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Constructing Braided Hopf Algebras in Monoidal Hom-category

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Abstract. In this paper, we first define the coquasitriangular monoidal Hom-Hopf algebras. Secondly, we present a method to construct braided monoidal Hom-Hopf algebras $\overline{B}$ and $\underline{B}$ in Yetter-Drinfeld category $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$ and $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ respectively. As applications, we study some special cases in both module and comodule form for (H, ξ_H) being quasitriangular and for (H, ξ_H) being coquasitriangular respectively. Finally, we give some applications and examples of braided monoidal Hom-Hopf algebras in this article.

1. Introduction

Hom-type algebras appeared first in physical contexts, in connection with twisted, discretized or deformed derivatives and corresponding generalizations, discretizations and deformations of vector fields and differential calculus. The paradigmatic examples are q-deformations of Witt and Virasoro algebras constructed in pioneering works (see [4,10,14]). In these examples, the authors used σ-derivations which leaded to a twisted Jacobi identity (see [11, 12]). Motivated by these examples and their generalizations, Larsson and Silvestrov in [13], introduced the notion of Hom-Lie algebras as a deforation of Lie algebras in which the Jacobi identity is twisted by a homomorphism. Later, the concepts of Hom-algebras, Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras and Hom-Lie algebras were developed first in [19, 20].

The original definitions of Hom-bialgebra and Hom-Hopf algebra involve two different linear maps α and β, with α twisting the associativity condition and β the coassociativity condition. Afterwards, two directions of study were developed. One direction is to consider the class of bialgebras for which $\beta = \alpha$. This class of bialgebras are also called Hom-bialgebras and Hom-Hopf algebras (cf. [24, 25]). The other one is called monoidal Hom-bialgebras and monoidal Hom-Hopf algebras in monoidal Hom-category, initiated in [3], where the map α is assumed to be invertible and $\beta = \alpha^{-1}$. Hom-Long dimodule category

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(see [5]), Yetter-Drinfeld module category (see [6, 15]) and generalized Yetter-Drinfeld module (see [26]) have been studied for monoidal Hom-bialgebras and we will construct braided monoidal Hom-Hopf algebras in these categories.

Braided Hopf algebras (braided groups) are Hopf algebras in the braided category of Yetter-Drinfeld modules in [16–18]. Applications in physics include the spectrum generating quantum groups and the constructions of homogeneous quantum groups. Applications in pure mathematics include the proof of Schur's double centralizer theorems in [7,9], the complete classification of all pointed Hopf algebras of dimension p^2 or p^3 [1,2], and linearly recursive sequences [21]. In [23] Wang presented a method to construct braided Hopf algebras in Yetter-Drinfel'd category. There is a natural question arising: Can we use Wang's method to construct braided Hopf algebras in braided monoidal Hom-category?

We give an answer to this question in our paper, which is one motivation of this paper. Another motivation is due to [1, 2] and [8] in which the authors investigated braided Hopf algebras of order p and the trace formulae respectively. Then it is natural to ask whether there is an analogue of such properties for braided monoidal Hom-Hopf algebras, i.e., braided Hopf algebras in braided monoidal Hom-category.

This article is organized as follows. In Section 2, we will present the background material, including the related definitions on monoidal Hom-Hopf algebras. In Section 3, we will define the notion of the coquasitriangular monoidal Hom-Hopf algebra.

In Section 4, we will consider two braided monoidal Hom-categories $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$ and $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$, and define one twisted algebra $\overline{B}$ for a bialgebra B which is both in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$ and $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$. Then under suitable assumption, we show that $\overline{B}$ is a braided monoidal Hom-Hopf algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$. Similarly, it is proved that there exists another braided monoidal Hom-Hopf algebra $\underline{B}$ in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$.

Section 5 is concerned with the conditions under which $\overline{B}$ and $\underline{B}$ above respectively become braided monoidal Hom-Hopf algebras. At the end of the paper, we will give some applications and examples of braided monoidal Hom-Hopf algebras.

2. Preliminaries

Throughout this paper, let k be a fixed field. All vector spaces and tensor product are over k unless otherwise specified. We refer the readers to the books of Sweedler [22] for the related concepts on the general theory of Hopf algebras. Let (C, Δ) be a coalgebra. We use the notation for Δ as follows:

$$\Delta(c) = c_1 \otimes c_2, \quad \forall\, c \in C.$$

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ denote the usual monoidal category of k-vector spaces and linear maps between them. Recall from [3] that there is the *monoidal Hom-category*

$\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, \mathrm{id}), \widetilde{a}, \widetilde{l}, \widetilde{r})$, a new monoidal category, associated with $\mathcal{M}_k$ as follows:

- The objects of $\mathcal{H}(\mathcal{M}_k)$ are couples (M, ξ_M), where $M \in \mathcal{M}_k$ and $\xi_M \in \mathrm{Aut}_k(M)$, the set of all k-linear automomorphisms of M;

- The morphism $f\colon (M, \xi_M) \to (N, \xi_N)$ in $\mathcal{H}(\mathcal{M}_k)$ is the k-linear map $f\colon M \to N$ in $\mathcal{M}_k$ satisfying $\xi_N \circ f = f \circ \xi_M$, for any two objects $(M, \xi_M), (N, \xi_N) \in \mathcal{H}(\mathcal{M}_k)$;

- The tensor product is given by

$$(M, \xi_M) \otimes (N, \xi_N) = (M \otimes N, \xi_M \otimes \xi_N)$$

  for any $(M, \xi_M), (N, \xi_N) \in \mathcal{H}(\mathcal{M}_k)$;

- The tensor unit is given by (k, id);

- The associativity constraint $\widetilde{a}$ is given by the formula

$$\widetilde{a}_{M,N,L} = a_{M,N,L} \circ ((\xi_M \otimes \mathrm{id}) \otimes \varsigma^{-1}) = (\xi_M \otimes (\mathrm{id} \otimes \varsigma^{-1})) \circ a_{M,N,L},$$

  for any objects $(M, \xi_M), (N, \xi_N), (L, \varsigma) \in \mathcal{H}(\mathcal{M}_k)$;

- The left and right unit constraint $\widetilde{l}$ and $\widetilde{r}$ are given by

$$\widetilde{l}_M = \xi_M \circ l_M = l_M \circ (\mathrm{id} \otimes \xi_M), \quad \widetilde{r}_M = \xi_M \circ r_M = r_M \circ (\xi_M \otimes \mathrm{id})$$

  for all $(M, \xi_M) \in \mathcal{H}(\mathcal{M}_k)$.

We now recall the following notions used later.

Definition 2.1. [3] A unital monoidal Hom-associative algebra is a vector space A together with an element $1_A \in A$ and linear maps

$$m\colon A \otimes A \to A; \quad a \otimes b \mapsto ab, \quad \xi_A \in \mathrm{Aut}_k(A)$$

such that

$$\xi_A(a)(bc) = (ab)\xi_A(c), \quad \xi_A(ab) = \xi_A(a)\xi_A(b),$$
$$a1_A = 1_A a = \xi_A(a), \quad \xi_A(1_A) = 1_A$$

for all $a, b, c \in A$.

Remark 2.2. Let (A, ξ_A) and $(A', \xi_{A'})$ be two monoidal Hom-algebras. A monoidal Hom-algebra map $f\colon (A, \xi_A) \to (A', \xi_{A'})$ is a linear map such that $f \circ \xi_A = \xi_{A'} \circ f$, $f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

Definition 2.3. [3] A counital monoidal Hom-coassociative coalgebra is an object (C, ξ_C) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta\colon C \to C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon\colon C \to k$ such that

$$\xi_C^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \xi_C^{-1}(c_2), \quad \Delta(\xi_C(c)) = \xi_C(c_1) \otimes \xi_C(c_2),$$
$$c_1\varepsilon(c_2) = \xi_C^{-1}(c) = \varepsilon(c_1)c_2, \quad \varepsilon(\xi_C(c)) = \varepsilon(c)$$

for all $c \in C$.

Remark 2.4. Let (C, ξ_C) and $(C', \xi_{C'})$ be two monoidal Hom-coalgebras. A monoidal Hom-coalgebra map $f\colon (C, \xi_C) \to (C', \xi_{C'})$ is a linear map such that $f \circ \xi_C = \xi_{C'} \circ f$, $\Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$.

Definition 2.5. [3] A monoidal Hom-bialgebra $H = (H, \xi_H, m, 1_H, \Delta, \varepsilon)$ is a bialgebra in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that $(H, \xi_H, m, 1_H)$ is a monoidal Hom-algebra and $(H, \xi_H, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are morphisms of algebras, that is, for all $h, g \in H$,

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1_H) = 1_H \otimes 1_H, \quad \varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad \varepsilon(1_H) = 1.$$

Definition 2.6. [3] A monoidal Hom-bialgebra (H, ξ_H) is called a monoidal Hom-Hopf algebra if there exists a morphism (called antipode) $S\colon H \to H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e., $S \circ \xi_H = \xi_H \circ S$), which is the convolution inverse of the identity morphism id_H (i.e., $S * \mathrm{id} = 1_H \circ \varepsilon = \mathrm{id} * S$). Explicitly, for all $h \in H$,

$$S(h_1)h_2 = \varepsilon(h)1_H = h_1S(h_2).$$

Remark 2.7. The antipode of monoidal Hom-Hopf algebras has almost all the properties of antipode of Hopf algebras such as

$$S(hg) = S(g)S(h), \quad S(1_H) = 1_H, \quad \Delta(S(h)) = S(h_2) \otimes S(h_1), \quad \varepsilon \circ S = \varepsilon$$

for all $h, g \in H$. That is, S is a monoidal Hom-anti-(co)algebra homomorphism. Since ξ_H is bijective and commutes with S, we can also have that the inverse ξ_H^{-1} commutes with S, that is, $S \circ \xi_H^{-1} = \xi_H^{-1} \circ S$.

In the following, we recall the notions of actions on monoidal Hom-algebras and coactions on monoidal Hom-coalgebras.

Definition 2.8. [3] Let (A, ξ_A) be a monoidal Hom-algebra. A left (A, ξ_A)-Hom-module consists of an object (M, ξ_M) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi\colon A \otimes M \to M$, $\psi(a \otimes m) = a \cdot m$ such that

$$\xi_A(a) \cdot (b \cdot m) = (ab) \cdot \xi_M(m), \quad \xi_M(a \cdot m) = \xi_A(a) \cdot \xi_M(m), \quad 1_A \cdot m = \xi_M(m)$$

for all $a, b \in A$ and $m \in M$.

Monoidal Hom-algebra (A, ξ_A) can be considered as a Hom-module on itself by the Hom-multiplication. Let (M, ξ_M) and (N, ξ_N) be two left (A, ξ_A)-Hom-modules. A morphism $f\colon M \to N$ is called left (A, ξ_A)-linear if $f(a \cdot m) = a \cdot f(m)$, $f \circ \xi_M = \xi_N \circ f$. We denote the category of left (A, ξ_A)-Hom modules by $\widetilde{\mathcal{H}}({}_A\mathcal{M}_k)$.

Definition 2.9. [3] Let (C, ξ_C) be a monoidal Hom-coalgebra. A right (C, ξ_C)-Hom-comodule is an object (M, ξ_M) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $\rho_M\colon M \to M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$ such that

$$\xi_M^{-1}(m_{(0)}) \otimes \Delta_C(m_{(1)}) = (m_{(0)(0)} \otimes m_{(0)(1)}) \otimes \xi_C^{-1}(m_{(1)}),$$
$$\rho_M(\xi_M(m)) = \xi_M(m_{(0)}) \otimes \xi_C(m_{(1)}), \quad m_{(0)}\varepsilon(m_{(1)}) = \xi_M^{-1}(m)$$

for all $m \in M$.

