{R}} J(y)[u(x-y,t) - u(x,t)]\,dy \\ \qquad\qquad + f\left(u(x,t), \displaystyle\int_{\mathbb{R}} k(y)u(x-y,t-\tau)\,dy\right), \\ u(x,s) = \varphi(x,s), \quad x \in \mathbb{R},\ -\tau \le s \le 0, \end{cases} \tag{3.1}$$

where $\varphi(\cdot,s) \in C([-\tau,0],X)$ and X is defined as follows

$$X = \{u(x) \mid u(x)\colon \mathbb{R} \to \mathbb{R} \text{ is bounded and uniformly continuous}\}.$$

It is clearly to see that X is a Banach space with the usual supremum norm $|\cdot|$. Define X^+ as $X^+ = \{u \in X \mid u(x) \ge 0, x \in \mathbb{R}\}$. It is easy to see that X^+ is a closed cone of X with respect to the general partial ordering. Define $Au(x) := d\int_{\mathbb{R}} J(y)[u(x-y) - u(x)]\,dy - \beta u(x)$ be a bounded linear operator on X, where β is a given positive constant satisfying $\beta \ge \max_{(u,v)\in[0,K]^2} |\partial_1 f(u,v)|$, we can obtain that $T(t) = e^{tA}$ is a uniformly continuous semigroup of A on X. Indeed, according to Corollary 1.4 in [30] and the assumption (J), it is easy to show that $T_1(t) = e^{tB}$ is a strong positive semigroup in the sense $T_1(t)\colon X^+ \to X^+$ and its operator norm is less than 1 for any $t > 0$, where $Bu(x) = d\int_{\mathbb{R}} J(y)u(x-y)\,dy$ is a bounded positive linear operator on X. In [30], we can find more details of the operator semigroup.

Hence, by the first section in [13], it then follows to obtain that the unique mild solution of the following initial value problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\displaystyle\int_{\mathbb{R}} J(y)[u(x-y,t) - u(x,t)]\,dy - \beta u(x,t), \\ u(x,s) = \varphi(x) \in X, \end{cases} \tag{3.2}$$

is given by $u(x,t) = T(t)\varphi(x)$.

Let $C = C([-\tau,0];X)$ be the Banach space of continuous functions from $[-\tau,0]$ into X with the supremum norm and $C^+ = \{\varphi \in C \mid \varphi(s) \in X^+, s \in [-\tau,0]\}$. Then C^+ is a positive cone of C. For any continuous function $w(\cdot)\colon [-\tau,b) \to X$, $b > 0$, we define $w_t \in C$, $t \in [0,b)$, by $w_t(s) = w(t+s)$, $s \in [-\tau,0]$. We can easily see that $t \to w_t$ is a continuous function from $[0,b)$ to C.

For any $\psi \in [0,K]_C = \{\psi \in C \mid \varphi(x,s) \in [0,K], x \in \mathbb{R}, s \in [-\tau,0]\}$, we define

$$F(\psi)(x) = f\left(\psi(x,s), \int_{\mathbb{R}} k(y)\psi(x-y,s-\tau)\,dy\right) + \beta\psi(x,s), \quad x \in \mathbb{R}.$$

By the Lipschitz continuity of $f(\cdot,\cdot)$ on $[0,K]^2$, we can obtain that $F(\psi) \in X$ and $F\colon [0,K]_C \to X$ is Lipschitz continuous.

By the theory of abstract functional differential equations in [22,23] and the analyticity of semigroup $T(t)$ in [13,30], we can obtain the following results of solutions for the initial value problem (3.1).

Theorem 3.1. *Assume that* (J), (K), *and* (A1)–(A3) *hold. Then for any* $\varphi(\cdot,s) \in [0,K]_C$, $s \in [-\tau,0]$, (3.1) *has a unique mild solution* $u(x,t)$ *defined for all* $(x,t) \in \mathbb{R} \times (0,\infty)$. *The form of the solution to* (3.1) *is as follows*

$$u(x,t) = T(t)\varphi(x,0) + \int_0^t T(t-r)F(u_r)(x)\,dr. \tag{3.3}$$

Proof. Under the theory of abstract functional differential equation, a mild solution of (3.1) is a solution to its associated integral equation

$$\begin{cases} u(x,t) = T(t)\varphi(x,0) + \displaystyle\int_0^t T(t-r)F(u_r)(x)\,dr, & t > 0, \\ u(x,s) = \varphi(x,0). \end{cases}$$

As we have mentioned, $F\colon [0,K]_C \to X$ is Lipschitz continuous. We further claim that F is quasi-monotone on $[0,K]_C$ in the sense that

$$\lim_{h\to 0^+} \frac{1}{h}\operatorname{dist}(\psi_1(0) - \psi_2(0) + h[F(\psi_1) - F(\psi_2)]; X^+) = 0, \tag{3.4}$$

for all $\psi_1, \psi_2 \in [0,K]_C$ with $\psi_1 \geq \psi_2$. Indeed, it follows from the condition (A1) and the choice of β that

$$
\begin{aligned}
F(\psi_1)(\cdot) - F(\psi_2)(\cdot) &= f\left(\psi_1(\cdot,0), \int_{\mathbb{R}} k(y)\psi_1(\cdot - y, -\tau)\,dy\right) \\
&\quad - f\left(\psi_2(\cdot,0), \int_{\mathbb{R}} k(y)\psi_2(\cdot - y, -\tau)\,dy\right) \\
&\quad + \beta(\psi_1(\cdot,0) - \psi_2(\cdot,0)) \\
&\geq f\left(\psi_1(\cdot,0), \int_{\mathbb{R}} k(y)\psi_2(\cdot - y, -\tau)\,dy\right) \\
&\quad - f\left(\psi_2(\cdot,0), \int_{\mathbb{R}} k(y)\psi_2(\cdot - y, -\tau)\,dy\right) \\
&\quad + \beta(\psi_1(\cdot,0) - \psi_2(\cdot,0)) \\
&\geq -(L_1 - \beta)(\psi_1(\cdot,0) - \psi_2(\cdot,0)) \\
&\geq 0,
\end{aligned}
\tag{3.5}
$$

where $L_1 = \max_{(u,v)\in[0,K]^2} |\partial_1 f(u,v)|$. Hence, for any $h > 0$, we have

$$\psi_1(0) - \psi_2(0) + h[F(\psi_1) - F(\psi_2)] \geq (1 - (L_1 - \beta)h)(\psi_1(0) - \psi_2(0)) \geq 0.$$

