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Solutions for a $p(x)$-Kirchhoff Type Problem with a Non-smooth Potential in $\mathbb{R}^N$

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Abstract. This paper is concerned with a class of $p(x)$-Kirchhoff type problem in $\mathbb{R}^N$. By the theories of nonsmooth critical point and variable exponent Sobolev spaces, we establish the existence and multiplicity of solutions to the $p(x)$-Kirchhoff type problem under weaker hypotheses on the nonsmooth potential at zero (at infinity, respectively). Some recent results in the literature are generalized and improved.

1. Introduction

In this paper, we investigate the existence and multiplicity of solutions to a class of $p(x)$-Kirchhoff type problem with a nonsmooth potential

$$\begin{cases} -M(t)\left(\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - V(x)|u|^{p(x)-2}u\right) \in \partial F(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases} \tag{1.1}$$

Here, $W^{1,p(x)}(\mathbb{R}^N)$ is the variable exponent Sobolev space, $N \geq 1$, $M(t)$ is a continuous function with $t := \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,\mathrm{d}x$, $F\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a locally Lipshitz not necessarily smooth potential function. We denote $\partial F(x,u)$ the partial generalized gradient of $F(x,\cdot)$ at the point u. $p(x)$ and $V(x)$ satisfy the following assumptions:

(H$_1$) The function $p\colon \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and

$$1 < p^- = \inf_{x\in\mathbb{R}^N} p(x) \leq \sup_{x\in\mathbb{R}^N} p(x) = p^+ < N;$$

(H$_2$) $V(x) \in C(\mathbb{R}^N)$, $V^- = \inf_{x\in\mathbb{R}^N} V(x) > 0$, $\mu(V^{-1}(-\infty, M_1]) < +\infty$ for all $M_1 \in \mathbb{R}$.

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Here μ is the Lebesgue measure on $\mathbb{R}^N$. Note that if $V \in C(\mathbb{R}^N, (0, +\infty))$ is coercive, namely

$$\lim_{|x|\to\infty} V(x) = +\infty,$$

then (H_2) is satisfied.

The operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called to be $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian, for example, it is inhomogeneous and in general it does not have the first eigenvalue. The study of various mathematical problems with variable exponent growth condition has caused great interest in recent years, and raised many difficult mathematical problems. Problems with variable exponent growth conditions appear in electrorheological fluids [37,40], stationary thermorheological viscous flows of non-Newtonian fluids [2,3] and image processing [7,22] and so on. The more details can be found in [24,38,41].

The problem (1.1) is a variant type of a class of Dirichlet problem of Kirchhoff type. Indeed, if the right-hand side function F is continuously differentiable with respect to the real variable u, $V(x) = 0$, $p(x) = 2$ and $M(t) = a + bt$ in bounded domain, then problem (1.1) reduces to the following Dirichlet problem:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

which is related to the stationary analogue of the following equation

$$\begin{cases} u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.3}$$

Such problems are viewed as being nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2\,\mathrm{d}x)\Delta u$, which means that the problems (1.2) and (1.3) are no longer a pointwise identity and are very different from classical elliptic equations. We know that such problems are proposed by Kirchhoff in [25] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Problem (1.2) caused much attention only after lions [30] proposed an abstract framework to the problem. Some interesting and important results can be found in [6,17,29,31,34,35] and references therein. Especially, Dai and Hao [11] studied the following $p(x)$-Kirchhoff-type problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.4}$$

where f is a continuous function. By means of a direct variational approach and the theory of the variable exponent Sobolev spaces, they established conditions ensuring the existence and multiplicity of solutions for problem (1.4).

As is well known, many free boundary problems and obstacle problems may be reduced to partial differential equations with nonsmooth potentials. The area of nonsmooth analysis is closely related with the development of a critical point theory for nondifferentiable functions, in particular, for locally Lipschitz continuous functions based on Clarke's generalized gradient [8]. It provides an appropriate mathematical framework to extend the classic critical point theory for C^1-functionals in a natural way, and to meet specific needs in applications, such as in nonsmooth mechanics and engineering. For a comprehensive understanding, we refer to the monographs of [19,32,33] and References [13,18,21,23,28,39]. More precisely, if $M(t) = 1$, there exist several existence results for the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + V(x)|u|^{p(x)-2}u \in \partial F(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.5}$$

Qian and Shen [36] established conditions ensuring the existence and multiplicity of solutions for problem (1.5) with $V(x) = 0$ via the theory of nonsmooth critical point theory and the properties of $W_0^{1,p(x)}(\Omega)$. Dai and Liu [12] obtained the existence of at least three solutions for problem (1.5) with $\partial F(x,u)$ replaced by $\lambda\partial F(x,u)$ and $V(x) = 0$ via a version of the nonsmooth three critical points theorem. Ge et al. [20], using a variational method combined with suitable truncation techniques, proved the existence of at least five solutions under the suitable conditions for problem (1.5) with $V(x) = 0$. For the case of unbounded domain, there exist few results for problem (1.5) on $\mathbb{R}^N$. Dai [9] derived the existence of infinitely many radially symmetric solutions for the problem (1.5) on $\mathbb{R}^N$ under suitable hypotheses by applying a nonsmooth variational principle with $V(x) = 1$. Besides, if $p(x) \equiv p$ (a constant), Kristály [27] studied the following differential inclusion problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u \in \alpha(x)\partial F(u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.6}$$

where $2 \le N < p < +\infty$, $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radially symmetric. Under suitable oscillatory assumptions on the potential F at zero or at infinity, they showed the existence of infinitely many, radially symmetric solutions of (1.6).

Being influenced by the reading of the above cited papers, we will study the existence and multiplicity of solutions for problem (1.1), where $V(x)$ satisfies the assumption (H_2). For the functions M and F, we assume that

(M_1) $M(t)\colon [0,+\infty) \to (m_0,+\infty)$ is a continuous and increasing function with $m_0 > 0$;

(M_2) $\exists 0<\mu<1$ such that

$$\widehat{M}(t) \geq (1-\mu) M(t) t,$$

where $\widehat{M}(t)=\int_0^t M(\tau)\,\mathrm{d}\tau$;

(F_1) $F(\cdot, u)$ is measurable for all $u \in \mathbb{R}$;

(F_2) $F(x, \cdot)$ is locally Lipshitz for a.a. $x \in \mathbb{R}^N$;

(F_3) For all $\omega \in \partial F(x, u)$, a.a. $x \in \mathbb{R}^N$

$$\lim_{|u| \to +\infty} \frac{\omega}{|u|^{q(x)-1}}=0, \quad \text{and} \quad \lim_{|u| \to 0} \frac{\omega}{|u|^{p(x)-1}}=0,$$

where $p_+ \leq q \ll p^*$;

(F_4) $F(x, u) \geq 0$ and $F(x, u)>0$ for all $u \neq 0$;

(F_5) $\exists \theta>\frac{p^+}{1-\mu}$ such that

$$\theta F(x, u)+F^{\circ}(x, u ;-u) \leq 0$$

for all $u \in \mathbb{R}$ and a.a. $x \in \mathbb{R}^N$ (F° is introduced in Definition 2.2);

(F_6) $F(x,-u)=F(x, u)$ for a.a. $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}$.