(C, ξ_C) is a Hom-comodule on itself via the Hom-comultiplication. Let (M, ξ_M) and (N, ξ_N) be two right (C, ξ_C)-Hom-comodules. A morphism $g\colon M \to N$ is called right (C, ξ_C)-colinear if $g \circ \mu = \nu \circ g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$. The category of right (C, γ)-Hom-comodules is denoted by $\widetilde{\mathcal{H}}(\mathcal{M}^C)$.

Definition 2.10. [6] Let (H, ξ_H) be a monoidal Hom-bialgebra. A monoidal Hom-algebra (B, ξ_B) is called a left H-Hom-module algebra, if (B, ξ_B) is a left H-Hom-module with action $\cdot$ obeying the following axioms:

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_B = \varepsilon(h)1_B$$

for all $a, b \in B$, $h \in H$.

Definition 2.11. [15] Let (H, ξ_H) be a monoidal Hom-bialgebra. A monoidal Hom-algebra (B, ξ_B) is called a right H-Hom-comodule algebra, if (B, ξ_B) is a right H-Hom-comodule with coaction ρ obeying the following axioms:

$$\rho(ab) = a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}, \quad \rho_l(1_B) = 1_B \otimes 1_H$$

for all $a, b \in B$, $h \in H$.

Definition 2.12. [15] Let (H, ξ_H) be a monoidal Hom-bialgebra. A monoidal Hom-coalgebra (B, ξ_B) is called a left H-Hom-module coalgebra, if (B, ξ_B) is a left H-Hom-module with coaction $\cdot$ obeying the following axioms:

$$\Delta(h \cdot b) = h_1 \cdot b_1 \otimes h_2 \cdot b_2, \quad \varepsilon_B(h \cdot b) = \varepsilon_H(h)\varepsilon_B(b)$$

for all $a, b \in B$, $h \in H$.

Definition 2.13. [15] Let (H, ξ_H) be a monoidal Hom-bialgebra. A monoidal Hom-coalgebra (B, ξ_B) is called a right H-Hom-comodule coalgebra, if (B, ξ_B) is a right H-Hom-comodule with coaction ρ obeying the following axioms:

$$b_{1(0)} \otimes b_{2(0)} \otimes b_{1(1)} b_{2(1)} = b_{(0)1} \otimes b_{(0)2} \otimes b_{(1)}, \quad \varepsilon_B(b_{(0)}) b_{(1)} = \varepsilon_B(b) 1_H$$

for all $a, b \in B$, $h \in H$.

Definition 2.14. [6, 15] Let (H, m, Δ, ξ_H) be a monoidal Hom-bialgebra. A left-right Yetter-Drinfeld Hom-module over (H, ξ_H) is the object $(M, \cdot, \rho, \xi_M)$ which is both in $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ and $\widetilde{\mathcal{H}}(\mathcal{M}^H)$ obeying the compatibility condition:

$$h_1 \cdot m_{(0)} \otimes h_2 m_{(1)} = (\xi_H(h_2) \cdot m)_{(0)} \otimes (\xi_H^{-1}(\xi_H(h_2) \cdot m)_{(1)}) h_1. \tag{2.1}$$

Remark 2.15. (1) The category of all left-right Yetter-Drinfeld Hom-modules is denoted by $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H)$ with understanding morphism.

(2) If (H, ξ_H) is a monoidal Hom-Hopf algebra with a bijective antipode S, then the above equality is equivalent to

$$\rho(h \cdot m) = \xi_H(h_{21}) \cdot m_{(0)} \otimes (h_{22} \xi_H^{-1}(m_{(1)})) S^{-1}(h_1)$$

for all $h \in H$ and $m \in M$.

There exist two prebraided monoidal structures on $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H)$ as follows. Let (V, ξ_V), $(W, \xi_W) \in \widetilde{\mathcal{H}}({}_H\mathcal{YD}^H)$. For $v \otimes w \in V \otimes W$ and $h \in (H, \xi_H)$, one structure is defined by the following structure:

$$h \rightharpoonup (v \otimes w) = h_2 \cdot v \otimes h_1 \cdot w, \tag{2.2}$$

$$\delta(v \otimes w) = v^{(0)} \otimes w^{(0)} \otimes v^{(1)} w^{(1)}, \tag{2.3}$$

$$\tau'_{V,W}(v \otimes w) = v^{(1)} \cdot \xi_W^{-1}(w) \otimes \xi_V(v^{(0)}), \tag{2.4}$$

and $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$ denotes the category $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H)$ which is equipped with the above prebraided monoidal structure. Then $(V \otimes W, \rightharpoonup, \delta)$ is in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$.

The other one is given by the following structure:

$$h \rightharpoondown (v \otimes w) = h_1 \cdot v \otimes h_2 \cdot w, \tag{2.5}$$

$$\delta(v \otimes w) = v_{(0)} \otimes w_{(0)} \otimes w_{(1)} v_{(1)}, \tag{2.6}$$

$$\tau''_{V,W}(v \otimes w) = \xi_W(w_{(0)}) \otimes w_{(1)} \cdot \xi_V^{-1}(v), \tag{2.7}$$

and $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ denotes the category $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H)$ with the above prebraided monoidal structure. Then $(V \otimes W, \rightharpoondown, \rho)$ is in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$.

Definition 2.16. [5] Let (A, ξ_A) and (H, ξ_H) be two monoidal Hom-Hopf algebras. A generalized Long dimodule is a quadruple $(M, \xi_M, \rightharpoonup, \rho)$, where $(M, \xi_M, \rightharpoonup)$ is a left H-module, (M, ξ_M, ρ) is a right H-comodule such that the following compatibility condition holds:

$$\delta(h \rightharpoonup b) = \xi_H^{-1}(h) \rightharpoonup b^{(0)} \otimes \xi_A(b^{(1)}) \tag{2.8}$$

for all $h \in H$ and $b \in M$. The category of H-Hom-bimodules over (H, ξ_H) will be denoted by $\widetilde{\mathcal{H}}({}_H\mathcal{L}^A)$ with morphisms being H-linear and H-colinear. Especially, when $A = H$ we get a Long dimodule category $\widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$.

Definition 2.17. A quasitriangular monoidal Hom-Hopf algebra is a triple (H, ξ_H, R), where (H, ξ_H) is a monoidal Hom-Hopf algebra over k and $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ is invertible such that the following conditions are satisfied ($r = R$):

(QT1) $\Delta(R^{(1)}) \otimes R^{(2)} = R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$,
(QT2) $R^{(1)} \otimes \Delta(R^{(2)}) = R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$,
(QT3) $\Delta^{\mathrm{cop}}(R) = R\Delta(h)R^{-1}$,
(QT4) $(\xi_H \otimes \xi_H) \circ R = R$,

where $\Delta^{\mathrm{cop}}(h) = h_2 \otimes h_1$ for all $h \in H$. If $R^{-1} = R^{(2)} \otimes R^{(1)}$, then (H, ξ_H, R) is called triangular.

3. Coquasitriangular monoidal Hom-Hopf algebras

Definition 3.1. A coquasitriangular monoidal Hom-Hopf algebra is a triple $(H, \xi_H, \langle \cdot \mid \cdot \rangle)$ where (H, ξ_H) is a monoidal Hom-Hopf algebra over k and $\langle \cdot \mid \cdot \rangle : H \otimes H \to k$ is a k-bilinear form which is convolution invertible such that the following conditions hold:
(CQT1) $\langle h \mid gl \rangle = \langle h_1 \mid l \rangle \langle h_2 \mid g \rangle$,
(CQT2) $\langle hg \mid l \rangle = \langle h \mid l_1 \rangle \langle g \mid l_2 \rangle$,
(CQT3) $\langle h_1 \mid g_1 \rangle h_2 g_2 = g_1 h_1 \langle h_2 \mid g_2 \rangle$,
(CQT4) $\langle \cdot \mid \cdot \rangle \circ (\xi_H \otimes \xi_H) = \langle \cdot \mid \cdot \rangle$,
If $\langle h_1 \mid g_1 \rangle \langle g_2 \mid h_2 \rangle = \varepsilon(g)\varepsilon(h)$ then $(H, \xi_H, \langle \cdot \mid \cdot \rangle)$ is called cotriangular.

Example 3.2. Recall from Example 2.5 in [26] that $(H_4 = k\{1, g, x, gx = -xg = y\}, \xi, \Delta, \varepsilon, S)$ is a monoidal Hom-Hopf algebra, where the algebraic structure is given as follows:

$\circ$	1	g	x	y
1	1	g	cx	cy
g	g	1	cy	cx
x	cx	$-cy$	0	0
y	cy	$-cx$	0	0

$$\xi(1) = 1, \quad \xi(g) = g, \quad \xi(x) = cx, \quad \xi(y) = cy,$$
$$\Delta(g) = g \otimes g, \quad \Delta(x) = c^{-1}(x \otimes 1) + c^{-1}(g \otimes x), \quad \Delta(y) = c^{-1}(y \otimes g) + c^{-1}(1 \otimes y),$$
$$\varepsilon(g) = 1, \quad \varepsilon(x) = \varepsilon(y) = 0, \quad S(g) = g, \quad S(x) = -y, \quad S(y) = x$$

for all $0 \neq c \in k$. Then $(H_4, \xi, \sigma_\alpha)$ has a uniquely coquasitriangular structure, where σ_m is given by

σ_α	1	g	x	y
1	1	1	0	0
g	1	-1	0	0
x	0	0	α	α
y	0	0	$-\alpha$	α

for $\alpha \in k$ and $c^2 = 1$.

Definition 3.3. A monoidal Hom-Hopf pairing (B, H) means a triple (B, H, τ), where (B, ξ_B) and (H, ξ_H) are monoidal Hom-Hopf algebras and $\tau \colon B \times H \to k$ is a convolution invertible bilinear form satisfying:

(DP1) $\tau(ab, h) = \tau(a, h_1)\tau(b, h_2)$,
(DP2) $\tau(a, hl) = \tau(a_1, h)\tau(a_2, l)$,
(DP3) $\tau \circ (\xi_H \otimes \xi_H) = \tau$,
(DP4) $\tau(1_B, h) = \varepsilon(h)1$,
(DP5) $\tau(a, 1_H) = \varepsilon(a)1$,

for any $a, b \in B$, $h \in H$.

It is easy to see that (DP1) and (DP2) yield

(DP1)$'$ $\tau^{-1}(ab, h) = \tau^{-1}(a, h_2)\tau^{-1}(b, h_1)$,
(DP2)$'$ $\tau(a, hl) = \tau(a_1, l)\tau(a_2, h)$,

for $a, b \in B$, $h, l \in H$.

Definition 3.4. Let (B, ξ_B) and (H, ξ_H) be two monoidal Hom-bialgebras. A bilinear form $\tau \colon B \otimes H \to k$ is called a skew pairing if

(SP1) $\tau(bc, h) = \tau(b, h_1)\tau(c, h_2)$,
(SP2) $\tau(b, gh) = \tau(b_1, h)\tau(b_2, g)$,
(SP3) $\tau(b, h) = \tau(\xi_B(b), \xi_H(h))$,
(SP4) $\tau(1_B, h) = \varepsilon(h)$, $\tau(b, 1_H) = \varepsilon(b)$

for all $b, c \in B$ and $g, h \in H$.

Let (C, ξ_C) be a monoidal Hom-coalgebra. The opposite coalgebra $(C^{\mathrm{cop}}, \xi_C)$ is (C, ξ_C) as a k-module with comultiplication given by $\Delta^{\mathrm{cop}}(c) = c_2 \otimes c_1$ for $c \in C$. Suppose that (H, ξ_H) is a monoidal Hom-Hopf algebra with bijective antipode S (this holds if H is quasitriangular or coquasitriangular). Then H^{op} and H^{cop} are both monoidal Hom-Hopf algebras with antipode S^{-1}.