Then the existence and uniqueness of $u(x,t,\varphi)$ follows from the theory of abstract functional differential equations in [22]. Moreover, by a semigroup theory argument given in the proof of Theorem 1 in [22], it follows that $u(x,t,\varphi)$ is a classical solution of (3.1) for $t \geq -\tau$. □

Definition 3.2. Assume that $u(\cdot,t) \in X$ for $t \in [-\tau, b)$, $0 < b \leq \infty$, and $u(\cdot,t)$ is continuous in $t \in [-\tau, b)$, then $u(x,t)$ is called a supersolution (subsolution) of (3.3) on $[0,b)$ if

$$u(x,t) \geq (\leq) T(t-s)u(x,s) + \int_s^t T(t-r)F(u_r)(x)\,dr \tag{3.6}$$

for all $-\tau \leq s < t < b$. If u is both a supersolution and a subsolution on $[0,b)$, then it is a mild solution of (3.3).

Remark 3.3. Assume that there is a function $u(x,t) \in C(\mathbb{R} \times [-\tau, b], \mathbb{R})$, $b > 0$ satisfying the following inequalities

$$
\begin{cases}
\dfrac{\partial u(x,t)}{\partial t} \geq (\leq) d\displaystyle\int_{\mathbb{R}} J(y)[u(x-y,t) - u(x,t)]\,dy \\
\qquad\qquad + f\left(u(x,t), \displaystyle\int_{\mathbb{R}} k(y)u(x-y,t-\tau)\,dy\right), \\
u(x,s) \geq (\leq) \varphi(x,s), \quad x \in \mathbb{R},\ -\tau \leq s \leq 0.
\end{cases}
\tag{3.7}
$$

Then, by the positivity of the linear semigroup $T(t)$, it easily follows that (3.6) holds. Hence, $u(x,t)$ is a supersolution (subsolution) of (3.1) on $[0,b)$.

Now we establish the following comparison theorem for (3.1).

Theorem 3.4. *Assume that* (J), (K), *and* (A1)–(A3) *hold. Then for any pair of supersolution* $u(x,t)$ *and subsolution* $w(x,t)$ *of* (3.1) *with* $0 \le u(x,t),\ w(x,t) \le K$, $x \in \mathbb{R}$, $t \in [-\tau, \infty)$, *and* $u(x,s) \ge w(x,s)$, $x \in \mathbb{R}$, $s \in [-\tau, 0]$, *there holds* $u(x,t) \ge w(x,t)$ *for* $x \in \mathbb{R}$, $t \ge 0$, *and*

$$u(x,t) - w(x,t) \ge e^{-(L_1-\beta)(t-t_0)} T(t-t_0)(u(x,t_0) - w(x,t_0)), \quad t > t_0 \ge 0. \tag{3.8}$$

Proof. Clearly, $u^+ = K$ and $u^- = 0$ are an ordered pair of super and subsolution of (3.1). For simplicity, let $\phi(x,s) = u(x,s)$, $\varphi(x,s) = w(x,s)$, $x \in \mathbb{R}$, $s \in [-\tau, 0]$. Then $\phi, \varphi \in [0,K]_C$ with $\phi \ge \varphi$. By Corollary 5 in [22], we have that

$$K \ge u(x,t,\phi) \ge u(x,t,\varphi) \ge 0, \quad x \in \mathbb{R},\ t \ge 0. \tag{3.9}$$

By Corollary 5 in [22] with $u^+(x,t) = K$ and $u^-(x,t) = w(x,t)$, $u^+(x,t) = u(x,t)$ and $u^-(x,t) = 0$, respectively, we get that

$$w(x,t) \le u(x,t,\varphi) \le K, \quad x \in \mathbb{R},\ t \ge 0, \tag{3.10}$$

and

$$0 \le u(x,t,\phi) \le u(x,t), \quad x \in \mathbb{R},\ t \ge 0. \tag{3.11}$$

Combining (3.9)–(3.11), we have that $u(x,t) \ge w(x,t)$ for all $x \in \mathbb{R}$, $t \ge 0$.

Next we prove the inequality (3.8) in the theorem. Let $v(x,t) = u(x,t) - w(x,t)$, then $v(x,t) \ge 0$, $x \in \mathbb{R}$, $t \ge 0$. By Definition 3.2 and (3.5), it then follows that, for all $t \ge t_0 \ge 0$,

$$\begin{aligned} v(x,t) &\ge T(t-t_0)v(x,t_0) + \int_{t_0}^{t} T(t-r)(F(u_r)(x) - F(w_r)(x))\,dr \\ &\ge T(t-t_0)v(x,t_0) - (L_1 - \beta)\int_{t_0}^{t} T(t-r)v_r(x)\,dr. \end{aligned} \tag{3.12}$$

Define $z(t) = e^{-(L_1-\beta)(t-t_0)}T(t-t_0)v(t_0)$, $t \ge 0$, it is easy to show that $z(t)$ satisfies

$$z(t) = T(t-t_0)z(t_0) - (L_1 - \beta)\int_{t_0}^{t} T(t-r)z(r)\,dr.$$

Then by Proposition 3 in [22], we have that $v(x,t) \ge z(t) = e^{-(L_1-\beta)(t-t_0)}T(t-t_0)v(t_0)$ for all $t \ge t_0$.