Remark 1.1. From hypotheses (F_4) and (F_5) it is easy to see that $F(x, 0)=0$.

Our main results are as follows:

Theorem 1.2. *If hypotheses* (H_1), (H_2), (M_1), (M_2) *and* (F_1)-(F_5) *hold, then problem* (1.1) *has at least one nontrivial solution.*

Theorem 1.3. *If hypotheses* (H_1), (H_2), (M_1), (M_2) *and* (F_1)-(F_6) *hold, then problem* (1.1) *has a sequence of weak solutions* $\{\pm u_k\}$ *such that* $I(\pm u_k) \to +\infty$ *as* $k \to +\infty$.

To the best of our knowledge, it seems that Theorems 1.2 and 1.3 are the first existence and multiplicity results for problem (1.1) with a nonsmooth potential function. In the present paper, we extend the main results of [11] to a class of non-differentiable functionals in unbounded domain. Compared with the previous works, the main difficulties lie in the appearance of the nonlocal term, non-differentiable functional and the lack of compactness due to the unboundedness of the domain. To deal with the difficulty caused by the non-compactness we will employ the Bartsch-Wang condition established in [4] to recover the compact embedding. Furthermore, the lack of differentiability of the nonlinearity causes several technical difficulties. This implies that the variational methods for C^1 functions are not suitable in our case. Therefore we will use a variational approach based on the nonsmooth critical point theory due to Clarke [8] and Chang [5] to obtain the existence and multiplicity of solutions for problem (1.1) under certain conditions.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge. In Section 3, we prove our main results. In Section 4, we deal with a special $p(x)$-Kirchhoff type problem and obtain some corollaries.

2. Preliminaries

We firstly give some basic notations.

- $\rightharpoonup$ means weak convergence and $\to$ strong convergence.
- c_i $(i = 1, 2, \ldots)$ denote the estimated constants (the exact value may be different).
- $(X, \|\cdot\|)$ denotes a (real) Banach space and $(X^*, \|\cdot\|_*)$ its topological dual.
- if $\inf_{x\in\mathbb{R}^N}(h_1(x) - h_2(x)) > 0$, we denote by $h_2(\cdot) \ll h_1(\cdot)$.
- $h^- = \inf_{x\in\mathbb{R}^N} h(x)$ and $h^+ = \inf_{x\in\mathbb{R}^N} h(x)$.

We recall some results on variable exponent Lebesgue-Sobolev spaces and list some properties of that spaces. For more details the reader is referred to [14–16, 26] and the references therein.

Let $p \in L^\infty(\mathbb{R}^N)$ and $p^- > 1$. The variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^N)$ is defined by

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u\colon \mathbb{R}^N \to \mathbb{R} \;\middle|\; u \text{ is measurable and } \int_{\mathbb{R}^N} |u|^{p(x)}\,\mathrm{d}x < \infty \right\}$$

endowed with the norm

$$\|u\|_{p(x)} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Then, we define the variable exponent Sobolev space

$$W^{1,p(x)}(\mathbb{R}^N) = \left\{ u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \right\}$$

with the norm

$$\|u\| = \|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)},$$

or equivalently

$$\|u\| = \|u\|_{1,p(x)} = \inf\left\{ \lambda > 0 : \int_{\Omega} \left(\left|\frac{u(x)}{\lambda}\right|^{p(x)} + \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} \right) \mathrm{d}x \le 1 \right\}$$

for all $u \in W^{1,p(x)}(\mathbb{R}^N)$. From the Proposition 2.1 of [14] we obtain that $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Next, we consider the following linear subspace

$$E = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} |\nabla u|^{p(x)} + V(x)|u|^{p(x)}\,\mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\|_E = \inf\left\{ \lambda > 0 \;\middle|\; \int_{\mathbb{R}^N} \left|\frac{\nabla u}{\lambda}\right|^{p(x)} + V(x)\left|\frac{u}{\lambda}\right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Then, $(E, \|\cdot\|_E)$ is continuously embedded into $W^{1,p(x)}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(E, \|\cdot\|_E)$ is also a separable reflexive Banach space.

Definition 2.1. A function $I\colon X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every $\nu, \eta \in U$

$$|I(\nu) - I(\eta)| \le L\,\|\nu - \eta\|.$$

Definition 2.2. Let $I\colon X \to \mathbb{R}$ be a locally Lipschitz function. The generalized derivative of I in u along the direction ν is defined by

$$I^0(u;\nu) = \limsup_{\eta \to u, \tau \to 0^+} \frac{I(\eta + \tau\nu) - I(\eta)}{\tau},$$

where $u, \nu \in X$.

It is easy to see that the function $\nu \mapsto I^0(u;\nu)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*-compact set $\partial I(u) \subset X^*$, defined by

$$\partial I(u) = \left\{ u^* \in X^* : \langle u^*, \nu \rangle_X \le I^0(u;\nu) \text{ for all } v \in X \right\}.$$

If $I \in C^1(X)$, then

$$\partial I(u) = \left\{ I'(u) \right\}.$$

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

Definition 2.3. We say that I satisfies the nonsmooth $(\mathrm{PS})_{\mathrm{c}}$ if any sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c \quad \text{and} \quad m^I(u_n) \to 0 \text{ as } n \to +\infty,$$

has a strongly convergent subsequence, where $m^I(u_n) = \inf_{u_n^* \in \partial I(x,u_n)} \|u_n^*\|_{X^*}$.

For $p \in L^\infty(\mathbb{R}^N)$ with $p^- > 1$, let $p'(x)\colon \mathbb{R}^N \to \mathbb{R}$ be such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, a.e. $x \in \mathbb{R}^N$. We have the following generalized Hölder's inequalities.