Example 3.5. Let (B, H, τ) be a skew-pairing monoidal Hom-Hopf algebras. Then $(B^{\mathrm{cop}}, H, \tau)$ and $(B, H^{\mathrm{op}}, \tau)$ are monoidal Hom-Hopf pairings.

Example 3.6. Let $(H, \xi_H, \langle \cdot \mid \cdot \rangle)$ be a coquasitriangular monoidal Hom-Hopf algebra. Then $(H^{\mathrm{cop}}, H, \langle \cdot \mid \cdot \rangle)$ and $(H, H^{\mathrm{op}}, \langle \cdot \mid \cdot \rangle)$ are monoidal Hom-Hopf pairings.

Example 3.7. Let (H, ξ_H) be a finite-dimensional monoidal Hom-Hopf algebra. Then $(H^*, H, \langle \cdot \mid \cdot \rangle)$ is a monoidal Hom-Hopf pairing, where H^* is the dual monoidal Hom-Hopf algebra and $\langle \cdot \mid \cdot \rangle$ is the evaluation map.

Dually, we define a dual R-Hom-Hopf algebra.

Definition 3.8. A dual R-Hom-Hopf algebra is a triple (B, H, R), where (B, ξ_B) and (H, ξ_H) are two monoidal Hom-Hopf algebras and $R = R^{(1)} \otimes R^{(2)} \in B \otimes H$ is an invertible element such that the following identities hold ($r = R$):

(QT1) $\Delta(R^{(1)}) \otimes R^{(2)} = R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$,
(QT2) $R^{(1)} \otimes \Delta(R^{(2)}) = R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$

for all $h \in H$. It is not hard to check that

$$R^{(1)} \otimes R^{(2)} = S_H(R^{(1)}) \otimes S_H(R^{(2)}) = S_H^2(R^{(1)}) \otimes R^{(2)} = R^{(1)} \otimes S_H^2(R^{(2)})$$

and

$$R^{-1} = S_H(R^{(1)}) \otimes R^{(2)} = R^{(1)} \otimes S_H(R^{(2)}).$$

Example 3.9. Let (B, H, R) be an R-Hom-Hopf algebras. Then (B, H^{cop}, R) and (B^{op}, H, R) are dual R-Hom-Hopf algebras.

Example 3.10. Let (H, R) be a quasitriangular monoidal Hom-Hopf algebra. Then (H^{op}, H, R) and (H, H^{cop}, R) are dual R-Hom-Hopf algebras.

Example 3.11. Let (H, ξ_H) be a finite-dimensional monoidal Hom-Hopf algebra. Let $\{h_i\}$ and $\{h_i^*\}$ be dual bases of (H, ξ_H). Then (H, H^*, R) is a dual R-Hom-Hopf algebra, where $R = \sum_{i=1}^{n} h_i \otimes h_i^*$.

4. Braided monoidal Hom-bialgebras in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H)$

Let (H,ξ_H) be a monoidal Hom-Hopf algebra and $(A,\xi_A,\cdot,\delta_A)$ a monoidal Hom-algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H_1)$ where $\cdot$ and δ_A is a left H-module structure and a right H-comodule structure on A respectively. We define $(A^\diamond,\xi_A)=(A,\xi_A)$ as linear space, with a twisted multiplication given by

$$a\diamond b=(b^{(1)}\cdot\xi_A^{-1}(a))\xi_A(b^{(0)}).$$

Proposition 4.1. *$(A^\diamond,\xi_A,\diamond)$ is an associative monoidal Hom-algebra.*

Proof. It is easy to see that 1_A is a unit of $A^\diamond$. As to associativity of $*$, one has

$$\begin{aligned}
(a\diamond b)\diamond\xi_A(c) &= (\xi_H(c^{(1)})\cdot((\xi_H^{-1}(b^{(1)})\cdot\xi_A^{-2}(a))b^{(0)}))\xi_A^2(c^{(0)})\\
&\overset{(2.2)}{=} ((c^{(1)}{}_2\xi_H^{-1}(b^{(1)})\cdot\xi_A^{-1}(a))(\xi_H(c^{(1)}{}_1)\cdot b^{(0)}))\xi_A^2(c^{(0)})\\
&\overset{(2.1)}{=} ((((c^{(1)}{}_2\cdot\xi_A^{-2}(b))^{(1)}c^{(1)}{}_1)\cdot\xi_A^{-1}(a))(\xi_H^2(c^{(1)}{}_2)\cdot b)^{(0)})\xi_A^2(c^{(0)})\\
&= (((c^{(1)}\cdot\xi_A^{-1}(b))^{(1)}\xi_H(c^{(0)(1)}))\cdot a)(\xi_A((c^{(1)}\cdot\xi_A^{-1}(b))^{(0)})\xi_A^2(c^{(0)(0)}))\\
&\overset{(2.3)}{=} (((c^{(1)}\cdot\xi_A^{-1}(b))\xi_A(c^{(0)}))^{(1)}\cdot a)\xi_A(((c^{(1)}\cdot\xi_A^{-1}(b))\xi_A(c^{(0)}))^{(0)})\\
&= \xi_A(a)\diamond((c^{(1)}\cdot\xi_A^{-1}(b))\xi_A(c^{(0)}))\\
&= \xi_A(a)\diamond(b\diamond c).
\end{aligned}$$

This concludes the proof. □

Remark 4.2. That $(A,\xi_A,\cdot,\delta_A)$ is a monoidal Hom-algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H_1)$ is not a necessary condition for $(A^\diamond,\xi_A,\diamond)$ to be an associative Hom-algebra. This can be seen by (4.10) and the proof of Theorem 4.4.

Similarly, for any $(A,\xi_A,\cdot,\rho)\in\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H_2)$ we define $(A^{\circledast},\xi_A)=(A,\xi_A)$ as linear space with a twisted multiplication defined by

$$a\circledast b=\xi_A(a_{(0)})(a_{(1)}\cdot\xi_A^{-1}(b)),$$

and we have the following proposition.

Proposition 4.3. *$(A^{\circledast},\xi_A,\circledast)$ is an associative monoidal Hom-algebra.*

Proof. Similar to that of Theorem 4.1. □

Let $(B,\xi_B,\rightharpoonup,\delta)$ be a monoidal Hom-algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H_1)$ and $(B,\xi_B,\neg,\rho)$ a monoidal Hom-algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H_2)$ such that $(B,\xi_B,\rightharpoonup,\rho)$ and $(B,\xi_B,\neg,\delta)$ are in $\widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$.

Now we assume that the following Conditions (A) are satisfied:

Conditions (A):

$$h \rightharpoonup (l \rightharpoondown b) = \xi_H(l) \rightharpoondown (\xi_H^{-1}(h) \rightharpoonup b), \tag{4.1}$$

$$(h_1 \rightharpoondown b_1) \otimes (h_2 \rightharpoonup b_2) = \varepsilon(h)\xi_B(b_1) \otimes \xi_B(b_2), \tag{4.2}$$

$$\Delta(h \rightharpoonup b) = (\xi_H^{-1}(h) \rightharpoonup b_1) \otimes \xi_B(b_2), \tag{4.3}$$

$$\Delta(h \rightharpoondown b) = \xi_B(b_1) \otimes (\xi_H^{-1}(h) \rightharpoondown b_2), \tag{4.4}$$

$$(b_{(0)}{}^{(0)} \otimes \xi_H^{-1}(b_{(1)})) \otimes b_{(0)}{}^{(1)} = (b^{(0)}{}_{(0)} \otimes b^{(0)}{}_{(1)}) \otimes \xi_H^{-1}(b^{(1)}), \tag{4.5}$$

$$(b_{1(0)} \otimes b_2{}^{(0)}) \otimes b_2{}^{(1)} b_{1(1)} = (\xi_B^{-1}(b_1) \otimes \xi_B^{-1}(b_2)) \otimes 1_H, \tag{4.6}$$

$$(b^{(0)}{}_1 \otimes b^{(0)}{}_2) \otimes \xi_H^{-1}(b^{(1)}) = (b_1{}^{(0)} \otimes \xi_B^{-1}(b_2)) \otimes b_1{}^{(1)}, \tag{4.7}$$

$$\xi_B^{-1}(b_1) \otimes (b_{2(0)} \otimes b_{2(1)}) = (b_{(0)1} \otimes b_{(0)2}) \otimes \xi_H^{-1}(b_{(1)}), \tag{4.8}$$

for any $b \in (B, \xi_B)$ and $h, l \in (H, \xi_H)$.

Then we define

$$h \to b = \xi_H(h_1) \rightharpoonup (h_2 \rightharpoondown \xi_B^{-1}(b)), \tag{4.9}$$

$$a \star b = (b^{(1)} \to \xi_B^{-1}(a))\xi_B(b^{(0)}), \tag{4.10}$$

$$\chi_B(b) = \xi_B(b_{(0)}{}^{(0)}) \otimes \xi_H^{-1}(b_{(1)}) b_{(0)}{}^{(1)}, \tag{4.11}$$

for any $h \in H$ and $a, b \in B$.

It is not hard to verify that $(B, \xi_B, \to)$ is a left H-Hom-module, that (B, ξ_B, χ_B) is a right H-Hom-comodule, and that $(B, \xi_B, \to, \chi_B)$ is an object in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H)$. But $(B, \xi_B, m_B, \to)$ is not a monoidal Hom-algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$. In fact, we have

$$\begin{aligned}
h \to (ab) &\overset{(4.9)(2.5)}{=} \xi_H(h_1) \rightharpoonup ((h_{21} \rightharpoondown \xi_B^{-1}(a))(h_{22} \rightharpoondown \xi_B^{-1}(b))) \\
&\overset{(2.2)}{=} (\xi_H^2(h_{121}) \rightharpoonup (\xi_H(h_{122}) \rightharpoondown \xi_B^{-1}(a)))(\xi_H(h_{11}) \rightharpoonup (h_{22} \rightharpoondown \xi_B^{-1}(b))) \\
&= (\xi_H(h_{12}) \to a)(\xi_H(h_{11}) \rightharpoonup (h_{22} \rightharpoondown \xi_B^{-1}(b))) \\
&\neq (h_1 \to a)(h_2 \to b),
\end{aligned}$$

and this proves that $(B, \xi_B, \to, \delta_B)$ is not an H-Hom-module algebra. Thus we cannot apply Proposition 4.1 to $(B, \xi_B, m_B, \to)$. However, one can calculate:

$$\begin{aligned}
& \xi_B(a) \star (b \star c) \\
\overset{(4.10)(2.3)}{=}\; & (((c^{(1)} \to \xi_B^{-1}(b))^{(1)} \xi_H(c^{(0)(1)})) \to a)(\xi_B((c^{(1)} \to \xi_B^{-1}(b))^{(0)}) \xi_B^2(c^{(0)(0)})) \\
=\; & (((\xi_H(c^{(1)}{}_2) \to \xi_B^{-1}(b))^{(1)} \xi_H(c^{(1)}{}_1)) \to a)(\xi_B((\xi_H(c^{(1)}{}_1) \to \xi_B^{-1}(b))^{(0)}) \xi_B(c^{(0)})) \\
\overset{(4.9)}{=}\; & (((\xi_H^2(c^{(1)}{}_{12}) \rightharpoonup (c^{(1)}{}_2 \rightharpoondown \xi_B^{-2}(b)))^{(1)} \xi_H^2(c^{(1)}{}_{11})) \to a)(\xi_B((\xi_H^2(c^{(1)}{}_{12}) \\
& \rightharpoonup (c^{(1)}{}_2 \rightharpoondown \xi_B^{-2}(b)))^{(0)}) \xi_B(c^{(0)})) \\
\overset{(2.1)}{=}\; & (\xi_H(\xi_H(c^{(1)}{}_{12})(c^{(1)}{}_2 \rightharpoondown \xi_B^{-2}(b))^{(1)}) \to a)(\xi_B(\xi_H(c^{(1)}{}_{11})
\end{aligned}$$