So we can obtain that

$$u(x,t) - w(x,t) \ge e^{-(L_1-\beta)(t-t_0)}T(t-t_0)(u(x,t_0) - w(x,t_0)).$$

This completes the proof. □

Remark 3.5. By Theorem 3.4, it follows that if $\varphi^+(x,t)$ and $\varphi^-(x,t)$ are a pair of supersolution and subsolution of (3.1) and $\varphi^+(x,0) \not\equiv \varphi^-(x,0)$, then for any $t>0$,

$$\varphi^+(x,t) - \varphi^-(x,t) \geq e^{-(L_1-\beta)t}T(t)(\varphi^+(x,0) - \varphi^-(x,0)) > 0.$$

In particular, if $u(x,t,\varphi)$ is a solution of (3.1) with initial value $\varphi \in [0,K]_C$ and $\varphi(x,0)$ ($\not\equiv$ constant) is nondecreasing one, then $\frac{\partial}{\partial x}u(x,t) > 0$, for any $t \geq -\tau$, $x \in \mathbb{R}$.

Lemma 3.6. *Assume that assumptions* (J), (K), *and* (A1)–(A3) *hold. Let* $\phi(x+ct)$ *be a nondecreasing traveling wave solution of* (1.5), *then* $\phi'(\xi) > 0$ *for* $\xi \in \mathbb{R}$.

4. Asymptotic stability of traveling wave fronts

In this section, for $c > c_*$, we will use the squeezing technique to get the stability of the unique traveling wave front. First we construct a pair of super and subsolutions which depend on the traveling wave front.

Lemma 4.1. *Assume that* (J), (K), *and* (A1)–(A4) *hold and* $\phi(x+ct)$ *is a traveling wave front of* (1.5) *given by Theorem* 2.1. *Then there exist three positive numbers* β_0 *(which is independent of* ϕ*),* σ_0 *and* $\delta \in (0,1)$ *such that for any* $\epsilon \in (0,\delta]$ *and* $\xi^\pm \in \mathbb{R}$*, the following continuous functions* $u^\pm$ *defined by*

$$u^+(x,t) = \min\left\{(1+\epsilon e^{-\beta_0 t})\phi(x+ct+\xi^+ - \sigma_0\epsilon e^{-\beta_0 t}), K\right\}, \tag{4.1}$$

and

$$u^-(x,t) = (1-\epsilon e^{-\beta_0 t})\phi(x+ct+\xi^- + \sigma_0\epsilon e^{-\beta_0 t}), \tag{4.2}$$

are a pair of supersolution and subsolution of (1.5), *respectively.*

Proof. We only prove that $u^+(x,t)$ is a supersolution of (1.5) since the proof for $u^-(x,t)$ is analogous. Fix $\delta \in (0,1)$, by the assumption (A4), we can choose $\beta_0 > 0$, $\kappa > 0$ small enough such that $\delta e^{\beta_0 \tau} \leq 2\delta$,

$$\partial_2 f(0,\mu)e^{-\beta_0\tau} > \partial_2 f(\phi(\xi),\eta), \tag{4.3}$$

and

$$\begin{aligned} &K\bigg\{\bigg[\partial_1 f\left(\gamma, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right) + \partial_2 f(0,\mu) - \beta_0 \\ &\qquad - \partial_1 f\left(\zeta, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right)\bigg] e^{-\beta_0\tau} - \partial_2 f(\phi(\xi),\eta)\bigg\} \\ &> \kappa\bigg\{\bigg[\partial_1 f\left(\gamma, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right) + \partial_2 f(0,\mu) \\ &\qquad - \partial_1 f\left(\zeta, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right)\bigg] e^{-\beta_0\tau} - \partial_2 f(\phi(\xi),\eta)\bigg\}, \end{aligned} \tag{4.4}$$

where $\gamma \in (0,\phi(\xi))$, $\mu \in (0,\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy)$, $\zeta \in (\phi(\xi),(1+\delta)\phi(\xi))$, and $\eta \in (\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy, (1+\delta e^{\beta_0\tau})\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy)$.

According to $\lim_{\xi\to\infty}\phi(\xi)=K$, we can take $M_1>0$ sufficiently large so that

$$\phi(\xi) \ge K - \frac{\kappa}{2} \quad \text{for } \xi \ge M_1. \tag{4.5}$$

By (2.1), we can take $M_2>0$ sufficiently large such that

$$\frac{1}{2} < \phi(\xi)e^{-\lambda_1\xi} < \frac{3}{2}, \quad \phi'(\xi)e^{-\lambda_1\xi} > \frac{1}{2}\lambda_1 \quad \text{for } \xi \le -M_2. \tag{4.6}$$

Denote $\rho := \min\{\phi'(\xi) : -M_2 \le \xi \le M_1\} > 0$. Finally, choose $\sigma_0>0$ sufficiently large so that

$$\begin{aligned}\sigma_0 > \max\Bigg\{ &\frac{3e^{\beta_0\tau}}{\lambda_1\beta_0}\left[\left(\beta_0 + \max_{(u,v)\in[0,K]^2}\partial_1 f(u,v)\right)e^{-\beta_0\tau} + \max_{(u,v)\in[0,K]^2}\partial_2 f(u,v)\right],\\ &\frac{e^{\beta_0\tau}K}{\rho\beta_0}\left[\left(\beta_0 + \max_{(u,v)\in[0,K]^2}\partial_1 f(u,v)\right)e^{-\beta_0\tau} + \max_{(u,v)\in[0,K]^2}\partial_2 f(u,v)\right]\Bigg\}.\end{aligned} \tag{4.7}$$

Define

$$B^+ = \left\{(x,t) \mid (1+\epsilon e^{-\beta_0 t})\phi(x+ct+\xi^+ - \sigma_0\epsilon e^{-\beta_0 t}) > K\right\}$$

and

$$B^- = \left\{(x,t) \mid (1+\epsilon e^{-\beta_0 t})\phi(x+ct+\xi^+ - \sigma_0\epsilon e^{-\beta_0 t}) \le K\right\}.$$

When $(x,t)\in B^+$, it is easy to show that $u^+(x,t)=K$. By the assumption (A1), we can get that

$$\begin{aligned} L[u^+](x,t) := {}& \frac{\partial u^+(x,t)}{\partial t} - d\int_{\mathbb{R}} J(y)[u^+(x-y,t)-u^+(x,t)]\,dy\\ &- f\left(u^+(x,t), \int_{\mathbb{R}} k(y)u^+(x-y,t-\tau)\,dy\right)\\ \ge {}& 0. \end{aligned}$$