Proposition 2.4. [14] (i) *For any* $u \in L^{p(x)}(\mathbb{R}^N)$ *and* $v \in L^{p'(x)}(\mathbb{R}^N)$ *we have*

$$\left| \int_{\mathbb{R}^N} uv \,\mathrm{d}x \right| \leq 2 \left\| u \right\|_{p(x)} \left\| v \right\|_{p'(x)};$$

(ii) *If* $\frac{1}{p(x)} + \frac{1}{q(x)} + \frac{1}{r(x)} = 1$, *then for any* $u \in L^{p(x)}(\mathbb{R}^N)$, $v \in L^{q(x)}(\mathbb{R}^N)$ *and* $w \in L^{r(x)}(\mathbb{R}^N)$

$$\begin{aligned} \int_{\mathbb{R}^N} |uvw| \,\mathrm{d}x &\leq \left(\frac{1}{p^-} + \frac{1}{q^-} + \frac{1}{r^-} \right) \left\| u \right\|_{p(x)} \left\| v \right\|_{q(x)} \left\| w \right\|_{r(x)} \\ &\leq 3 \left\| u \right\|_{p(x)} \left\| v \right\|_{q(x)} \left\| w \right\|_{r(x)}. \end{aligned}$$

Proposition 2.5. *The function* $\rho \colon W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ *defined by*

$$\rho(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) \mathrm{d}x,$$

has the following properties:

(i) *If* $\|u\| \geq 1$, *then* $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$*;*

(ii) *If* $\|u\| \leq 1$, *then* $\|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$*.*

In particular, if $\|u\| = 1$ *then* $\rho(u) = 1$*. Moreover,* $\|u_n\| \to 0$ *if and only if* $\rho(u_n) \to 0$*.*

Remark 2.6. It is easy to see that with the norm $\|\cdot\|_E$, Proposition 2.5 remains valid.

Proposition 2.7. [8] (i) $(-h)^\circ(u; z) = h^\circ(u; -z)$ *for all* $u, z \in X$*;*

(ii) $h^\circ(u; z) = \max \{ \langle u^*, z \rangle_X : u^* \in \partial h(u) \}$ *for all* $u, z \in X$*;*

(iii) *Let* $j \colon X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}$, $j^\circ(u; z)$ *coincides with* $\langle j'(u), z \rangle_X$ *and* $(h + j)^\circ(u; z) = h^\circ(u; z) + \langle j'(u), z \rangle_X$ *for all* $u, z \in X$*;*

(iv) *(Lebourg's mean value theorem) Let* u *and* v *be two points in* X*. Then there exists a point* ξ *in the open segment between* u *and* v*, and a* $u_\xi^* \in \partial h(\omega)$ *such that*

$$h(u) - h(v) = \left\langle u_\xi^*, u - v \right\rangle_X;$$

(v) *(Second chain rule) Let* Y *be a Banach space and* $j \colon Y \to X$ *a continuously differentiable function. Then* $h \circ j$ *is locally Lipschitz and*

$$\partial(h \circ j)(y) \subseteq \partial h(j(y)) \circ j'(y) \quad \text{for all } y \in Y;$$

(vi) *If* $h_1, h_2 \colon X \to \mathbb{R}$ *are locally Lipschitz, then*

$$\partial(h_1 + h_2)(u) \subseteq \partial h_1(u) + \partial h_2(u).$$

From Lemma 2.6 in [1], we have the following theorem.

Theorem 2.8. *If $V(x)$ satisfies* (H_2), *then*

(i) *we have a compact embedding $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$;*

(ii) *for any measurable function $q\colon \mathbb{R}^N \to \mathbb{R}$ with $p < q \ll p^* = \frac{Np(x)}{N-p(x)}$, we have a compact embedding $E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.*

The next theorem is the nonsmooth version of the classic Mountain Theorem, which comes from Theorem 2.1.3 in [19].

Theorem 2.9. *Let X be a Banach space, and $h\colon X \to \mathbb{R}$ be a locally Lipschitz function with $h(0) = 0$. Suppose that there exist a point $e \in X$ and constants $\rho, \eta > 0$ such that*

(i) *$h(u) \geq \eta$ for all $u \in X$ with $\|u\| = \rho$;*

(ii) *$\|e\| > \rho$ and $h(e) \leq 0$;*

(iii) *h satisfies* $(\mathrm{PS})_{\mathrm{c}}$ *with*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} h(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1]) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then $c \geq \eta$ and $c \in \mathbb{R}$ is a critical value of h.

3. Existence and multiplicity of solutions

In this section, we prove our main results. We firstly give some notions. Consider the following function I defined on $W^{1,p(x)}(\mathbb{R}^N)$

$$\begin{aligned} I(u) &= \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,\mathrm{d}x\right) - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\ &= \Phi(u) - \Psi(u), \end{aligned} \tag{3.1}$$

where $\Phi(u) = \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,\mathrm{d}x\right)$ and $\Psi(u) = \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x$.

Definition 3.1. We say that $u \in W^{1,p(x)}(\mathbb{R}^N)$ is a weak solution of problem (1.1), if for all $v \in W^{1,p(x)}(\mathbb{R}^N)$

$$\begin{aligned} &M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,\mathrm{d}x\right) \int_{\mathbb{R}^N}\Big(|\nabla u|^{p(x)-2}\nabla u\nabla v + V(x)|u|^{p(x)-2}uv\Big)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \omega v\,\mathrm{d}x \end{aligned}$$

where $\omega \in \partial F(x,u)$. Then, the critical points of I are weak solutions of problem (1.1).

The following three lemmas play an important role in our proofs.

Lemma 3.2. *Assume that* (H_1), (H_2), (M_1), (M_2) *and* (F_1)-(F_4) *hold. If* $\{u_n\} \subset E$ *is a bounded sequence with* $m^I(u_n) \to 0$*, then* $\{u_n\} \subset E$ *has a convergent sequence.*

Proof. Since $\{u_n\} \subset E$ is bounded and the embedding

$$E \hookrightarrow L^r(\mathbb{R}^N)$$

is compact for all $p(x) \le r \ll p^*(x)$, passing to a subsequence, we assume

$$u_n \rightharpoonup u \quad \text{in } E \tag{3.2}$$

and

$$u_n \to u \quad \text{in } L^r(\mathbb{R}^N). \tag{3.3}$$

For $u_n^* \in \partial I(u_n)$, $u^* \in \partial I(u)$, $\omega_n \in \partial F(x, u_n)$ and $\omega \in \partial F(x, u)$ we have

$$\begin{aligned}
\langle u_n^* - u^*, u_n - u\rangle &= \langle \Phi(u_n) - \Phi(u), u_n - u\rangle - \langle \omega_n - \omega, u_n - u\rangle \\
&= M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right) \mathrm{d}x\right) \\
&\quad \times \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(u_n - u) + V(x)|u_n|^{p(x)-2}u_n(u_n - u)\right) \mathrm{d}x \\
&\quad - M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right) \mathrm{d}x\right) \\
&\quad \times \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(u_n - u) + V(x)|u|^{p(x)-2}u(u_n - u)\right) \mathrm{d}x \\
&\quad - \langle \omega_n - \omega, u_n - u\rangle \\
&= M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right) \mathrm{d}x\right) \\
&\quad \times \left(\int_{\mathbb{R}^N} \left\langle |\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u, \nabla(u_n - u)\right\rangle \mathrm{d}x\right. \\
&\qquad \left. + \int_{\mathbb{R}^N} \left\langle V(x)\left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right), (u_n - u)\right\rangle \mathrm{d}x\right) \\
&\quad + \left[M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right) \mathrm{d}x\right)\right. \\
&\qquad \left. - M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right) \mathrm{d}x\right)\right] \\
&\quad \times \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(u_n - u) + V(x)|u|^{p(x)-2}u(u_n - u)\right) \mathrm{d}x \\
&\quad - \langle \omega_n - \omega, u_n - u\rangle .
\end{aligned}$$