$$
\begin{aligned}
& \rightharpoonup (c^{(1)}{}_2 \rightharpoondown \xi_B^{-2}(b))^{(0)})\xi_B(c^{(0)})) \\
= \quad & (((\xi_H(c^{(1)}{}_{12})(c^{(1)}{}_2 \rightharpoondown \xi_B^{-2}(b))^{(1)}) \rightarrow \xi_B^{-1}(a))(\xi_H^2(c^{(1)}{}_{11}) \\
& \rightharpoonup \xi_B((c^{(1)}{}_2 \rightharpoondown \xi_B^{-2}(b))^{(0)})))\xi_B^2(c^{(0)}) \\
\overset{(2.8)}{=} \quad & (((\xi_H(c^{(1)}{}_{12})\xi_H^{-1}(b^{(1)})) \rightarrow \xi_B^{-1}(a))(\xi_H^2(c^{(1)}{}_{11}) \rightharpoonup (c^{(1)}{}_2 \rightharpoondown \xi_B^{-1}(b^{(0)}))))\xi_B^2(c^{(0)}) \\
\overset{(4.9)}{=} \quad & ((\xi_H^3(c^{(1)}{}_{121}) \rightharpoonup (\xi_H^2(c^{(1)}{}_{122}) \rightharpoondown (\xi_H^{-2}(b^{(1)}) \rightarrow \xi_B^{-3}(a))))(\xi_H^2(c^{(1)}{}_{11}) \\
& \rightharpoonup (c^{(1)}{}_2 \rightharpoondown \xi_B^{-1}(b^{(0)}))))\xi_B^2(c^{(0)}) \\
= \quad & ((\xi_H^2(c^{(1)}{}_{12}) \rightharpoonup (\xi_H(c^{(1)}{}_{21}) \rightharpoondown (\xi_H^{-2}(b^{(1)}) \rightarrow \xi_B^{-3}(a))))(\xi_H^2(c^{(1)}{}_{11}) \\
& \rightharpoonup (\xi_H(c^{(1)}{}_{22}) \rightharpoondown \xi_B^{-1}(b^{(0)}))))\xi_B^2(c^{(0)}) \\
\overset{(2.2)(2.5)}{=} \quad & (\xi_H^2(c^{(1)}{}_1) \rightharpoonup (\xi_H(c^{(1)}{}_2) \rightharpoondown ((\xi_H^{-2}(b^{(1)}) \rightarrow \xi_B^{-3}(a))\xi_B^{-1}(b^{(0)}))))\xi_B^2(c^{(0)}) \\
\overset{(4.9)}{=} \quad & (\xi_H(c^{(1)}) \rightarrow ((\xi_H^{-1}(b^{(1)}) \rightarrow \xi_B^{-2}(a))b^{(0)}))\xi_B^2(c^{(0)}) \\
\overset{(4.10)}{=} \quad & (a \star b) \star \xi_B(c),
\end{aligned}
$$

and this proves that $(B, \xi_B, \star)$ is an associative monoidal Hom-algebra with the unit 1_B.

Theorem 4.4. *Let (H, ξ_H) be a monoidal Hom-Hopf algebra and (B, ξ_B) a monoidal Hom-bialgebra. Assume that $(B, \xi_B, \rightharpoonup, \delta)$ is a monoidal Hom-algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$ and $(B, \xi_B, \rightharpoondown, \rho)$ is a monoidal Hom-algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ such that both objects are in $\widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$. If Conditions* (A) *hold, then $(\overline{B}, \xi_B)$ is a bialgebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$, where $\overline{B} = B$ is a linear space and the coalgebra structure of $(\overline{B}, \xi_B)$ coincides with that of (B, ξ_B) and the multiplication is given by* (4.10)*. The module and comodule structures are given by* (4.9) *and* (4.11).

Proof. We show that $(B, \xi_B, \rightarrow, \star)$ is a left H-Hom-module algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ as follows.

$$
\begin{aligned}
& h \rightarrow (a \star b) \\
\overset{(4.10)(2.5)}{=} \quad & \xi_H(h_1) \rightharpoonup ((h_{21} \rightharpoondown (\xi_H^{-1}(b^{(1)}) \rightarrow \xi_B^{-2}(a)))(h_{22} \rightharpoondown b^{(0)})) \\
\overset{(2.2)(4.1)}{=} \quad & (\xi_H(h_{21}) \rightharpoondown (h_{12} \rightharpoonup (\xi_H^{-1}(b^{(1)}) \rightarrow \xi_B^{-2}(a))))(\xi_H(h_{22}) \rightharpoondown (h_{11} \rightharpoonup b^{(0)})) \\
\overset{(4.9)}{=} \quad & (\xi_H(h_{21}) \rightharpoondown (h_{12} \rightharpoonup (b^{(1)}{}_1 \rightharpoonup (\xi_H^{-1}(b^{(1)}{}_2) \rightharpoondown \xi_B^{-3}(a)))))(\xi_H(h_{22}) \\
& \rightharpoondown (h_{11} \rightharpoonup b^{(0)})) \\
= \quad & (\xi_H(h_{21}) \rightharpoondown (\xi_H^{-1}(h_{12})b^{(0)(1)} \rightharpoonup (\xi_H^{-1}(b^{(1)}) \rightharpoondown \xi_B^{-2}(a))))(\xi_H(h_{22}) \\
& \rightharpoondown \xi_B(\xi_H^{-1}(h_{11}) \rightharpoonup b^{(0)(0)})) \\
\overset{(2.1)}{=} \quad & (\xi_H(h_{21}) \rightharpoondown (\xi_H^{-1}((h_{12} \rightharpoonup b^{(0)})^{(1)}h_{11}) \rightharpoonup (\xi_H^{-1}(b^{(1)}) \rightharpoondown \xi_B^{-2}(a))))(\xi_H(h_{22}) \\
& \rightharpoondown \xi_B((h_{12} \rightharpoonup b^{(0)})^{(0)})) \\
\overset{(4.1)}{=} \quad & (((\xi_H(h_{121}) \rightharpoonup b^{(0)})^{(1)}h_{11}) \rightharpoonup ((h_{122}\xi_H^{-1}(b^{(1)})) \rightharpoondown \xi_B^{-1}(a)))(h_2 \\
& \rightharpoondown \xi_B((\xi_H(h_{121}) \rightharpoonup b^{(0)})^{(0)}))
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(2.1)}{=} && ((\xi_B((\xi_H(h_{122}) \rightharpoonup \xi_B^{-1}(b))^{(0)})^{(1)}h_{11}) \rightharpoonup ((\xi_H^{-1}((\xi_H(h_{122}) \rightharpoonup \xi_B^{-1}(b))^{(1)})h_{121})\\
& && \rightharpoondown \xi_B^{-1}(a)))(h_2 \rightharpoondown \xi_B(\xi_B((\xi_H(h_{122}) \rightharpoonup \xi_B^{-1}(b))^{(0)})^{(0)}))\\
&= && ((\xi_H((h_{21} \rightharpoonup \xi_B^{-1}(b))^{(1)1})h_{11}) \rightharpoonup (((h_{21} \rightharpoonup \xi_B^{-1}(b))^{(1)2}\xi_H^{-1}(h_{12})) \rightharpoondown \xi_B^{-1}(a)))\\
& && (\xi_H(h_{22}) \rightharpoondown \xi_B((h_{21} \rightharpoonup \xi_B^{-1}(b))^{(0)}))\\
&\overset{(4.9)}{=} && (((h_{21} \rightharpoonup \xi_B^{-1}(b))^{(1)}\xi_H^{-1}(h_1)) \rightarrow a)(\xi_H(h_{22}) \rightharpoondown \xi_B((h_{21} \rightharpoonup \xi_B^{-1}(b))^{(0)}))\\
&\overset{(2.8)}{=} && (((h_{22} \rightharpoondown \xi_B^{-1}(h_{21} \rightharpoonup \xi_B^{-1}(b)))^{(1)}\xi_H^{-1}(h_1)) \rightarrow a)\\
& && \xi_B^2((h_{22} \rightharpoondown \xi_B^{-1}(h_{21} \rightharpoonup \xi_B^{-1}(b)))^{(0)})\\
&\overset{(4.1)}{=} && (((h_{21} \rightharpoonup (\xi_H^{-1}(h_{22}) \rightharpoondown \xi_B^{-2}(b)))^{(1)}\xi_H^{-1}(h_1)) \rightarrow a)\\
& && \xi_B^2((h_{21} \rightharpoonup (\xi_H^{-1}(h_{22}) \rightharpoondown \xi_B^{-2}(b)))^{(0)})\\
&\overset{(4.9)}{=} && (((\xi_H^{-1}(h_2) \rightarrow \xi_B^{-1}(b))^{(1)}\xi_H^{-1}(h_1)) \rightarrow a)\xi_B^2((\xi_H^{-1}(h_2) \rightarrow \xi_B^{-1}(b))^{(0)})\\
&\overset{(4.10)}{=} && (h_1 \rightarrow a) \star (h_2 \rightarrow b),
\end{aligned}
$$

as required.

Then we check that $(B, \xi_B, \star, \chi_B)$ is a right H-Hom-module algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ according to the equation (2.6). In fact, one has

$$
\begin{aligned}
\chi_B(a)\chi_B(b) &= (\xi_B({a_{(0)}}^{(0)}) \star \xi_B({b_{(0)}}^{(0)})) \otimes (\xi_H^{-1}(b_{(1)}){b_{(0)}}^{(1)})(\xi_H^{-1}(a_{(1)}){a_{(0)}}^{(1)})\\
&= (\xi_H({{b_{(0)}}^{(1)}}_1) \rightarrow {a_{(0)}}^{(0)})\xi_B({b_{(0)}}^{(0)}) \otimes (\xi_H^{-1}(b_{(1)})\xi_H({{b_{(0)}}^{(1)}}_2))(\xi_H^{-1}(a_{(1)}){a_{(0)}}^{(1)}),
\end{aligned}
$$