When $(x,t)\in B^-$, $u^+(x,t) = (1+\epsilon e^{-\beta_0 t})\phi(x+ct+\xi^+-\sigma_0\epsilon e^{-\beta_0 t})$. For any $\epsilon\in(0,\delta]$ and $t\ge 0$, set $\xi = x+ct+\xi^+-\sigma_0\epsilon e^{-\beta_0 t}$, by (1.6) and Lemma 3.6, we have

$$\begin{aligned} &L[u^+](x,t)\\ &= -\epsilon\beta_0 e^{-\beta_0 t}\phi(\xi) + (c+\sigma_0\epsilon\beta_0 e^{-\beta_0 t})(1+\epsilon e^{-\beta_0 t})\phi'(\xi)\\ &\quad - d\int_{\mathbb{R}} J(y)(1+\epsilon e^{-\beta_0 t})(\phi(\xi-y)-\phi(\xi))\,dy\\ &\quad - f\left((1+\epsilon e^{-\beta_0 t})\phi(\xi), (1+\epsilon e^{-\beta_0(t-\tau)})\int_{\mathbb{R}} k(y)\phi\left(\xi-y-c\tau-\sigma_0\epsilon e^{-\beta_0 t}(e^{\beta_0\tau}-1)\right)dy\right)\\ &\ge -\epsilon\beta_0 e^{-\beta_0 t}\phi(\xi) + \sigma_0\epsilon\beta_0 e^{-\beta_0 t}\phi'(\xi) + (1+\epsilon e^{-\beta_0 t}) f\left(\phi(\xi), \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right)\\ &\quad - f\left((1+\epsilon e^{-\beta_0 t})\phi(\xi), (1+\epsilon e^{-\beta_0(t-\tau)})\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right). \end{aligned}$$

Since by the mean value theorem, it is easy to show that

$$\begin{aligned} f\left(\phi(\xi), \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right) &= f\left(\phi(\xi), \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right) - f(0,0) \\ &= \partial_1 f\left(\gamma, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right)\phi(\xi) \\ &\quad + \partial_2 f(0,\mu)\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy, \end{aligned}$$

and

$$\begin{aligned} & f\left(\phi(\xi), \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right) \\ & - f\left((1+\epsilon e^{-\beta_0 t})\phi(\xi), (1+\epsilon e^{-\beta_0(t-\tau)})\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right) \\ = & -\left\{\partial_1 f\left(\zeta, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right)\phi(\xi)\epsilon e^{-\beta_0 t}\right. \\ & \left. + \partial_2 f((1+\epsilon e^{-\beta_0 t})\phi(\xi),\eta)\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\epsilon e^{-\beta_0(t-\tau)}\right\}, \end{aligned}$$

where $\gamma \in (0,\phi(\xi))$, $\mu \in (0, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy)$, $\zeta \in (\phi(\xi), (1+\epsilon e^{-\beta_0 t})\phi(\xi))$, and $\eta \in (\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy, (1+\epsilon e^{-\beta_0(t-\tau)})\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy)$. Then we have that

$$\begin{aligned} & \epsilon^{-1}e^{\beta_0(t-\tau)}L[u^+](x,t) \\ \geq & -\beta_0 e^{-\beta_0\tau}\phi(\xi) + \sigma_0\beta_0 e^{-\beta_0\tau}\phi'(\xi) \\ & + e^{-\beta_0\tau}\left\{\partial_1 f\left(\gamma, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right)\phi(\xi)\right. \\ & \quad + \partial_2 f(0,\mu)\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy \\ & \quad - \partial_1 f\left(\zeta, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right)\phi(\xi) \\ & \quad \left. - \partial_2 f((1+\epsilon e^{-\beta_0 t})\phi(\xi),\eta)\int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy e^{\beta_0\tau}\right\}. \end{aligned} \tag{4.8}$$

We need to distinguish the following three cases.

Case 1. When $\xi \geq M_1$, it follows from (4.8), (4.3) and (4.4) that

$$\begin{aligned} & \epsilon^{-1}e^{\beta_0(t-\tau)}L[u^+](x,t) \\ \geq & -\beta_0 e^{-\beta_0\tau}K + \left[\partial_2 f(0,\mu)e^{-\beta_0\tau} - \partial_2 f((1+\epsilon e^{-\beta_0 t})\phi(\xi),\eta)\right]\left(K - \frac{\kappa}{2}\right) \\ & + e^{-\beta_0\tau}\left(K - \frac{\kappa}{2}\right)\left[\partial_1 f\left(\gamma, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right)\right. \\ & \qquad \left. - \partial_1 f\left(\zeta, \int_{\mathbb{R}} k(y)\phi(\xi-y-c\tau)\,dy\right)\right] \\ \geq & \ 0. \end{aligned}$$

Case 2. When $\xi \le -M_2$, by (4.8), (4.6) and (4.7), we can obtain that

$$\begin{aligned}
&\epsilon^{-1} e^{\beta_0(t-\tau)} e^{-\lambda_1 \xi} L[u^+](x,t) \\
&\ge -\beta_0 e^{-\beta_0 \tau} \phi(\xi) e^{-\lambda_1 \xi} + \sigma_0 \beta_0 e^{-\beta_0 \tau} \phi'(\xi) e^{-\lambda_1 \xi} - \max_{(u,v)\in[0,K]^2} \partial_1 f(u,v) \phi(\xi) e^{-\lambda_1 \xi} e^{-\beta_0 \tau} \\
&\quad - \max_{(u,v)\in[0,K]^2} \partial_2 f(u,v) \int_{\mathbb{R}} k(y) \phi(\xi - y - c\tau)\, dy e^{-\lambda_1 \xi} \\
&\ge \frac{1}{2} \lambda_1 \sigma_0 \beta_0 e^{-\beta_0 \tau} - \frac{3}{2} \left[\beta_0 e^{-\beta_0 \tau} + \max_{(u,v)\in[0,K]^2} \partial_1 f(u,v) e^{-\beta_0 \tau} + \max_{(u,v)\in[0,K]^2} \partial_2 f(u,v) \right] \\
&\ge 0.
\end{aligned}$$