Recall the elementary inequalities

$$\left\langle |x|^{p-2}x-|y|^{p-2}y, x-y\right\rangle \geq c_p \times \begin{cases} |x-y|^p, & \text{if } p\geq 2,\\ \dfrac{|x-y|^2}{(1+|x|+|y|)^{2-p}}, & \text{if } 1<p<2,\end{cases}$$

where $c_p>0$ is a constant, and $x,y\in\mathbb{R}^N$. Then,

$$\begin{aligned}
\langle u_n^*-u^*, u_n-u\rangle \geq{}& c_p M\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u_n|^{p(x)}+V(x)|u_n|^{p(x)}\right)\mathrm{d}x\right)\\
&\times\int_{\mathbb{R}^N}\left(|\nabla(u_n-u)|^{p(x)}+V(x)|u_n-u|^{p(x)}\right)\mathrm{d}x\\
&+\left[M\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u_n|^{p(x)}+V(x)|u_n|^{p(x)}\right)\mathrm{d}x\right)\right.\\
&\quad\left.-M\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+V(x)|u|^{p(x)}\right)\mathrm{d}x\right)\right]\\
&\times\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u_n-u)+V(x)|u|^{p(x)-2}u(u_n-u)\right)\mathrm{d}x\\
&-\langle\omega_n-\omega, u_n-u\rangle\\
\geq{}& m_0c_p\int_{\mathbb{R}^N}\left(|\nabla(u_n-u)|^{p(x)}+V(x)|u_n-u|^{p(x)}\right)\mathrm{d}x\\
&+\left[M\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u_n|^{p(x)}+V(x)|u_n|^{p(x)}\right)\mathrm{d}x\right)\right.\\
&\quad\left.-M\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+V(x)|u|^{p(x)}\right)\mathrm{d}x\right)\right]\\
&\times\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u_n-u)+V(x)|u|^{p(x)-2}u(u_n-u)\right)\mathrm{d}x\\
&-\langle\omega_n-\omega, u_n-u\rangle.
\end{aligned}$$

One has

$$\begin{aligned}
&m_0c_p\int_{\mathbb{R}^N}\left(|\nabla(u_n-u)|^{p(x)}+V(x)|u_n-u|^{p(x)}\right)\mathrm{d}x\\
&\leq\langle u_n^*-u^*, u_n-u\rangle+\int_{\mathbb{R}^N}(\omega_n-\omega)(u_n-u)\,\mathrm{d}x\\
&\quad-\left[M\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u_n|^{p(x)}+V(x)|u_n|^{p(x)}\right)\mathrm{d}x\right)\right.\\
&\qquad\left.-M\left(\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+V(x)|u|^{p(x)}\right)\mathrm{d}x\right)\right]\\
&\quad\times\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u_n-u)+V(x)|u|^{p(x)-2}u(u_n-u)\right)\mathrm{d}x.
\end{aligned}$$

Set

$$\widehat{E}=\left\{u\in L^{p(x)}(\mathbb{R}^N):\nabla u\in L^{p(x)}(\mathbb{R}^N)\right\}$$

with the norm

$$\|u\|_{\widehat{E}} = \|\nabla u\|_{p(x)} .$$

Since the embedding $E \hookrightarrow \widehat{E}$ is continuous, we also have

$$u_n \rightharpoonup u \quad \text{in } \widehat{E}$$

from (3.2). So, from the boundedness of $\{u_n\}$ in E, and the continuity of $M(t)$, we have

$$\begin{aligned}
&\left[M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)}\right) \mathrm{d}x\right)\right. \\
&\quad \left. - M\left(\int_{\mathbb{R}^N} \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right) \mathrm{d}x\right)\right] \\
&\times \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(u_n - u) + V(x)|u|^{p(x)-2}u(u_n - u)\right) \mathrm{d}x \to 0
\end{aligned} \tag{3.4}$$

as $n \to +\infty$. Moreover, for any $\varepsilon > 0$, from hypotheses (F_3) and (F_4) there exists a $c_\varepsilon > 0$ such that

$$|\omega| \le \varepsilon|u|^{p(x)-1} + c_\varepsilon|u|^{q(x)-1} \tag{3.5}$$

for all $\omega \in \partial F(x,u)$. Then, from (3.3) and (3.5) one has

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\omega_n - \omega)(u_n - u)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} |\omega_n - \omega||u_n - u|\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \left(\varepsilon|u_n|^{p(x)-1} + c_\varepsilon|u_n|^{q(x)-1} + \varepsilon|u|^{p(x)-1} + c_\varepsilon|u|^{q(x)-1}\right)|u_n - u|\,\mathrm{d}x \\
&\le \varepsilon\left(\|u_n\|_{p(x)}^{p^+-1} + \|u_n\|_{p(x)}^{p^--1} + \|u\|_{p(x)}^{p^+-1} + \|u\|_{p(x)}^{p^--1}\right)\|u_n - u\|_{p(x)} \\
&\quad + c_\varepsilon\left(\|u_n\|_{q(x)}^{q^+-1} + \|u_n\|_{q(x)}^{q^--1} + \|u\|_{q(x)}^{q^+-1} + \|u\|_{q(x)}^{q^--1}\right)\|u_n - u\|_{q(x)} \to 0
\end{aligned} \tag{3.6}$$

as $n \to +\infty$.

Consequently, by $m^I(u_n) = \|u_n^*\|_{E^*} \to 0$, we obtain

$$\int_{\mathbb{R}^N} \left(|\nabla(u_n - u)|^{p(x)} + V(x)|u_n - u|^{p(x)}\right) \mathrm{d}x \to 0$$

from (3.5) and (3.6), i.e., $\|u_n - u\|_E \to 0$. This completes the proof. □

Lemma 3.3. *Suppose that F satisfies* (F_1)-(F_3)*. Then, $\Psi\colon W^{1,p(x)}(\mathbb{R}^N) \to \mathbb{R}$ defined by*

$$\Psi(u) = \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x$$

is locally Lipschitz. Moreover

$$\Psi^\circ(u;v) \le \int_{\mathbb{R}^N} F^\circ(x,u;v)\,\mathrm{d}x$$

for all $u, v \in E$.