on the other hand,

$$
\begin{aligned}
& && \chi_B(a \star b)\\
&\overset{(4.9)(4.11)}{=} && \xi_B({((b^{(1)} \rightarrow \xi_B^{-1}(a))\xi_B(b^{(0)}))_{(0)}}^{(0)}) \otimes \xi_H^{-1}(((b^{(1)} \rightarrow \xi_B^{-1}(a))\xi_B(b^{(0)}))_{(1)})\\
& && {((b^{(1)} \rightarrow \xi_B^{-1}(a))\xi_B(b^{(0)}))_{(0)}}^{(1)}\\
&\overset{(2.6)(2.3)}{=} && \xi_B({(b^{(1)} \rightarrow \xi_B^{-1}(a))_{(0)}}^{(0)}\xi_B({{b^{(0)}}_{(0)}}^{(0)})) \otimes \xi_H^{-1}(\xi_H({b^{(0)}}_{(1)})(b^{(1)} \rightarrow \xi_B^{-1}(a))_{(1)})\\
& && ({(b^{(1)} \rightarrow \xi_B^{-1}(a))_{(0)}}^{(1)}\xi_B({{b^{(0)}}_{(0)}}^{(1)}))\\
&\overset{(4.9)}{=} && \xi_B({((\xi_H({b^{(1)}}_1) \rightharpoonup ({b^{(1)}}_2 \rightharpoondown \xi_B^{-2}(a)))_{(0)}}^{(0)}\xi_B({{b^{(0)}}_{(0)}}^{(0)}))\\
& && \otimes\xi_H^{-1}(\xi_H({b^{(0)}}_{(1)})(\xi_H({b^{(1)}}_1) \rightharpoonup ({b^{(1)}}_2 \rightharpoondown \xi_B^{-2}(a)))_{(1)})\\
& && ({(\xi_H({b^{(1)}}_1) \rightharpoonup ({b^{(1)}}_2 \rightharpoondown \xi_B^{-2}(a)))_{(0)}}^{(1)}\xi_B({{b^{(0)}}_{(0)}}^{(1)}))\\
&\overset{(2.8)}{=} && \xi_B(({b^{(1)}}_1 \rightharpoonup ({b^{(1)}}_2 \rightharpoondown \xi_B^{-2}(a))_{(0)})^{(0)}\xi_B({{b^{(0)}}_{(0)}}^{(0)})) \otimes ({b^{(0)}}_{(1)}({b^{(1)}}_2 \rightharpoondown \xi_B^{-2}(a))_{(1)})\\
& && (({b^{(1)}}_1 \rightharpoonup ({b^{(1)}}_2 \rightharpoondown \xi_B^{-2}(a))_{(0)})^{(1)}\xi_B({{b^{(0)}}_{(0)}}^{(1)}))\\
&\overset{(4.5)}{=} && \xi_B((\xi_H({{b_{(0)}}^{(1)}}_1) \rightharpoonup (\xi_H({{b_{(0)}}^{(1)}}_2) \rightharpoondown \xi_B^{-2}(a))_{(0)})^{(0)}\xi_B({b_{(0)}}^{(0)(0)}))\\
& && \otimes(\xi_H^{-1}(b_{(1)})(\xi_H({{b_{(0)}}^{(1)}}_2) \rightharpoondown \xi_B^{-2}(a))_{(1)})((\xi_H({{b_{(0)}}^{(1)}}_1) \rightharpoonup (\xi_H({{b_{(0)}}^{(1)}}_2)
\end{aligned}
$$

$$
\begin{aligned}
& \rightharpoondown \xi_B^{-2}(a))_{(0)})^{(1)}\xi_H(b_{(0)}{}^{(0)(1)}))\\
= \quad & \xi_B((\xi_H^2(b_{(0)}{}^{(1)}{}_{12}) \rightharpoonup (\xi_H(b_{(0)}{}^{(1)}{}_2) \rightharpoondown \xi_B^{-2}(a))_{(0)})^{(0)}b_{(0)}{}^{(0)})\\
& \otimes(\xi_H^{-1}(b_{(1)})(\xi_H(b_{(0)}{}^{(1)}{}_2) \rightharpoondown \xi_B^{-2}(a))_{(1)})\\
& ((\xi_H^2(b_{(0)}{}^{(1)}{}_{12}) \rightharpoonup (\xi_H(b_{(0)}{}^{(1)}{}_2) \rightharpoondown \xi_B^{-2}(a))_{(0)})^{(1)}\xi_H^2(b_{(0)}{}^{(1)}{}_{11}))\\
\overset{(2.1)}{=} \quad & \xi_B((\xi_H(b_{(0)}{}^{(1)}{}_{11}) \rightharpoonup (\xi_H(b_{(0)}{}^{(1)}{}_2) \rightharpoondown \xi_B^{-2}(a))_{(0)}{}^{(0)}b_{(0)}{}^{(0)})\\
& \otimes(\xi_H^{-1}(b_{(1)})(\xi_H(b_{(0)}{}^{(1)}{}_2) \rightharpoondown \xi_B^{-2}(a))_{(1)})\\
& \xi_H(\xi_H(b_{(0)}{}^{(1)}{}_{12})(\xi_H(b_{(0)}{}^{(1)}{}_2) \rightharpoondown \xi_B^{-2}(a))_{(0)}{}^{(1)})\\
\overset{(4.5)}{=} \quad & \xi_B((\xi_H^{-1}(b^{(1)}{}_1) \rightharpoonup (\xi_H(b^{(1)}{}_{22}) \rightharpoondown \xi_B^{-2}(a))_{(0)}{}^{(0)}b^{(0)}{}_{(0)})\\
& \otimes\xi_H(b^{(0)}{}_{(1)})((\xi_H^{-1}((\xi_H(b^{(1)}{}_{22}) \rightharpoondown \xi_B^{-2}(a))_{(1)})b^{(1)}{}_{21})\\
& \xi_H((\xi_H(b^{(1)}{}_{22}) \rightharpoondown \xi_B^{-2}(a))_{(0)}{}^{(1)}))\\
\overset{(2.1)}{=} \quad & \xi_B((\xi_H^{-1}(b^{(1)}{}_1) \rightharpoonup (b^{(1)}{}_{21} \rightharpoondown \xi_B^{-2}(a)_{(0)})^{(0)})b^{(0)}{}_{(0)})\\
& \otimes\xi_H(b^{(0)}{}_{(1)})((b^{(1)}{}_{22}\xi_B^{-2}(a)_{(1)})\xi_H((b^{(1)}{}_{21} \rightharpoondown \xi_B^{-2}(a)_{(0)})^{(1)}))\\
\overset{(2.8)}{=} \quad & \xi_B((b^{(1)}{}_{11} \rightharpoonup (\xi_H^{-1}(b^{(1)}{}_{12}) \rightharpoondown \xi_B^{-2}(a_{(0)}{}^{(0)})))b^{(0)}{}_{(0)})\\
& \otimes\xi_H(b^{(0)}{}_{(1)})((\xi_H^{-1}(b^{(1)}{}_2)\xi_H^{-2}(a_{(1)}))a_{(0)}{}^{(1)})\\
\overset{(4.9)}{=} \quad & (b^{(1)}{}_1 \rightarrow a_{(0)}{}^{(0)})\xi_B(b^{(0)}{}_{(0)}) \otimes (b^{(0)}{}_{(1)}b^{(1)}{}_2)(\xi_H^{-1}(a_{(1)})a_{(0)}{}^{(1)})\\
\overset{(4.5)}{=} \quad & (\xi_H(b_{(0)}{}^{(1)}{}_1 \rightarrow a_{(0)}{}^{(0)})\xi_B(b_{(0)}{}^{(0)}) \otimes (\xi_H^{-1}(b_{(1)})\xi_H(b_{(0)}{}^{(1)}{}_2))(\xi_H^{-1}(a_{(1)})a_{(0)}{}^{(1)}),
\end{aligned}
$$

as required.

It is easy to see that $(B, \xi_B, \Delta_B, \rightarrow)$ is a Hom-module coalgebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ by the conditions (4.2), (4.3) and (4.4). And by the formulae (4.6)–(4.8), $(B, \xi_B, \Delta_B, \chi_B)$ is a left H-Hom-comodule coalgebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$.

Finally, using the braiding τ'' in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ (see (2.7)), we have a braided tensor product $\overline{B} \otimes \overline{B}$: $(a \otimes b)(c \otimes d) = a\xi_B(c_{(0)}) \otimes (c_{(1)} \rightharpoondown \xi_B^{-1}(b))d$ for any $a, b, c, d \in B$. We will show that $\Delta_B \colon \overline{B} \to \overline{B} \otimes \overline{B}$ is an algebra map. We compute:

$$
\begin{aligned}
\Delta_B(a \star b) \overset{(4.10)}{=} \quad & (b^{(1)} \rightarrow \xi_B^{-1}(a))_1\xi_B(b^{(0)}{}_1) \otimes (b^{(1)} \rightarrow \xi_B^{-1}(a))_2\xi_B(b^{(0)}{}_2)\\
= \quad & (b^{(1)}{}_1 \rightarrow \xi_B^{-1}(a_1))\xi_B(b^{(0)}{}_1) \otimes (b^{(1)}{}_2 \rightarrow \xi_B^{-1}(a_2))\xi_B(b^{(0)}{}_2)\\
\overset{(4.9)}{=} \quad & (\xi_H(b^{(1)}{}_{11}) \rightharpoonup (b^{(1)}{}_{12} \rightharpoondown \xi_B^{-2}(a_1)))\xi_B(b^{(0)}{}_1) \otimes (\xi_H(b^{(1)}{}_{21})\\
& \rightharpoonup (b^{(1)}{}_{22} \rightharpoondown \xi_B^{-2}(a_2)))\xi_B(b^{(0)}{}_2)\\
\overset{(4.1)}{=} \quad & (b^{(1)}{}_1 \rightharpoonup (\xi_H(b^{(1)}{}_{211}) \rightharpoondown \xi_B^{-2}(a_1)))\xi_B(b^{(0)}{}_1) \otimes (\xi_H(b^{(1)}{}_{22})\\
& \rightharpoondown (\xi_H(b^{(1)}{}_{212}) \rightharpoonup \xi_B^{-2}(a_2)))\xi_B(b^{(0)}{}_2)\\
\overset{(4.2)}{=} \quad & (b^{(1)}{}_1 \rightharpoonup \varepsilon_H(b^{(1)}{}_{21})\xi_B^{-1}(a_1))\xi_B(b^{(0)}{}_1) \otimes (\xi_H(b^{(1)}{}_{22}) \rightharpoondown \xi_B^{-1}(a_2))\xi_B(b^{(0)}{}_2)\\
= \quad & (b^{(1)}{}_1 \rightharpoonup \xi_B^{-1}(a_1))\xi_B(b^{(0)}{}_1) \otimes (b^{(1)}{}_2 \rightharpoondown \xi_B^{-1}(a_2))\xi_B(b^{(0)}{}_2).
\end{aligned}
$$