Case 3. For $\xi \in [-M_2, M_1]$, according to the chosen of σ_0, it is easy to show that

$$\begin{aligned}
\epsilon^{-1} e^{\beta_0(t-\tau)} L[u^+](x,t) &\ge -\beta_0 e^{-\beta_0 \tau} K + \sigma_0 \beta_0 e^{-\beta_0 \tau} \rho - \max_{(u,v)\in[0,K]^2} \partial_1 f(u,v) K e^{-\beta_0 \tau} \\
&\quad - \max_{(u,v)\in[0,K]^2} \partial_2 f(u,v) K \\
&\ge 0.
\end{aligned}$$

Combining these three cases, we can obtain that $L[u^+](x,t) \ge 0$ for all $x \in \mathbb{R}$ and $t \ge 0$. Therefore, $u^+(x,t)$ is a supersolution of (1.5).

This completes the proof. □

Remark 4.2. By the proof of the supersolution, we can know that β_0 and σ_0 are uniform in $\epsilon \in (0, \delta]$, which will be very useful in the following.

Next, we give the main result of this paper.

Theorem 4.3. *Assume that* (J), (K), *and* (A1)–(A4) *hold. Let* $c > c_*$ *and* ϕ *be the traveling wave front as given in Theorem* 2.1. *Suppose that there exists a constant* $\rho_0 > 0$ *such that the initial data* $\varphi \in [0, K]_C$ *of* (3.1) *satisfies* $\liminf_{x\to\infty} \varphi(x, 0) > 0$ *and*

$$\lim_{x\to-\infty} \max_{s\in[-\tau,0]} \left| \varphi(x,s) e^{-\lambda_1(x+cs)} - \rho_0 \right| = 0. \tag{4.9}$$

Then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x + ct + \xi_0)| = 0,$$

where $\xi_0 = \frac{1}{\lambda_1} \ln \rho_0$.

Before proving the theorem, we give some useful lemmas under the conditions of Theorem 4.3. In the following, we set $\xi = x + ct$.

Lemma 4.4. *For any* $\varepsilon > 0$, *there exists* $\xi(\varepsilon)$ *such that*

$$\sup_{t\ge-\tau} u(\xi - 2\varepsilon - ct, t) < \phi(\xi + \xi_0) < \inf_{t\ge-\tau} u(\xi + 2\varepsilon - ct, t), \tag{4.10}$$

for all $\xi \le \xi(\varepsilon)$.

Proof. Due to (4.9), we get that there exists $x_1(\varepsilon) < 0$ such that

$$\varphi(x-\varepsilon, s) < e^{\lambda_1(x+cs+\xi_0)} < \varphi(x+\varepsilon, s) \quad \text{for all } x \le x_1(\varepsilon) \text{ and } s \in [-\tau, 0].$$

Let $\phi^-(\xi) = \max\left\{0, e^{\lambda_1(\xi+\xi_0)} - qe^{\eta\lambda_1(\xi+\xi_0)}\right\}$, where $\eta = \frac{1}{2}\left(1 + \min\left\{1+\sigma_1, 1+\sigma_2, \frac{\lambda_2}{\lambda_1}\right\}\right)$, $q \ge e^{-(\eta-1)\lambda_1(x_1(\varepsilon)+\xi_0-c\tau)}$. By (2.3), it is easy to see that $\phi^-(\xi)$ is a subsolution of (3.1). As $e^{\lambda_1(x+\xi_0+cs)} - qe^{\eta\lambda_1(x+\xi_0+cs)} < 0$ for all $x > x_1(\varepsilon)$ and $s \in [-\tau, 0]$, we have that $\varphi(x+\varepsilon, s) \ge \max\left\{0, e^{\lambda_1(x+\xi_0+cs)} - qe^{\eta\lambda_1(x+\xi_0+cs)}\right\}$ for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$. By the comparison principle, we show that

$$u(x+\varepsilon, t) \ge e^{\lambda_1(x+\xi_0+ct)} - qe^{\eta\lambda_1(x+\xi_0+ct)} \quad \text{for all } x \in \mathbb{R} \text{ and } t \ge -\tau.$$

As $\lim_{\xi\to-\infty} \phi(\xi)e^{-\lambda_1\xi} = 1$, there exists $x_2(\varepsilon) < 0$ such that

$$e^{\lambda_1(\xi+\xi_0+\varepsilon)} - qe^{\eta\lambda_1(\xi+\xi_0+\varepsilon)} > \phi(\xi+\xi_0) \quad \text{for all } \xi \le x_2(\varepsilon).$$

Consequently, for all $\xi \le \min\{x_1(\varepsilon), x_2(\varepsilon)\}$, we have

$$\inf_{t\ge-\tau} u(\xi+2\varepsilon-ct, t) \ge e^{\lambda_1(\xi+\xi_0+\varepsilon)} - qe^{\eta\lambda_1(\xi+\xi_0+\varepsilon)} > \phi(\xi+\xi_0).$$

Let $\phi^+(\xi) = \min\left\{K, e^{\lambda_1(\xi+\xi_0)} + qe^{\eta\lambda_1(\xi+\xi_0)}\right\}$. Then by (2.3), $\phi^+(\xi)$ is a supersolution of (3.1). Since $e^{\lambda_1\xi} + qe^{\eta\lambda_1\xi} > K$ for $\xi > -\frac{1}{\eta\lambda_1}\ln\frac{q}{K}$, we can take q large enough so that $e^{\lambda_1(x+\xi_0+cs)} + qe^{\eta\lambda_1(x+\xi_0+cs)} > K$ for all $x > x_1(\varepsilon)$ and $s \in [-\tau, 0]$. As $\varphi(x-\varepsilon, s) < e^{\lambda_1(x+\xi_0+cs)} < e^{\lambda_1(x+\xi_0+cs)} + qe^{\eta\lambda_1(x+\xi_0+cs)}$ for all $x \le x_1(\varepsilon)$ and $s \in [-\tau, 0]$, we have that $\varphi(x-\varepsilon, s) \le \min\left\{K, e^{\lambda_1(x+\xi_0+cs)} + qe^{\eta\lambda_1(x+\xi_0+cs)}\right\}$ for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$. Therefore, the comparison principle gives

$$u(x-\varepsilon, t) \le \min\left\{K, e^{\lambda_1(x+\xi_0+ct)} + qe^{\eta\lambda_1(x+\xi_0+ct)}\right\} \quad \text{for all } x \in \mathbb{R} \text{ and } t \ge -\tau.$$