Proof. Let $u_1, u_2 \in E$ be fixed elements. Applying Lebourg's mean value theorem, there exists a $\omega_\xi \in \partial F(x,\xi)$ such that

$$F(x,u_1) - F(x,u_2) = \omega_\xi(u_1 - u_2),$$

where ξ is between u_1 and u_2. From (3.5) and the above equation, we have that

$$\begin{aligned}
|\Psi(u_1) - \Psi(u_2)| &\le \int_{\mathbb{R}^N} |\omega_\xi||u_1 - u_2|\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \Big(\varepsilon|u_1|^{p(x)-1} + c_\varepsilon|u_1|^{q(x)-1} + \varepsilon|u_2|^{p(x)-1} + c_\varepsilon|u_2|^{q(x)-1}\Big)|u_1 - u_2|\,\mathrm{d}x \\
&\le \varepsilon\left(\|u_1\|_{p(x)}^{p^+-1} + \|u_1\|_{p(x)}^{p^--1} + \|u_2\|_{p(x)}^{p^+-1} + \|u_2\|_{p(x)}^{p^--1}\right)\|u_1 - u_2\|_{p(x)} \\
&\quad + c_\varepsilon\left(\|u_1\|_{q(x)}^{q^+-1} + \|u_1\|_{q(x)}^{q^--1} + \|u_2\|_{q(x)}^{q^+-1} + \|u_2\|_{q(x)}^{q^--1}\right)\|u_1 - u_2\|_{q(x)} \\
&\le \varepsilon c_3\left(\|u_1\|_E^{p^+-1} + \|u_1\|_E^{p^--1} + \|u_2\|_E^{p^+-1} + \|u_2\|_E^{p^--1}\right)\|u_1 - u_2\|_E \\
&\quad + c_\varepsilon c_4\left(\|u_1\|_E^{q^+-1} + \|u_1\|_E^{q^--1} + \|u_2\|_E^{q^+-1} + \|u_2\|_E^{q^--1}\right)\|u_1 - u_2\|_E.
\end{aligned}$$

From this relation, it follows that $\Psi(u)$ is a locally Lipschitz function on E.

Now, we fix $u, v \in E$. Since F is continuous, $F^\circ(x,u(x);v(x))$ can be expressed as the upper limit of

$$\frac{F(x, z + tv(x)) - F(x,z)}{t},$$

where $t \to 0^+$ and $z \to u$. Since E is a Banach space, there exist functions $z_n \in E$ and numbers $t_n \to 0^+$ such that

$$z_n \to u \quad \text{in } E$$

and

$$\Psi^\circ(u;v) = \lim_{n\to\infty} \frac{\Psi(z_n + t_n v) - \Psi(z_n)}{t_n}.$$

Without loss of generality, we suppose $z_n(x) \to u(x)$ for a.a. $x \in \mathbb{R}^N$, as $n \to \infty$. From (3.5), we have

$$|\omega(x,u)| \le \varepsilon|u|^{p(x)-1} + c_\varepsilon|u|^{q(x)-1} \tag{3.7}$$

for all $\omega(x,u) \in \partial F(x,u)$. We define $g_n \colon \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned}
g_n(x) = &-\frac{F(x, z_n + t_n v) - F(x, z_n)}{t_n} \\
&+ |v|\left[\varepsilon\left(|z_n|^{p(x)-1} + |z_n + t_n v|^{p(x)-1}\right) + c_\varepsilon\left(|z_n|^{q(x)-1} + |z_n + t_n v|^{q(x)-1}\right)\right].
\end{aligned}$$

According to (3.7) it is easy to see that $g_n(x)$ is measurable and non-negative. From Fatou's lemma, we have

$$A = \int_{\mathbb{R}^N} \limsup_{n\to\infty}\,[-g_n(x)]\,\mathrm{d}x \ge \limsup_{n\to\infty} \int_{\mathbb{R}^N} [-g_n(x)]\,\mathrm{d}x = B.$$

Set $g_n = -C_n + D_n$, where

$$C_n = \frac{F(x, z_n + t_n v) - F(x, z_n)}{t_n}$$

and

$$D_n = |v| \left[\varepsilon \left(|z_n|^{p(x)-1} + |z_n + t_n v|^{p(x)-1} \right) + c_\varepsilon \left(|z_n|^{q(x)-1} + |z_n + t_n v|^{q(x)-1} \right) \right].$$

Let $d_n = \int_{\mathbb{R}^N} D_n \,\mathrm{d}x$. Then, $B = \limsup_{n\to\infty} \left(\int_{\mathbb{R}} C_n \,\mathrm{d}x - d_n \right)$. From Hölder's inequality, we have the following estimation

$$\begin{aligned}
&\left| d_n - 2\int_{\mathbb{R}} |v| \left(\varepsilon |u|^{p(x)-1} + c_\varepsilon |u|^{q(x)-1} \right) \mathrm{d}x \right| \\
&\le 3\varepsilon (p^+ - 1) 2^{p^+-2} \|v\|_{p(x)} \left\{ \left[\|z_n - u\|_{p(x)} \left(\|z_n\|_{p(x)}^{p^- -2} + \|z_n\|_{p(x)}^{p^+ -2} + \|u\|_{p(x)}^{p^+ -2} + \|u\|_{p(x)}^{p^- -2} \right) \right] \right. \\
&\quad + \left(\|z_n - u\|_{p(x)} + t_n \|v\|_{p(x)} \right) \\
&\qquad \times \left. \left[\left(\|z_n\|_{p(x)} + t_n \|v\|_{p(x)} \right)^{p^+-2} + \left(\|z_n\|_{p(x)} + t_n \|v\|_{p(x)} \right)^{p^- -2} + \|u\|_{p(x)}^{p^+ -2} + \|u\|_{p(x)}^{p^- -2} \right] \right\} \\
&\quad + 3 c_\varepsilon (q^+ - 1) 2^{p^+-2} \|v\|_{q(x)} \\
&\quad \times \left\{ \left[\|z_n - u\|_{q(x)} \left(\|z_n\|_{q(x)}^{q^- -2} + \|z_n\|_{q(x)}^{q^+ -2} + \|u\|_{q(x)}^{q^+ -2} + \|u\|_{q(x)}^{q^- -2} \right) \right] \right. \\
&\qquad + \left(\|z_n - u\|_{q(x)} + t_n \|v\|_{q(x)} \right) \\
&\qquad \times \left. \left[\left(\|z_n\|_{q(x)} + t_n \|v\|_{q(x)} \right)^{q^+-2} + \left(\|z_n\|_{q(x)} + t_n \|v\|_{q(x)} \right)^{q^- -2} + \|u\|_{q(x)}^{q^+ -2} + \|u\|_{q(x)}^{q^- -2} \right] \right\}.
\end{aligned}$$