and

$$\begin{aligned}
& a_1 \star \xi_B(\xi_B({b_{1(0)}}^{(0)})) \otimes ((\xi_H^{-1}(b_{1(1)}){b_{1(0)}}^{(1)}) \to \xi_B^{-1}(a_2)) \star b_2 \\
= \;& (\xi_B^2({b_{1(0)}}^{(0)})^{(1)} \to \xi_B^{-1}(a_1))\xi_B(\xi_B^2({b_{1(0)}}^{(0)})^{(0)}) \otimes ({b_2}^{(1)} \to \xi_B^{-1}((\xi_H^{-1}(b_{1(1)}){b_{1(0)}}^{(1)}) \\
& \to \xi_B^{-1}(a_2)))\xi_B({b_2}^{(0)}) \\
= \;& (\xi_B^2({b_{1(0)}}^{(0)(1)}) \to \xi_B^{-1}(a_1))\xi_B^3({b_{1(0)}}^{(0)(0)}) \otimes ((\xi_H^{-2}({b_2}^{(1)}b_{1(1)}){b_{1(0)}}^{(1)}) \\
& \to \xi_B^{-1}(a_2))\xi_B({b_2}^{(0)}) \\
\overset{(4.5)}{=} \;& (\xi_B({b_1}^{(0)(1)}) \to \xi_B^{-1}(a_1))\xi_B^3({b_1}^{(0)(0)}{}_{(0)}) \otimes (((\xi_H^{-2}({b_2}^{(1)}){b_1}^{(0)(0)}{}_{(1)})\xi_H^{-1}({b_1}^{(1)})) \\
& \to \xi_B^{-1}(a_2))\xi_B({b_2}^{(0)}) \\
= \;& (\xi_B({b_1}^{(1)}{}_1) \to \xi_B^{-1}(a_1))\xi_B^2({b_1}^{(0)}{}_{(0)}) \otimes (((\xi_H^{-2}({b_2}^{(1)})\xi_H^{-1}({b_1}^{(0)}{}_{(1)})){b_1}^{(1)}{}_2) \\
& \to \xi_B^{-1}(a_2))\xi_B({b_2}^{(0)}) \\
\overset{(4.9)}{=} \;& (\xi_B^2({b_1}^{(1)}{}_{11}) \rightharpoonup (\xi_H({b_1}^{(1)}{}_{12}) \rightharpoondown \xi_B^{-2}(a_1)))\xi_B^2({b_1}^{(0)}{}_{(0)}) \otimes (((\xi_H^{-1}({b_2}^{(1)}{}_1){b_1}^{(0)}{}_{(1)1}) \\
& \xi_H({b_1}^{(1)}{}_{21})) \rightharpoonup (((\xi_H^{-2}({b_1}^{(1)}{}_2)\xi_H^{-1}({b_1}^{(0)}{}_{(1)2})){b_1}^{(1)}{}_{22}) \rightharpoondown \xi_B^{-2}(a_2)))\xi_B({b_2}^{(0)}) \\
= \;& (\xi_B^2({b_1}^{(1)}{}_{11}) \rightharpoonup (\xi_H({b_1}^{(1)}{}_{12}) \rightharpoondown \xi_B^{-2}(a_1)))\xi_B^2({b_1}^{(0)}{}_{(0)}) \otimes (({b_2}^{(1)}{}_1\xi_H({b_1}^{(0)}{}_{(1)1})) \\
& \rightharpoonup (\xi_H({b_1}^{(1)}{}_{21}) \rightharpoonup (((\xi_H^{-3}({b_2}^{(1)}{}_2)\xi_H^{-2}({b_1}^{(0)}{}_{(1)2}))\xi_H^{-1}({b_1}^{(1)}{}_{22})) \rightharpoondown \xi_B^{-3}(a_2))))\xi_B({b_2}^{(0)}) \\
\overset{(4.1)}{=} \;& (\xi_B^2({b_1}^{(1)}{}_{11}) \rightharpoonup (\xi_H({b_1}^{(1)}{}_{12}) \rightharpoondown \xi_B^{-2}(a_1)))\xi_B^2({b_1}^{(0)}{}_{(0)}) \otimes (({b_2}^{(1)}{}_1\xi_H({b_1}^{(0)}{}_{(1)1})) \\
& \rightharpoonup (((\xi_H^{-2}({b_2}^{(1)}{}_2)\xi_H^{-1}({b_1}^{(0)}{}_{(1)2})){b_1}^{(1)}{}_{22}) \rightharpoondown ({b_1}^{(1)}{}_{21} \rightharpoonup \xi_B^{-3}(a_2))))\xi_B({b_2}^{(0)}) \\
= \;& (\xi_B({b_1}^{(1)}{}_1) \rightharpoonup (\xi_H^2({b_1}^{(1)}{}_{211}) \rightharpoondown \xi_B^{-2}(a_1)))\xi_B^2({b_1}^{(0)}{}_{(0)}) \otimes (({b_2}^{(1)}{}_1\xi_H({b_1}^{(0)}{}_{(1)1})) \\
& \rightharpoonup (((\xi_H^{-2}({b_2}^{(1)}{}_2)\xi_H^{-1}({b_1}^{(0)}{}_{(1)2})){b_1}^{(1)}{}_{22}) \rightharpoondown (\xi_H({b_1}^{(1)}{}_{212}) \rightharpoonup \xi_B^{-3}(a_2))))\xi_B({b_2}^{(0)}) \\
\overset{(4.2)}{=} \;& (\xi_B({b_1}^{(1)}{}_1) \rightharpoonup \xi_B^{-1}(a_1))\xi_B^2({b_1}^{(0)}{}_{(0)}) \otimes (({b_2}^{(1)}{}_1\xi_H({b_1}^{(0)}{}_{(1)1})) \\
& \rightharpoonup (((\xi_H^{-2}({b_2}^{(1)}{}_2)\xi_H^{-1}({b_1}^{(0)}{}_{(1)2}))\xi_H^{-1}({b_1}^{(1)}{}_2)) \rightharpoondown \xi_B^{-2}(a_2)))\xi_B({b_2}^{(0)}) \\
\overset{(4.7)}{=} \;& (b^{(1)}{}_1 \rightharpoonup \xi_B^{-1}(a_1))\xi_B^2(b^{(0)}{}_{1(0)}) \otimes ((\xi_H({b^{(0)}{}_2}^{(1)}{}_1)\xi_H(b^{(0)}{}_{1(1)1})) \\
& \rightharpoonup (((\xi_H^{-1}({b^{(0)}{}_2}^{(1)}{}_2)\xi_H^{-1}(b^{(0)}{}_{1(1)2}))\xi_H^{-2}(b^{(1)}{}_2)) \rightharpoondown \xi_B^{-2}(a_2)))\xi_B^2({b^{(0)}{}_2}^{(0)}) \\
= \;& (b^{(1)}{}_1 \rightharpoonup \xi_B^{-1}(a_1))\xi_B^2(b^{(0)}{}_{1(0)}) \otimes (\xi_H({b^{(0)}{}_2}^{(1)}b^{(0)}{}_{1(1)})_1 \rightharpoonup (({b^{(0)}{}_2}^{(1)}b^{(0)}{}_{1(1)})_2 \\
& \rightharpoondown (\xi_H^{-2}(b^{(1)}{}_2) \rightharpoondown \xi_B^{-3}(a_2))))\xi_B^2({b^{(0)}{}_2}^{(0)}) \\
\overset{(4.9)}{=} \;& (b^{(1)}{}_1 \rightharpoonup \xi_B^{-1}(a_1))\xi_B^2(b^{(0)}{}_{1(0)}) \otimes (({b^{(0)}{}_2}^{(1)}b^{(0)}{}_{1(1)}) \\
& \to (\xi_H^{-1}(b^{(1)}{}_2) \rightharpoondown \xi_B^{-2}(a_2)))\xi_B^2({b^{(0)}{}_2}^{(0)}) \\
\overset{(4.6)}{=} \;& (b^{(1)}{}_1 \rightharpoonup \xi_B^{-1}(a_1))\xi_B(b^{(0)}{}_1) \otimes (b^{(1)}{}_2 \rightharpoondown \xi_B^{-1}(a_2))\xi_B(b^{(0)}{}_2).
\end{aligned}$$

Hence $(B, \xi_B, \Delta, \star)$ is a monoidal Hom-bialgebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}^H)$, concluding the proof. □

Similarly, we can make another definition as follows:

$$h \succ b = \xi_H(h_1) \rightharpoondown (h_2 \rightharpoonup \xi_B^{-1}(b)), \tag{4.12}$$

$$a \,\overline{\star}\, b = \xi_B(a_{(0)})(a_{(1)} \succ \xi_B^{-1}(b)), \tag{4.13}$$

$$\zeta_B(b) = \xi_B({b_{(0)}}^{(0)}) \otimes {b_{(0)}}^{(1)}\xi_H^{-1}(b_{(1)}). \tag{4.14}$$

In what follows, we replace (4.2) and (4.6) in Conditions (A) by the following relations

$$(h_2 \rightharpoondown b_1) \otimes (h_1 \rightharpoonup b_2) = \varepsilon(h)\xi_B(b_1) \otimes \xi_B(b_2), \tag{4.15}$$

$$b_{1(0)} \otimes {b_2}^{(0)} \otimes b_{1(1)}{b_2}^{(1)} = \xi_B^{-1}(b_1) \otimes \xi_B^{-1}(b_2) \otimes 1_H. \tag{4.16}$$

Theorem 4.5. *Let (H, ξ_H) be a monoidal Hom-Hopf algebra and (B, ξ_B) a monoidal Hom-bialgebra. Assume that $(B, \xi_B, \rightharpoonup, \delta)$ is a monoidal Hom-algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$ and $(B, \xi_B, \rightharpoondown, \rho)$ is a monoidal Hom-algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ such that both objects are in $\widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$. If the conditions* (4.1),(4.3)–(4.5), (4.7), (4.8), (4.15), (4.16) *hold, then $(\underline{B}, \xi_B)$ is a bialgebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$, where $\underline{B} = B$ is a linear space, the coalgebra structure coincides with that of (B, ξ_B) and the multiplication is given by* (4.13). *The module and comodule structures are defined by* (4.12) *and* (4.14).

Proof. Similar to that of Theorem 4.4. □

Remark 4.6. The left Yetter-Drinfeld modules constitute the braided category $\widetilde{\mathcal{H}}({}_H^H\mathcal{YD})$, see [15]. Similarly, the right Yetter-Drinfeld modules constitute $\widetilde{\mathcal{H}}(\mathcal{YD}_H^H)$. We have natural identification of braided categories

$$\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H) = \widetilde{\mathcal{H}}({}_{H^{\mathrm{cop}}}^{H^{\mathrm{cop}}}\mathcal{YD}), \quad \widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H) = \widetilde{\mathcal{H}}(\mathcal{YD}_{H^{\mathrm{op}}}^{H^{\mathrm{op}}}).$$

Replace H by H^{op}, and identify $H^{\mathrm{op,cop}}$ with H via $S\colon H \xrightarrow{\cong} H^{\mathrm{op,cop}}$. Thus, if M is an object in $\widetilde{\mathcal{H}}({}_{H^{\mathrm{op}}}\mathcal{YD}_1^{H^{\mathrm{op}}})$ with structures $(h^{\mathrm{op}}, m) \mapsto h^{\mathrm{op}}m$, $H^{\mathrm{op}} \otimes M \to M$ and $m \mapsto m^{(0)} \otimes m^{(1)}$, $M \to M \otimes H$, then it becomes an object in $\widetilde{\mathcal{H}}({}_H^H\mathcal{YD})$ with the structures given by

$$hm = S(h)^{\mathrm{op}}m, \ \lambda(m) = S^{-1}(m^{(1)}) \otimes m^{(0)} \in H \otimes M.$$

Theorem 4.4 is translated as follows.

Let (B, ξ_B) be a monoidal Hom-bialgebra. Suppose further that (B, ξ_B) is an algebra object in $\widetilde{\mathcal{H}}({}_H^H\mathcal{YD})$ and also in $\widetilde{\mathcal{H}}(\mathcal{YD}_H^H)$. Suppose that each pair of structures indicated by

$${}_HB_H, \quad {}^HB^H, \quad {}_HB^H, \quad {}^HB_H$$

commutes with each other, i.e., (B, ξ_B) is an H-Hom-bimodule, H-Hom-bicomodule, $(B, \xi_B) \in \widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$, $(B, \xi_B) \in \widetilde{\mathcal{H}}({}^H\mathcal{L}_H)$. Denote the left and the right H-Hom-comodule structures on (B, ξ_B) by

$$\lambda(b) = b^{(-1)} \otimes b^{(0)}, \quad \rho(b) = b^{(0)} \otimes b^{(1)}, \quad (\forall\, b \in B),$$

respectively, and suppose further that

$$\xi_H^{-1}(h)b_1 \otimes \xi_B(b_2) = \xi_B(b_1) \otimes b_2\xi_H^{-1}(h), \qquad \xi_B^{-1}(b_1) \otimes \lambda(b_2) = \rho(b_1) \otimes \xi_B^{-1}(b_2),$$
$$\Delta(hb) = hb_1 \otimes b_2, \qquad \Delta(bh) = b_1 \otimes b_2 h,$$
$$\xi_H^{-1}(b^{(-1)}) \otimes \Delta(b^{(0)}) = \lambda(b_1) \otimes \xi_B(b_2), \qquad \xi_B^{-1}(b_1) \otimes \rho(b_2) = \Delta(b^{(0)}) \otimes b^{(1)},$$

where $b \in B$, $h \in H$. Then the coalgebra B becomes a bialgebra in $\widetilde{\mathcal{H}}(\mathcal{YD}_H^H)$, the new structures given as follows:

$$\begin{aligned}
b \leftarrow h &:= (S_H^{-1}(h_1)\xi_B^{-1}(b))\xi_H(h_2),\\
a \star b &:= (\xi_B^{-1}(a) \leftarrow S_H(b^{(-1)}))\xi_B(b^{(0)})\\
&= (\xi_H(b^{(0)(-1)})\xi_B^{-1}(c))(S_H(b^{(-1)})\xi_H(b^{(0)(0)})),\\
b &\mapsto \xi_B(b^{(0)(0)}) \otimes S_H\xi_H^{-1}(b^{(-1)})b^{(0)(1)},\ B \to B \otimes H,
\end{aligned}$$

where $a, b \in B$, $h \in H$.