Since

$$\lim_{\xi\to-\infty} \frac{e^{\lambda_1(\xi-\varepsilon)} + qe^{\eta\lambda_1(\xi-\varepsilon)}}{\phi(\xi)} = \lim_{\xi\to-\infty} \frac{e^{-\lambda_1\varepsilon} + qe^{(\eta-1)\lambda_1\xi-\eta\lambda_1\varepsilon}}{\phi(\xi)e^{-\lambda_1\xi}} = e^{-\lambda_1\varepsilon} < 1,$$

there exists $x_3(\varepsilon) < 0$ such that $e^{\lambda_1(\xi+\xi_0-\varepsilon)} + qe^{\eta\lambda_1(\xi+\xi_0-\varepsilon)} < \phi(\xi+\xi_0)$ for all $\xi \le x_3(\varepsilon)$. Hence, for all $\xi \le \min\{x_1(\varepsilon), x_3(\varepsilon)\}$, we have

$$\sup_{t\ge-\tau} u(\xi-2\varepsilon-ct, t) \le e^{\lambda_1(\xi+\xi_0-\varepsilon)} + qe^{\eta\lambda_1(\xi+\xi_0-\varepsilon)} < \phi(\xi+\xi_0).$$

Finally, we can choose $\xi(\varepsilon) = \min\{x_1(\varepsilon), x_2(\varepsilon), x_3(\varepsilon)\}$, such that for all $\xi \le \xi(\varepsilon)$, the assertion of the lemma follows. $\square$

Lemma 4.5. *For all $\xi \in \mathbb{R}$ and $t \geq 1+\tau$, there exist constants $\delta \in (0,1)$, $\beta_0 > 0$, $\sigma_0 > 0$ and $z_0 > 0$ such that*

$$\begin{aligned} &(1-\delta e^{-\beta_0(t-1-\tau)})\phi(\xi+\xi_0-z_0+\delta\sigma_0 e^{-\beta_0(t-1-\tau)}) \\ &\leq u(\xi-ct,t) \\ &\leq (1+\delta e^{-\beta_0 t})\phi(\xi+\xi_0+z_0-\delta\sigma_0 e^{-\beta_0 t}). \end{aligned} \tag{4.11}$$

Proof. Since $\liminf_{x\to+\infty}\varphi(x,0) > 0$, there exist $\delta_1 > 0$ and $x_4 > 0$ such that

$$\varphi(x,0) > \delta_1 \quad \text{for all } x > x_4.$$

Choose a positive integer N such that $N > \frac{1}{2}(x_4 - [\xi(N) - c(1+\tau)])$, where $\xi(N)$ is given in Lemma 4.4. If $x \geq \xi(N) - c(1+\tau)$, then $x + 2N > x_4$. Hence, by the strong positivity of the semigroup $T(t)$, we have that

$$u(x+2N, 1+\tau+s) \geq \varphi(x+2N, 0) > \delta_1 \geq (1-\delta)K$$

for all $x \geq \xi(N) - c(1+\tau)$, $s \in [-\tau, 0]$ and some $\delta < 1$.

In view of (4.10), $u(\xi+2N-c(1+\tau+s), 1+\tau+s) \geq \phi(\xi+\xi_0)$ for all $\xi \leq \xi(N)$, hence, $u(x+2N, 1+\tau+s) \geq \phi(x+c(1+\tau+s)+\xi_0)$ for all $x \leq \xi(N) - c(1+\tau+s)$ and $s \in [-\tau, 0]$.

Thus for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$, we have

$$\begin{aligned} u(x+2N, 1+\tau+s) &\geq (1-\delta)\phi(x+c(1+\tau+s)+\xi_0) \\ &\geq (1-\delta e^{-\beta_0 s})\phi(x+c(1+\tau+s)+\xi_0-\sigma_0\delta e^{\beta_0\tau}+\sigma_0\delta e^{-\beta_0 s}). \end{aligned}$$

Then by (4.2), it is easy to show that

$$u(x+2N, 1+\tau+t) \geq (1-\delta e^{-\beta_0 t})\phi(x+c(1+\tau+t)+\xi_0-\sigma_0\delta e^{\beta_0\tau}+\sigma_0\delta e^{-\beta_0 t}),$$

hence,

$$u(\xi-c(1+\tau+t), 1+\tau+t) \geq (1-\delta e^{-\beta_0 t})\phi(\xi-2N+\xi_0-\sigma_0\delta e^{\beta_0\tau}+\sigma_0\delta e^{-\beta_0 t}). \tag{4.12}$$

Again, in view of (4.10), $u(\xi-2N-cs, s) < \phi(\xi+\xi_0)$ for all $\xi \leq \xi(N)$, and hence, $u(x-2N, s) < \phi(x+cs+\xi_0)$ for all $x \leq \xi(N) - cs$ and $s \in [-\tau, 0]$. For δ given in the lower bound estimate, if we choose large $x_5 > 0$ such that $\phi(\xi(N)+x_5+\xi_0) \geq \frac{K}{1+\delta}$, then we can obtain that

$$u(x-2N, s) \leq K \leq (1+\delta)\phi(x+cs+x_5+\xi_0)$$

for all $x \geq \xi(N) - cs$ and $s \in [-\tau, 0]$. Thus, for all $x \in \mathbb{R}$ and $s \in [-\tau, 0]$,

$$\begin{aligned} u(x-2N, s) &\leq (1+\delta)\phi(x+cs+x_5+\xi_0) \\ &\leq (1+\delta e^{-\beta_0 s})\phi(x+cs+x_5+\xi_0+\sigma_0\delta e^{\beta_0\tau}-\sigma_0\delta e^{-\beta_0 s}), \end{aligned}$$

then, according to (4.1), we can obtain that

$$u(x-2N,t) \le (1+\delta e^{-\beta_0 t})\phi(x+ct+x_5+\xi_0+\sigma_0\delta e^{\beta_0\tau}-\sigma_0\delta e^{-\beta_0 t}),$$

that is,

$$u(\xi-ct,t) \le (1+\delta e^{-\beta_0 t})\phi(\xi+2N+x_5+\xi_0+\sigma_0\delta e^{\beta_0\tau}-\sigma_0\delta e^{-\beta_0 t}). \tag{4.13}$$

Finally, by setting $z_0 = 2N + x_5 + \sigma_0\delta e^{\beta_0\tau}$, we can get (4.11) from (4.12) and (4.13).