From Theorem 2.8, $\|z_n - u\|_{p(x)} \to 0$ and $t_n \to 0^+$, we infer that the sequence $\{d_n\}$ is convergent, with its limit being

$$\lim_{n\to\infty} d_n = 2\int_{\mathbb{R}^N} |v| \left(\varepsilon |u|^{p(x)-1} + c_\varepsilon |u|^{q(x)-1} \right) \mathrm{d}x.$$

Then, we derive

$$\begin{aligned}
B &= \limsup_{n\to\infty} \int_{\mathbb{R}^N} [-g_n(x)] \,\mathrm{d}x = \limsup_{n\to\infty} \frac{\Psi(z_n + t_n v) - \Psi(z_n)}{t_n} - \lim_{n\to\infty} d_n \\
&= \Psi^\circ(u; v) - \lim_{n\to\infty} d_n.
\end{aligned}$$

Furthermore, $A \le A_1 - A_2$, where

$$A_1 = \int_{\mathbb{R}^N} \limsup_{n\to\infty} C_n(x) \,\mathrm{d}x, \quad A_2 = \int_{\mathbb{R}^N} \liminf_{n\to\infty} D_n(x) \,\mathrm{d}x = \lim_{n\to\infty} d_n.$$

Then

$$\begin{aligned}
A_1 &= \int_{\mathbb{R}^N} \limsup_{n\to\infty} \frac{F(x, z_n + t_n v) - F(x, z_n)}{t_n} \,\mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \limsup_{z\to u, t\to 0^+} \frac{F(x, z + tv) - F(x, z)}{t} \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} F^\circ(x, u; v) \,\mathrm{d}x.
\end{aligned}$$

Thus, we complete the proof of Lemma 3.3. $\square$

Lemma 3.4. *If hypotheses* (H_1), (H_2), (M_1), (M_2) *and* (F_1)-(F_5) *hold, then I satisfies the nonsmooth* $(\mathrm{PS})_{\mathrm{c}}$.

Proof. Suppose that $\{u_n\} \subset E$ be a sequence from E such that

$$|I(u_n)| \leq c_5 \tag{3.8}$$

and

$$m^I(u_n) \to 0 \tag{3.9}$$

as $n \to +\infty$. Assume $\|u\|_E > 1$ for convenience. From Lemma 3.2, we only need to show that $\{u_n\}$ is bounded in E. For every $n \in \mathbb{N}$ there exists $u_n^* \in \partial I(u_n)$ such that

$$m^I(u_n) = \|u_n^*\|_{E^*}\,.$$

Clearly, (3.9) implies that

$$I^\circ(u_n; u_n) \geq \langle u_n^*, u_n \rangle_E \geq -\|u_n^*\|_{E^*} \|u_n\|_E \geq -\theta \|u_n\|_E$$

for n large enough. From Lemma 3.3, (3.8), (M_1), (M_2) and (F_5), for n large enough, we have

$$\begin{aligned}
c_1 + 1 + \|u_n\|_E &\geq I(u_n) - \frac{1}{\theta} I^\circ(u_n; u_n) \\
&= \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} \right) \mathrm{d}x \right) - \int_{\mathbb{R}^N} F(x, u_n)\, \mathrm{d}x \\
&\quad - \frac{1}{\theta} M \left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} \right) \mathrm{d}x \right) \\
&\qquad \times \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} \right) \mathrm{d}x - \frac{1}{\theta} \Psi^\circ(u_n; -u_n) \\
&\geq \left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) M \left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} \right) \mathrm{d}x \right) \\
&\qquad \times \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + V(x)|u_n|^{p(x)} \right) \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \left(F(x, u_n) + \frac{1}{\theta} F^\circ(x, u_n; -u_n) \right) \mathrm{d}x \\
&\geq \left(\frac{1-\mu}{p^+} - \frac{1}{\theta} \right) m_0 \|u\|_E^{p^-},
\end{aligned}$$

where $u_n^* \in \partial I(u_n)$ and $\omega_n \in \partial F(x, u_n)$. Noting that $p^- > 1$, we conclude that $\{\|u_n\|_E\}$ is bounded. The proof is completed. □

Proof of Theorem 1.2. From Lemma 3.3 and noting that $\Phi(u)$ is continuous, we obtain that the function $I(u)$ is locally Lipschitz on E.

Claim. There exist $\eta > 0$, $\rho > 0$ and $e \in E$ such that

$$I(u) \geq \eta \quad \text{for all } \|u\|_E = \rho \tag{3.10}$$

and

$$\|e\|_E > \rho, \quad I(e) < 0. \tag{3.11}$$

Firstly, it is easy to obtain

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + \left(V(x) - \frac{V^-}{2} \right) |u|^{p(x)} \right) \mathrm{d}x \geq \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) \mathrm{d}x. \tag{3.12}$$

Set $\varepsilon = \frac{m_0 V^-}{2}$ in (3.5). Then, there exists $c_6 > 0$ such that

$$|F(x,u)| \leq \frac{m_0 V^-}{2p^+} |u|^{p(x)} + c_6 |u|^{q(x)} \tag{3.13}$$

for a.a. $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}$. By virtue of (3.12), (3.13) and (M_1), if $\|u\|_E \leq 1$ we have

$$\begin{aligned}
I(u) &\geq \frac{m_0}{p^+} \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) \mathrm{d}x - \frac{m_0 V^-}{2p^+} \int_{\mathbb{R}^N} |u|^{p(x)} \, \mathrm{d}x - c_6 \int_{\mathbb{R}^N} |u|^{q(x)} \, \mathrm{d}x \\
&= \frac{m_0}{p^+} \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + \left(V(x) - \frac{V^-}{2} \right) |u|^{p(x)} \right) \mathrm{d}x - c_6 \int_{\mathbb{R}^N} |u|^{q(x)} \, \mathrm{d}x \\
&\geq \frac{m_0}{2p^+} \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) \mathrm{d}x - c_6 \int_{\mathbb{R}^N} |u|^{q(x)} \, \mathrm{d}x \\
&\geq \frac{m_0}{2p^+} \|u\|_E^{p^+} - c_7 \|u\|_E^{q^-}.
\end{aligned}$$

Since $p^+ \ll q$, there exist $\eta > 0$ and $\rho > 0$ such that (3.10) holds.

In order to prove (3.11), we firstly prove

$$t^\theta F(x,u) \leq F(x,tu) \quad \text{for all } t > 1 \text{ and all } u \in \mathbb{R}. \tag{3.14}$$

Fix any arbitrarily $u \in \mathbb{R}$. By virtue of the second chain rule, it follows that

$$\partial_t F(x,tu) \subseteq \partial F(x,tu) u$$

for all $t > 0$.