Similarly, Theorem 4.5 can be reformulated in a symmetric form, which will give a construction of bialgebra in $\widetilde{\mathcal{H}}({}_H^H\mathcal{YD})$. These reformulated statements look simpler than the original, although here one has to assume that the antipode S of H is bijective.

5. Braided monoidal Hom-Hopf algebras

Let (H, ξ_H) be a monoidal Hom-Hopf algebra with a bijective antipode S_H, and (B, ξ_B) a monoidal Hom-Hopf algebra with a bijective antipode S_B. In this section we give a sufficient condition for the braided monoidal Hom-bialgebras defined in Section 4 to be a braided monoidal Hom-Hopf algebra. At first, we assume that the following Conditions (B) are satisfied:

Conditions (B):

$$S_B(h \rightharpoonup b) = h \rightharpoondown S_B(b), \tag{5.1}$$
$$S_B(h \rightharpoondown b) = S_H^{-2}\xi_H^{-1}(h) \rightharpoonup S_B(b), \tag{5.2}$$
$$(S_B(b))^{(0)} \otimes (S_B(b))^{(1)} = S_B(b_{(0)}) \otimes S_H^{-2}(b_{(1)}), \tag{5.3}$$
$$(S_B(b))_{(0)} \otimes (S_B(b))_{(1)} = S_B(b^{(0)}) \otimes b^{(1)}. \tag{5.4}$$

Proposition 5.1. *In the situation of Theorem* 4.4*. Assume that Conditions* (B) *hold. Then* $(\overline{B}, \xi_{\overline{B}})$ *has antipode in the category* $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ *given by*

$$\overline{S}(b) = b^{(1)} \rightarrow S_B(b_{(0)}).$$

Proof. We need to prove that $\overline{S}$ is a morphism in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$. For this, we have

$$\begin{aligned}
\overline{S}(h \to b) &= (h \to b)_{(1)} \to S_B((h \to b)_{(0)}) \\
&\overset{(4.9)(2.8)}{=} \xi_H((h_2 \rightharpoondown \xi_B^{-1}(b))_{(1)}) \to S_B(h_1 \rightharpoonup (h_2 \rightharpoondown \xi_B^{-1}(b))_{(0)}) \\
&\overset{(2.1)(4.9)}{=} \xi_H((\xi_H(h_{222})\xi_H^{-1}(b_{(1)}))_1(S_H^{-1}\xi_H(h_{21}))_1) \\
&\quad \rightharpoonup (((\xi_H(h_{222})\xi_H^{-1}(b_{(1)}))_2(S_H^{-1}\xi_H(h_{21}))_2) \\
&\quad \rightharpoondown S_B(\xi_H^{-1}(h_1) \rightharpoonup (h_{221} \rightharpoondown \xi_B^{-2}(b_{(0)})))) \\
&\overset{(5.1)(5.2)}{=} \xi_H((\xi_H(h_{2221})\xi_H^{-1}(b_{(1)1}))S_H^{-1}\xi_H(h_{212})) \\
&\quad \rightharpoonup (((\xi_H(h_{2222})\xi_H^{-1}(b_{(1)2}))S_H^{-1}\xi_H(h_{211})) \\
&\quad \rightharpoondown (\xi_H^{-1}(h_1) \rightharpoondown (S_H^{-2}\xi_H^{-1}(h_{221}) \rightharpoonup S_B\xi_B^{-2}(b_{(0)})))) \\
&= ((\xi_H(h_{221})b_{(1)1})S_H^{-1}\xi_H(h_{211})) \\
&\quad \rightharpoonup (((h_{222}\xi_H^{-1}(b_{(1)2}))(S_H^{-1}\xi_H^{-1}(h_{12})\xi_H^{-1}(h_{11}))) \\
&\quad \rightharpoondown (S_H^{-2}(h_{212}) \rightharpoonup S_B\xi_B^{-1}(b_{(0)}))) \\
&= ((\xi_H(h_{221})b_{(1)1})S_H^{-1}\xi_H^{-1}(h_1)) \rightharpoonup ((\xi_H(h_{222})b_{(1)2}) \\
&\quad \rightharpoondown (S_H^{-2}\xi_H^{-1}(h_{21}) \rightharpoonup S_B\xi_B^{-1}(b_{(0)})) \\
&\overset{(4.1)}{=} ((h_{21}b_{(1)1})(S_H^{-1}\xi_H^{-1}(h_{11})S_H^{-2}\xi_H^{-1}(h_{12}))) \rightharpoonup ((h_{22}b_{(1)2}) \rightharpoondown S_B(b_{(0)})) \\
&\overset{(4.9)}{=} h \to \overline{S}(b),
\end{aligned}$$

and this proves that $\overline{S}$ is H-module map.

Also, one has

$$\begin{aligned}
&\chi_B \circ \overline{S}(b) \\
={}& \xi_B((b_{(1)} \to S_B(b_{(0)}))_{(0)}{}^{(0)}) \otimes \xi_H^{-1}((b_{(1)} \to S_B(b_{(0)}))_{(1)})(b_{(1)} \to S_B(b_{(0)}))_{(0)}{}^{(1)} \\
\overset{(4.9)(4.1)}{=}{}& \xi_B((\xi_H(b_{(1)2}) \rightharpoondown (b_{(1)1} \rightharpoonup S_B\xi_B^{-1}(b_{(0)})))_{(0)}{}^{(0)}) \otimes \xi_H^{-1}((\xi_H(b_{(1)2}) \\
& \rightharpoondown (b_{(1)1} \rightharpoonup S_B\xi_B^{-1}(b_{(0)})))_{(1)})(\xi_H(b_{(1)2}) \rightharpoondown (b_{(1)1} \rightharpoonup S_B\xi_B^{-1}(b_{(0)})))_{(0)}{}^{(1)} \\
\overset{(2.1)(2.8)}{=}{}& \xi_B((\xi_H^2(b_{(1)221}) \rightharpoondown (\xi_H^{-1}(b_{(1)1}) \rightharpoonup (S_B\xi_B^{-1}(b_{(0)}))_{(0)}))^{(0)}) \\
& \otimes((b_{(1)222}(S_B\xi_B^{-2}(b_{(0)}))_{(1)})S_H^{-1}(b_{(1)21}))(\xi_H^2(b_{(1)221}) \rightharpoondown (\xi_H^{-1}(b_{(1)1}) \\
& \rightharpoonup (S_B\xi_B^{-1}(b_{(0)}))_{(0)}))^{(1)} \\
\overset{(5.4)(2.8)}{=}{}& \xi_B(\xi_H(b_{(1)221}) \rightharpoondown (\xi_H^{-1}(b_{(1)1}) \rightharpoonup S_B\xi_B^{-1}(b_{(0)}{}^{(0)}))^{(0)}) \\
& \otimes((b_{(1)222}\xi_H^{-2}(b_{(0)}{}^{(1)}))S_H^{-1}(b_{(1)21}))\xi_B((\xi_H^{-1}(b_{(1)1}) \rightharpoonup S_B\xi_B^{-1}(b_{(0)}{}^{(0)}))^{(1)}) \\
\overset{(2.1)(5.3)}{=}{}& \xi_H^2(b_{(1)221}) \rightharpoondown (\xi_H(b_{(1)121}) \rightharpoonup S_B(b_{(0)}{}^{(0)(0)})) \\
& \otimes((b_{(1)222}\xi_H^{-2}(b_{(0)}{}^{(1)}))S_H^{-1}(b_{(1)21}))((b_{(1)122}S_H^{-2}\xi_H^{-1}(b_{(0)}{}^{(0)(1)}))S_H^{-1}(b_{(1)11})) \\
={}& \xi_H^2(b_{(1)221}) \rightharpoondown (b_{(1)12} \rightharpoonup S_B(b_{(0)}{}^{(0)}{}_{(0)}))
\end{aligned}$$

$$
\begin{aligned}
&\otimes(\xi_H(b_{(1)222})\xi_H^{-1}(b_{(0)}{}^{(1)}))((S_H^{-1}(b_{(1)212})b_{(1)211})\\
&(S_H^{-2}\xi_H^{-1}(b_{(0)}{}^{(0)}{}_{(1)})S_H^{-1}\xi_H^{-1}(b_{(1)11})))\\
\overset{(4.5)}{=}\ &\xi_H^2(b^{(0)}{}_{(1)21}) \rightharpoondown (\xi_H(b^{(0)}{}_{(1)12}) \rightharpoonup S_B(b^{(0)}{}_{(0)(0)})) \otimes (\xi_H(b^{(0)}{}_{(1)22})\xi_H^{-2}(b^{(1)}))\\
&(S_H^{-2}(b^{(0)}{}_{(0)(1)})S_H^{-1}\xi_H(b^{(0)}{}_{(1)11}))\\
=\ &\xi_H^3(b^{(0)}{}_{(1)212}) \rightharpoondown (\xi_H^2(b^{(0)}{}_{(1)211}) \rightharpoonup S_B\xi_B^{-1}(b^{(0)}{}_{(0)})) \otimes (\xi_H(b^{(0)}{}_{(1)22})\xi_H^{-2}(b^{(1)}))\\
&(S_H^{-2}\xi_H(b^{(0)}{}_{(1)11})S_H^{-1}\xi_H(b^{(0)}{}_{(1)12}))\\
=\ &\xi_H^2(b^{(0)}{}_{(1)12}) \rightharpoondown (\xi_H(b^{(0)}{}_{(1)11}) \rightharpoonup S_B\xi_B^{-1}(b^{(0)}{}_{(0)})) \otimes (\xi_H(b^{(0)}{}_{(1)2})\xi_H^{-1}(b^{(1)}))\\
\overset{(4.9)(4.11)}{=}\ &(\overline{S}\otimes \mathrm{id})\circ \chi_B(b),
\end{aligned}
$$

completing the $\overline{S}$ is a morphism in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$.