This completes the proof. □

Lemma 4.6. *There exists $M_0 > 0$ such that*

$$(1-\epsilon)\phi(\xi+3\epsilon\sigma_0 e^{\beta_0\tau}) \le \phi(\xi) \le (1+\epsilon)\phi(\xi-3\epsilon\sigma_0 e^{\beta_0\tau})$$

for all $\epsilon \in (0,\delta)$ and $\xi \ge M_0+\xi_0$.

Proof. In view of

$$\frac{d}{dx}\left[(1+x)\phi(\xi-3x\sigma_0 e^{\beta_0\tau})\right] = \phi(\xi-3x\sigma_0 e^{\beta_0\tau}) - 3\sigma_0 e^{\beta_0\tau}(1+x)\phi'(\xi-3x\sigma_0 e^{\beta_0\tau}),$$

if we choose $M_0 > 0$ large enough, we can get that

$$\frac{d}{dx}\left[(1+x)\phi(\xi-3x\sigma_0 e^{\beta_0\tau})\right] > 0 \quad \text{for all } x \in [-\delta,\delta] \text{ and } \xi \ge M_0+\xi_0.$$

Then the result of the lemma is clear. The proof is completed. □

Lemma 4.7. *Assume that z and M are fixed positive constants. Let $u^{\pm}$ be the solution of (3.1) on $\mathbb{R}\times(0,+\infty)$ with the initial value*

$$u^{\pm}(x,s) = \phi(x+cs+\xi_0 \pm z)\chi(x+cs+M) + \phi(x+cs+\xi_0 \pm 2z)[1-\chi(x+cs+M)], \tag{4.14}$$

respectively, where $x \in \mathbb{R}$ and $s \in [-\tau,0]$, $\chi(y) = \min\{\max\{0,-y\},1\}$ for all $y \in \mathbb{R}$. Then there exists a constant $\epsilon \in (0, \min\{\delta, ze^{-\beta_0\tau}/(3\sigma_0)\})$ such that

$$u^{+}(\xi-c(1+\tau+s),1+\tau+s) \le (1+\epsilon)\phi(\xi+\xi_0+2z-3\epsilon\sigma_0 e^{\beta_0\tau}), \tag{4.15}$$

and

$$u^{-}(\xi-c(1+\tau+s),1+\tau+s) \ge (1-\epsilon)\phi(\xi+\xi_0-2z+3\epsilon\sigma_0 e^{\beta_0\tau}) \tag{4.16}$$

for any $\xi \in [-M,\infty)$ and $s \in [-\tau,0]$.

Proof. By the definition of $\chi(y)$, we can get that

$$u^+(\cdot, s) \leq \phi(\cdot + cs + \xi_0 + 2z) \quad \text{on } \mathbb{R},$$

and

$$u^+(\cdot, s) < \phi(\cdot + cs + \xi_0 + 2z) \quad \text{on } (-\infty, -M-1].$$

From the strong positivity of $T(t)$, it is easy to show that

$$u^+(\xi - c(1+\tau+s), 1+\tau+s) < \phi(\xi + \xi_0 + 2z) \quad \text{for all } \xi \in \mathbb{R} \text{ and } s \in [-\tau, 0].$$

Since $u^+(x,t)$ and $\phi(x+ct)$ are continuous, then there exists $\epsilon \in (0, \min\left\{\delta, ze^{-\beta_0\tau}/(3\sigma_0)\right\})$ such that

$$u^+(\xi - c(1+\tau+s), 1+\tau+s) \leq \phi(\xi + \xi_0 + 2z - 3\epsilon\sigma_0 e^{\beta_0\tau})$$

for $\xi \in [-M, M_0 - 2z]$, where M_0 is defined in Lemma 4.6.

By Lemma 4.6, it is clear that

$$u^+(\xi - c(1+\tau+s), 1+\tau+s) < \phi(\xi + \xi_0 + 2z) \leq (1+\epsilon)\phi(\xi + \xi_0 + 2z - 3\epsilon\sigma_0 e^{\beta_0\tau})$$

for all $\xi \in [M_0 - 2z, +\infty)$. Therefore, (4.15) holds.

Similarly, we can also show that (4.16) holds. This completes the proof. □

Here we give the proof of Theorem 4.3.

Proof. We define constants $z^\pm$ as follows

$$\begin{aligned} &z^+ := \inf\left\{z \mid z \in A^+\right\}, \\ &A^+ := \left\{ z \geq 0 \;\middle|\; \limsup_{t\to+\infty} \sup_{\xi\in\mathbb{R}} (u(\xi - ct, t) - \phi(\xi + \xi_0 + 2z)) \leq 0 \right\}, \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} &z^- := \inf\left\{z \mid z \in A^-\right\}, \\ &A^- := \left\{ z \geq 0 \;\middle|\; \liminf_{t\to+\infty} \sup_{\xi\in\mathbb{R}} (u(\xi - ct, t) - \phi(\xi + \xi_0 - 2z)) \geq 0 \right\}. \end{aligned} \tag{4.18}$$

In view of (4.11), we see that $\frac{1}{2}z_0 \in A^\pm$. Hence, $z^\pm$ are well defined and $z^\pm \in [0, \frac{1}{2}z_0]$. Thus, to complete the proof, we need only show that $z^\pm = 0$.