Since $t \mapsto t^{-\theta} F(x,tu)$ $(t > 0)$ is locally Lipschitz, we have

$$\partial_t (t^{-\theta} F(x,tu)) = -\theta t^{-\theta-1} F(x,tu) + t^{-\theta} \partial_t F(x,tu)$$

for all $t > 0$. Therefore,

$$\partial_t (t^{-\theta} F(x,tu)) \subseteq t^{-\theta-1} \left[-\theta F(x,tu) + t \partial F(x,tu) u \right] \tag{3.15}$$

for all $t > 0$.

Next, set $t > 1$. From Lebourg's mean value theorem and (3.15), there exists a $\tau \in (1, t)$, such that

$$\begin{aligned} t^{-\theta}F(x,tu) - F(x,u) &\in \partial_t(\tau^{-\theta}F(x,\tau u))(t-1) \\ &\subseteq \tau^{-\theta-1}\left[-\theta F(x,\tau u) + \tau\partial F(x,\tau u)u\right](t-1). \end{aligned}$$

Thus, there exists $\xi^\tau \in \partial F(x, \tau u)$ such that

$$t^{-\theta}F(x,tu) - F(x,u) = -\tau^{-\theta-1}\left[\theta F(x,\tau u) + \xi^\tau(-\tau u)\right](t-1).$$

Employing (F_5), we have

$$\begin{aligned} t^{-\theta}F(x,tu) - F(x,u) &\geq -\tau^{-\theta-1}\left[\theta F(x,\tau u) + F^\circ(x,\tau u; -\tau u)\right](t-1) \\ &\geq 0, \end{aligned}$$

which deduces (3.14). When $t > t_0 > 0$, by (M_2) we can easily obtain

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{1-\mu}}} t^{\frac{1}{1-\mu}}, \tag{3.16}$$

where t_0 is an arbitrary positive constant. For $v \in E \setminus \{0\}$, choosing $t > 1$, by virtue of (3.14) and (3.16), one has

$$\begin{aligned} I(tv) &= \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|t\nabla v|^{p(x)} + V(x)|tv|^{p(x)}\right) \mathrm{d}x\right) - \int_{\mathbb{R}^N} F(x,tv)\,\mathrm{d}x \\ &\leq \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|t\nabla v|^{p(x)} + V(x)|tv|^{p(x)}\right) \mathrm{d}x\right) - t^\theta \int_{\mathbb{R}^N} F(x,v)\,\mathrm{d}x \\ &\leq \frac{c_8}{(p^-)^{\frac{1}{1-\mu}}} t^{\frac{p^+}{1-\mu}} \left(\int_{\mathbb{R}^N} \left(|\nabla v|^{p(x)} + V(x)|v|^{p(x)}\right) \mathrm{d}x\right)^{\frac{1}{1-\mu}} - t^\theta \int_{\mathbb{R}^N} F(x,v)\,\mathrm{d}x \\ &\to -\infty \end{aligned}$$

as $t \to +\infty$ (since $\theta > \frac{p^+}{1-\mu}$). Note that $I(0) = 0$. So from the nonsmooth mountain pass theorem, I possesses at least one nontrivial solution. □

We will use the following nonsmooth fountain theorem to prove Theorem 1.3.

Since E is a reflexive and separable Banach space, there exist $\{e_j\} \subset E$ and $\left\{e_j^*\right\} \subset E^*$ such that

$$E = \overline{\operatorname{span}\{e_j : j = 1, 2, \ldots\}}, \quad E^* = \overline{\operatorname{span}\left\{e_j^* : j = 1, 2, \ldots\right\}},$$

and

$$\left\langle e_i, e_j^* \right\rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

For convenience, we write $E_j = \operatorname{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^k E_j$ and $Z_k = \overline{\bigoplus_{j=k}^\infty E_j}$.

Definition 3.5. Assume that the compact group G acts diagonally on V^k

$$g(v_1, \ldots, v_k) = (gv_1, \ldots, gv_k),$$

where V is a finite dimensional space. The action of G is admissible if every continuous equivariant map $\partial U \to V^{k-1}$, where U is an open bounded invariant neighborhood of 0 in V^k, $k \geq 2$, has a zero.

Example 3.6. The antipodal action $G = \mathbb{Z}$ on $V = \mathbb{R}$ is admissible.

We consider the following situation:

(A$_1$) The compact group G acts isometrically on the Banach space $X = \overline{\bigoplus_{m\in\mathbb{N}} X_m}$, the space X_m are invariant and there exists a finite dimensional space V such that, for every $m \in \mathbb{N}$, $X_m \simeq V$ and the action of G on V is admissible.

The following lemma is very important when we use the fountain theorem to prove infinite solutions for problem (1.1).

Lemma 3.7. *If $p(x) \leq r(x) \ll p^*(x)$, then we have that*

$$\beta_k = \sup_{u\in Z_k, \|u\|_E=1} |u|_{r(x)} \to 0, \quad k \to \infty.$$

Proof. It is obvious that $0 < \beta_{k+1} \leq \beta_k$. So there exists $\beta \geq 0$ such that $\beta \to \beta$ as $k \to \infty$. We need to show $\beta = 0$. From the definition of β_k, for every $k \geq 0$ there exists $u_k \in Z_k$ such that $\|u_k\|_E = 1$, $0 \leq \beta - |u_k|_{r(x)} < \frac{1}{k}$. Then, there exists a subsequence of $\{u_k\}$, which still denote by u_k, such that

$$u_k \rightharpoonup u \text{ in } E, \quad \text{and} \quad \langle e_j^*, u\rangle = \lim_{k\to\infty} \langle e_j^*, u_k\rangle = 0,\ j = 1, 2, \ldots,$$

which means that $u = 0$ and $u_k \rightharpoonup 0$ in E. Since the Sobolev embedding $E \hookrightarrow L^{r(x)}(\mathbb{R}^N)$ is compact then $u_k \to 0$ in $L^{r(x)}(\mathbb{R}^N)$. Thus we obtain $\beta = 0$. □

The following lemma comes from Theorem 3.1 in [10].

Lemma 3.8. *Under assumption* (A$_1$)*, let $I\colon X \to \mathbb{R}$ be an invariant locally Lipschitz functional. If for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(A$_2$) $a_k = \max_{u\in Y_k, \|u\|=\rho_k} I(u) \leq 0$*;*

(A$_3$) $b_k = \inf_{u\in Z_k, \|u\|=r_k} I(u) \to \infty$*, $k \to \infty$;*

(A$_4$) *I satisfies the nonsmooth* $(\mathrm{PS})_\mathrm{c}$ *condition for every $c > 0$,*

then I has an unbounded sequence of critical values.