First we prove $z^+ = 0$ by a contradiction argument. Suppose $z^+ > 0$. Let $z = z^+$, $M = -\xi(\frac{z^+}{2})$ and ϵ be given in Lemma 4.7. Follows that $z^+ \in A^+$, there exists $T_1 \geq 0$ such that

$$u(\xi - c(T_1+s), T_1+s) - \phi(\xi + \xi_0 + 2z^+) \leq \widehat{\epsilon} \quad \text{for all } s \in [-\tau, 0],$$

where $\widehat{\epsilon}=\epsilon\phi(-M+\xi_0-3\epsilon\sigma_0 e^{\beta_0\tau})e^{(L_1-\beta)(1+\tau)}$. From (4.14), $u^+(x,s)=\phi(x+cs+\xi_0+2z^+)$ on $[-M-cs,+\infty)$. Then, for any $x\in[-M-cs,+\infty)$,

$$u(x-cT_1,T_1+s)\leq\phi(x+cs+\xi_0+2z^+)+\widehat{\epsilon}=u^+(x,s)+\widehat{\epsilon}.$$

For any $x\in(-\infty,-M-cs]=(-\infty,\xi(\frac{z^+}{2})-cs]$, by (4.10) and (4.14), we have that

$$u(x-cT_1,T_1+s)\leq\phi(x+cs+\xi_0+z^+)\leq u^+(x,s).$$

Therefore, by (3.8), we have that

$$\begin{aligned}u(x-cT_1,T_1+1+\tau+s)&\leq u^+(x,1+\tau+s)+\widehat{\epsilon}e^{-(L_1-\beta)(1+\tau)}\\&=u^+(x,1+\tau+s)+\epsilon\phi(-M+\xi_0-3\epsilon\sigma_0 e^{\beta_0\tau})\end{aligned}$$

for all $x\in\mathbb{R}$.

By Lemma 4.7, we obtain that for $\xi\geq-M$,

$$\begin{aligned}&u(\xi-c(T_1+1+\tau+s),T_1+1+\tau+s)\\&\leq u^+(\xi-c(1+\tau+s),1+\tau+s)+\epsilon\phi(-M+\xi_0-3\epsilon\sigma_0 e^{\beta_0\tau})\\&\leq(1+\epsilon)\phi(\xi+\xi_0+2z^+-3\epsilon\sigma_0 e^{\beta_0\tau})+\epsilon\phi(-M+\xi_0-3\epsilon\sigma_0 e^{\beta_0\tau})\\&\leq(1+2\epsilon)\phi(\xi+\xi_0+2z^+-3\epsilon\sigma_0 e^{\beta_0\tau}).\end{aligned}$$

For any $\xi\leq-M=\xi(\frac{z^+}{2})$, by (4.10) and the chosen of ϵ in Lemma 4.7, we have

$$u(\xi-c(T_1+1+\tau+s),T_1+1+\tau+s)\leq\phi(\xi+\xi_0+z^+)\leq\phi(\xi+\xi_0+2z^+-3\epsilon\sigma_0 e^{\beta_0\tau}).$$

Thus, we have that

$$\begin{aligned}&u(\xi-c(T_1+1+\tau+s),T_1+1+\tau+s)\\&\leq(1+2\epsilon)\phi(\xi+\xi_0+2z^+-3\epsilon\sigma_0 e^{\beta_0\tau})\\&\leq(1+2\epsilon e^{-\beta_0 s})\phi(\xi+\xi_0+2z^+-\epsilon\sigma_0 e^{\beta_0\tau}-2\epsilon\sigma_0 e^{-\beta_0 s})\end{aligned}$$

for any $\xi\in\mathbb{R}$, $s\in[-\tau,0]$.

By the comparison principle, we can get that

$$\begin{aligned}&u(\xi-c(T_1+1+\tau+t),T_1+1+\tau+t)\\&\leq(1+2\epsilon e^{-\beta_0 t})\phi(\xi+\xi_0+2z^+-\epsilon\sigma_0 e^{\beta_0\tau}-2\epsilon\sigma_0 e^{-\beta_0 t})\end{aligned}$$

for any $\xi\in\mathbb{R}$, $t\geq0$. This implies that $(z^+-\epsilon\sigma_0 e^{\beta_0\tau}/2)\in A^+$, it contradicts the definition of z^+. So we obtain $z^+=0$.

By the similar way, it is easy to get $z^-=0$. The proof is completed. □

5. An application

Example 5.1. For the typical nonlinearity $f(u,v) = rv(1-u)v$, since $f\colon \mathbb{R}^2 \to \mathbb{R}$ satisfies assumptions (A1)–(A4), where $K = 1$, we can obtain that under the assumptions (J) and (K), Theorem 4.3 still holds for (1.5) with the classical logistic nonlinearity.

Next, we consider the Nicholson's blowflies model.

Example 5.2. The following diffusive Nicholson's blowflies equation

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= d\left(\int_{\mathbb{R}} J(x-y)u(y,t)\,dy - u(x,t)\right) - ru \\ &\quad + rp\int_{\mathbb{R}} k(y)u(x-y,t-\tau)\,dy e^{-\int_{\mathbb{R}} k(y)u(x-y,t-\tau)\,dy}, \end{aligned} \tag{5.1}$$

where $r > 0$, $\tau > 0$ and $1 < p \le e$.

If $1 < p \le e$, $f(u,v) = -ru + rpve^{-v}$ satisfies assumptions (A1)–(A4). Therefore, we can get the following result.

Theorem 5.3. *Assume that* (J) *and* (K) *hold, let* $c > c^*$ *and* ϕ *be the traveling wave front of* (5.1) *connecting* 0 *and* $\ln p$. *Suppose that there exists a constant* $\rho_0 > 0$ *such that the initial data* $\varphi \in [0, \ln p]_C$ *of* (3.1) *with* $f(u,v) = -ru + rpve^{-v}$ *satisfies* $\liminf_{x\to\infty} \varphi(x,0) > 0$ *and* (4.9), *then* $\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x+ct+\xi_0)| = 0$, *where* $\xi_0 = \frac{1}{\lambda_1}\ln \rho_0$.

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