Proof of Theorem 1.3. From the Claim in the proof of Theorem 1.2, we have known that I is a locally Lipschitz function on E. Considering of (F_6), we can use the nonsmooth fountain theorem with the antipodal action of $\mathbb{Z}_2$ to prove Theorem 1.3. Furthermore, by Lemma 3.4, we already known that I satisfies the nonsmooth $(\mathrm{PS})_{\mathrm{c}}$. So we only need to check the conditions of (A_2) and (A_3).

Verification of (A_2). From Lemma 3.7, for $u \in Z_k$ with $\|u\|_E \geq 1$, we have

$$\int_{\mathbb{R}^N} |u|^{q(x)}\,\mathrm{d}x \leq \beta_k \|u\|_E^{q^+}. \tag{3.17}$$

Choose a constant $c_{11} > 0$ satisfying (3.13). Then, we consider the real function $H(r)\colon \mathbb{R} \to \mathbb{R}$,

$$H(r) = \frac{m_0}{2p^+} r^{p^-} - c_{11}\beta_k r^{q^+}.$$

By elementary calculus, it is easy to see that H attains its maximum value at

$$r_k = \left(\frac{2c_{11}p^+q^+\beta_k}{m_0 p^-}\right)^{\frac{1}{p^- - q^+}}.$$

The maximum value

$$\begin{aligned}
H(r_k) &= \frac{m_0}{2p^+}\left[\left(\frac{2c_{11}p^+q^+\beta_k}{m_0 p^-}\right)^{\frac{p^-}{p^- - q^+}} - \frac{2p^+}{m_0}c_{11}\beta_k\left(\frac{2c_{11}p^+q^+\beta_k}{m_0 p^-}\right)^{\frac{q^+}{p^- - q^+}}\right] \\
&= \frac{m_0}{2p^+}\left(\frac{2c_{11}p^+\beta_k}{m_0}\right)^{\frac{p^-}{p^- - q^+}}\left[\left(\frac{q^+}{p^-}\right)^{\frac{p^-}{p^- - q^+}} - \left(\frac{q^+}{p^-}\right)^{\frac{q^+}{p^- - q^+}}\right] \\
&= \frac{m_0}{2p^+}\left(\frac{2c_{11}p^+\beta_k}{m_0}\right)^{\frac{p^-}{p^- - q^+}}\left(\frac{q^+}{p^-}\right)^{\frac{p^-}{p^- - q^+}}\left(1 - \frac{p^-}{q^+}\right).
\end{aligned}$$

Since $p^- < q^+$ and $\beta_k \to 0$, we infer that

$$H(r_k) \to +\infty \quad \text{as } k \to +\infty. \tag{3.18}$$

We also have $r_k \to +\infty$. For $u \in Z_k$, $\|u\|_E = r_k$, employing (3.12), (3.13) and (3.17) we have

$$\begin{aligned}
I(u) &\geq \frac{m_0}{p^+}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)\mathrm{d}x - \frac{m_0 V^-}{2p^+}\int_{\mathbb{R}^N}|u|^{p(x)}\,\mathrm{d}x - c_{11}\int_{\mathbb{R}^N}|u|^{q(x)}\,\mathrm{d}x \\
&\geq \frac{m_0}{2p^+}\int_{\mathbb{R}^N}\left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right)\mathrm{d}x - c_{11}\int_{\mathbb{R}^N}|u|^{q(x)}\,\mathrm{d}x \\
&\geq \frac{m_0}{2p^+}\|u\|_E^{p^-} - c_{11}\beta_k\|u\|_E^{q^+} \\
&= H(r_k).
\end{aligned}$$

It follows from (3.18) that

$$b_k = \inf_{u\in Z_k, \|u\|_E = r_k} I(u) \to +\infty$$

as $k \to +\infty$.

Verification of (A_3). From (3.14), we have

$$F(x, tu) \geq t^\theta F(x, u)$$

for all $t > 1$. Therefore for any $v \in Y_k$ with $\|v\|_E = 1$ and $1 < t = \rho_k$, from (3.16) we have

$$\begin{aligned}
I(tv) &= \widehat{M}\left(\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|t\nabla v|^{p(x)} + V(x)|tv|^{p(x)}\right) \mathrm{d}x\right) - \int_{\mathbb{R}^N} F(x, tv)\, \mathrm{d}x \\
&\leq \frac{c_{12}}{(p^-)^{\frac{1}{1-\mu}}} \left(\int_{\mathbb{R}^N} \left(|t\nabla v|^{p(x)} + V(x)|tv|^{p(x)}\right) \mathrm{d}x\right)^{\frac{1}{1-\mu}} - t^\theta \int_{\mathbb{R}^N} F(x, v)\, \mathrm{d}x \\
&\leq \frac{c_{12}}{(p^-)^{\frac{1}{1-\mu}}} \rho_k^{\frac{p^+}{1-\mu}} \left(\int_{\mathbb{R}^N} \left(|\nabla v|^{p(x)} + V(x)|v|^{p(x)}\right) \mathrm{d}x\right)^{\frac{1}{1-\mu}} - \rho_k^\theta \int_{\mathbb{R}^N} F(x, v)\, \mathrm{d}x + c_{13}.
\end{aligned}$$

Since $\theta > \frac{p^+}{1-\mu}$ and $\dim Y_k = k$, setting $u = tv$, it is easy to see that $I(u) \to -\infty$ as $\|u\| \to +\infty$ for $u \in Y_k$. Then, the results of Theorem 1.3 are obtained by the nonsmooth fountain theorem. □

4. Corollaries for a special problem

In this section we will give some typical consequences of Theorems 1.2 and 1.3. We discuss the following special problem:

$$\text{(4.1)} \qquad \begin{cases} -\left(a + b\int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right) \mathrm{d}x\right) \\ \quad \times \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u - V(x)|u|^{p(x)-2}\right) \in \partial F(x, u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

where $a > 0$ and $b \geq 0$. Set $M(t) = a + bt$, $t = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right) \mathrm{d}x$. It is obvious that

$$M(t) \geq a > 0.$$

Taking $\mu = \frac{1}{2}$, we have

$$\widehat{M}(t) = \int_0^t M(s)\, \mathrm{d}s = at + \frac{1}{2}bt^2 \geq \frac{1}{2}(a + bt)t = (1-\mu)M(t)t.$$

So the hypotheses (M_1) and (M_2) are satisfied. Therefore, corresponding to Theorems 1.2 and 1.3, we obtain the following corollaries.

Corollary 4.1. *If hypotheses* (H_1), (H_2) *and* (F_1)-(F_5) *hold, then problem* (4.1) *has at least one nontrivial solution.*

Corollary 4.2. *If hypotheses* (H_1), (H_2) *and* (F_1)-(F_6) *hold, then problem* (4.1) *has a sequence of weak solutions* $\{\pm u_k\}_{k=1}^{\infty}$ *such that* $I(\pm u_k) \to +\infty$ *as* $k \to +\infty$*.*

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