Using (4.6), relation $\overline{S}(b_1)\star b_2 = \varepsilon(b)1_B$ holds. We also have

$$
\begin{aligned}
b_1\star \overline{S}(b_2) &= ((b_{2(1)} \to S_B(b_{2(0)}))^{(1)} \to \xi_B^{-1}(b_1))\xi_B((b_{2(1)} \to S_B(b_{2(0)})^{(0)})\\
&\overset{(4.9)(4.1)(2.8)}{=} (\xi_B((b_{2(1)1} \rightharpoonup S_B\xi_B^{-1}(b_{2(0)}))^{(1)}) \to b_1)\xi_B(b_{2(1)2}\\
&\quad \rightharpoondown (b_{2(1)1} \rightharpoonup S_B\xi_B^{-1}(b_{2(0)}))^{(0)})\\
&\overset{(2.1)(5.3)}{=} ((\xi_H(b_{2(1)22})S_H^{-2}\xi_H^{-1}(b_{2(0)(1)}))S^{-1}(b_{2(1)1}) \to b_1)(\xi_H(b_{2(1)2})\\
&\quad \rightharpoondown (\xi_H^2(b_{2(1)121}) \rightharpoonup S_B\xi_B^{-1}(b_{2(0)(0)})))\\
&= ((\xi_H^2(b_{2(1)212})(S_H^{-2}(b_{2(1)11})S_H^{-1}(b_{2(1)12})) \to b_1)(\xi_H^2(b_{2(1)22})\\
&\quad \rightharpoondown (\xi_H^2(b_{2(1)211}) \rightharpoonup S_B\xi_B^{-1}(b_{2(0)})))\\
&= (\xi_H^2(b_{2(1)21}) \rightharpoondown (\xi_H(b_{2(1)12}) \rightharpoonup \xi_H^{-1}(b_1)))(\xi_H^2(b_{2(1)22})\\
&\quad \rightharpoondown (\xi_H(b_{2(1)11}) \rightharpoonup S_B\xi_B^{-1}(b_{2(0)})))\\
&\overset{(4.4)(4.3)(4.8)}{=} b_{(1)} \to (b_{(0)1}S_B(b_{(0)2}))\\
&= \varepsilon(b)1_B.
\end{aligned}
$$

This completes the proof. □

Similarly, we postulate the following
Conditions (C):

$$S_B(h \rightharpoondown b) = h \rightharpoonup S_B(b), \tag{5.5}$$

$$S_B(h \rightharpoonup b) = S_H^{-2}\xi_H^{-1}(h) \rightharpoondown S_B(b), \tag{5.6}$$

$$(S_B(b))^{(0)} \otimes (S_B(b))^{(1)} = S_B(b_{(0)}) \otimes b_{(1)}, \tag{5.7}$$

$$(S_B(b))_{(0)} \otimes (S_B(b))_{(1)} = S_B(b^{(0)}) \otimes S_H^{-2}(b^{(1)}), \tag{5.8}$$

where S_H^{-2} means $(S_H^{-1})^2$. Thus we have the following result similar to Proposition 5.1.

Proposition 5.2. *In the situation of Theorem* 4.5*. Assume that Conditions* (C) *hold. If* (B,ξ_B) *has an antipode then* $(\underline{B},\xi_{\underline{B}})$ *has an antipode in the category* $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$ *given by*

$$\underline{S}(b) = b^{(1)} \succ S_B(b_{(0)}).$$

Proof. The proof is similar to that of Proposition 5.1. □

6. Applications and examples

In this section we give some braided Hopf algebras in the category $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ for a quasitriangular monoidal Hom-Hopf algebra (H,ξ_H) and in the category $\widetilde{\mathcal{H}}(\mathcal{M}^H)$ for a coquasitriangular monoidal Hom-Hopf algebra (H,ξ_H).

When (H,ξ_H,R) is quasitriangular, the category $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ of left H-modules is a braided monoidal category endowed with the following structures:

$$\begin{aligned}
\tau''(m\otimes n) &= \xi_H(R^{(2)})\cdot\xi_N^{-1}(n)\otimes\xi_H(R^{(1)})\cdot\xi_M^{-1}(m),\\
h\cdot(m\otimes n) &= h_1\cdot m\otimes h_2\cdot n
\end{aligned}$$

for all objects $(M,\xi_M),(N,\xi_N)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$, and $m\in M$, $n\in N$, $h\in H$. Moreover define $\rho\colon M\to M\otimes H$ by

$$\rho(m) = R^{(2)}\cdot\xi_M^{-2}(m)\otimes\xi_H(R^{(1)}).$$

It is easy to see that M becomes an object in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$. Thus $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ contains $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ as its subcategory. Now we denote $\widetilde{\mathcal{H}}({}_H\mathcal{M}_2)=\widetilde{\mathcal{H}}({}_H\mathcal{M})$.

Similarly, we have a braided monoidal subcategory $\widetilde{\mathcal{H}}({}_H\mathcal{M}_1)=\widetilde{\mathcal{H}}({}_H\mathcal{M})$ with the structures:

$$\begin{aligned}
\tau'(m\otimes n) &= \xi_H(R^{(1)})\cdot\xi_N^{-1}(n)\otimes\xi_H(R^{(2)})\cdot\xi_M^{-1}(m),\\
h\cdot(m\otimes n) &= (h_2\cdot m)\otimes(h_1\cdot n).
\end{aligned}$$

Moreover $\widetilde{\mathcal{H}}({}_H\mathcal{M}_1)$ is a subcategory of $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ under the coaction

$$\rho(m) = R^{(2)}\cdot\xi_M^{-2}(m)\otimes\xi_H(R^{(1)}).$$

When $(H,\xi_H,\langle\cdot\mid\cdot\rangle)$ is a coquasitriangular monoidal Hom-Hopf algebra, the category $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ contains this braided monoidal subcategory $\widetilde{\mathcal{H}}(\mathcal{M}_2^H)$ which is endowed with the following structure:

$$\begin{aligned}
h\cdot m &= \left\langle\xi_H^{-1}(h)\mid m_{(1)}\right\rangle\xi_M(m_{(0)}),\\
\tau''\colon m\otimes n &\mapsto \left\langle n_{(1)}\mid m_{(1)}\right\rangle n_{(0)}\otimes m_{(0)},\\
\rho(m\otimes n) &= m_{(0)}\otimes n_{(0)}\otimes n_{(1)}m_{(1)}
\end{aligned}$$

for any $m \in (M, \xi_M) \in \widetilde{\mathcal{H}}(\mathcal{M}_2^H)$ and $n \in (N, \xi_N) \in \widetilde{\mathcal{H}}(\mathcal{M}_2^H)$.

Similarly, we have a braided monoidal subcategory $\widetilde{\mathcal{H}}(\mathcal{M}_1^H)$ armed with the following structure:

$$
\begin{aligned}
h \cdot m &= \left\langle \xi_H^{-1}(h) \mid m^{(1)} \right\rangle \xi_M(m^{(0)}),\\
\tau' \colon m \otimes n &\mapsto \left\langle m^{(1)} \mid n^{(1)} \right\rangle n_{(0)} \otimes m_{(0)},\\
\rho(m \otimes n) &= m^{(0)} \otimes n^{(0)} \otimes m^{(1)} n^{(1)}
\end{aligned}
$$

for any $m \in (M, \xi_M) \in \widetilde{\mathcal{H}}(\mathcal{M}_1^H)$ and $n \in (N, \xi_N) \in \widetilde{\mathcal{H}}(\mathcal{M}_1^H)$.

In what follows, we construct two classes of braided monoidal Hom-Hopf algebra in the categories $\widetilde{\mathcal{H}}({}_H\mathcal{M}_1)$, $\widetilde{\mathcal{H}}({}_H\mathcal{M}_2)$, and $\widetilde{\mathcal{H}}(\mathcal{M}_1^H)$, $\widetilde{\mathcal{H}}(\mathcal{M}_2^H)$.

Let (H, ξ_H, R) be quasitriangular monoidal Hom-Hopf algebra and (B, H, τ) be a monoidal Hom-Hopf pairing. We define

$$
\begin{aligned}
h \rightharpoonup b &= \tau(S(b_1), \xi_H^{-1}(h)) \xi_B^2(b_2),\\
h \rightharpoondown b &= \tau(b_2, \xi_H^{-1}(h)) \xi_B^2(b_1)
\end{aligned}
$$

for all $b \in (B, \xi_B)$, $h \in (H, \xi_H)$. In $\widetilde{\mathcal{H}}(\mathcal{M}_2^H)$, it is natural that $\delta(b) = (R^{(2)} \rightharpoonup \xi_B^{-2}(b)) \otimes \xi_H(R^{(1)}) \overset{\text{def}}{=} b^{(0)} \otimes b^{(1)}$ and $\rho(b) = (R^{(2)} \rightharpoondown \xi_B^{-2}(b)) \otimes \xi_H(R^{(1)}) \overset{\text{def}}{=} b_{(0)} \otimes b_{(1)}$.

It is easy to verify that $(B, \xi_B, \rightharpoonup, \delta)$ is an algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$ and $(B, \xi_B, \rightharpoondown, \rho)$ is an algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$. Obviously, $(B, \xi_B, \rightharpoonup, \rho)$ and $(B, \xi_B, \rightharpoondown, \delta)$ are objects in Long dimodule category $\widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$.

Thus by (4.9)–(4.10) and Proposition 5.1, we obtain

$$
h \rightarrow b = \xi_H(h_1) \rightharpoonup (h_2 \rightharpoondown \xi_B^{-1}(b)) = \tau(S\xi_B(b_{11}) b_2, h) \xi_B^3(b_{12}), \tag{6.1}
$$

$$
\begin{aligned}
a \star b &= (b^{(1)} \rightarrow \xi_B^{-1}(a)) \xi_B(b^{(0)})\\
&= \tau(S(a_{11}) \xi_B^{-1}(a_2), \xi_H(R^{(1)})) \xi_B^3(a_{12}) \xi_B(b_2) \tau(S\xi_B^{-1}(b_1), R^{(2)}),
\end{aligned} \tag{6.2}
$$

$$
\overline{S}(b) = b^{(1)} \rightarrow S_B(b_{(0)}) = \tau(b_2, R^{(2)})(R^{(1)} \rightarrow S_B(b_1)) \tag{6.3}
$$

for all $a, b \in (B, \xi_B)$ and $h \in (H, \xi_H)$.

We now have the following:

Theorem 6.1. *Let (H, ξ_H, R) be quasitriangular. With the notations above, there exists a braided monoidal Hom-Hopf algebra $(\overline{B}, \xi_B)$ in $\widetilde{\mathcal{H}}({}_H\mathcal{M}_2)$, where $\overline{B} = B$ is a linear space with a module structure given by* (6.1)*. The coalgebra structure and unit of $(\overline{B}, \xi_{\overline{B}})$ coincide with that of (B, ξ_B). The multiplication is given by* (6.2) *and the antipode is given by* (6.3)*.*

Proof. First, in order to apply Theorem 4.4, we need to verify Conditions (B) hold. A routine computation shows that the conditions (4.1)–(4.4) are satisfied. Then by definition and (4.1), we have

$$
\begin{aligned}
&(b_{(0)}{}^{(0)} \otimes \xi_H^{-1}(b^{(1)})) \otimes \xi_H^{-1}(b_{(0)}{}^{(1)})\\
&= (R^{(2)} \rightharpoondown \xi_B^{-2}(b))^{(0)} \otimes R^{(1)} \otimes (R^{(2)} \rightharpoondown \xi_B^{-2}(b))^{(1)}\\
&= r^{(2)} \rightharpoonup (\xi_H^{-2}(R^{(2)}) \rightharpoondown \xi_B^{-4}(b)) \otimes R^{(1)} \otimes \xi_H(r^{(1)}) \qquad \text{(by (QT4) and (4.1))}\\
&= \xi_H^{-1}(R^{(2)}) \rightharpoondown \xi_H^{-2}(r^{(2)} \rightharpoonup \xi_B^{-2}(b)) \otimes R^{(1)} \otimes r^{(1)}\\
&= \xi_H^{-1}(R^{(2)}) \rightharpoondown \xi_H^{-2}(b^{(0)}) \otimes R^{(1)} \otimes \xi_H^{-1}(b^{(1)})\\
&= (b^{(0)}{}_{(0)} \otimes b^{(0)}{}_{(1)}) \otimes \xi_H^{-1}(b^{(1)}),
\end{aligned}
$$

and the formula (4.5) is proved.

The following computation

$$
\begin{aligned}
b_{1(0)} \otimes b_2{}^{(0)} \otimes b_2{}^{(1)} b_{1(1)} &= (r^{(2)} \rightharpoondown \xi_B^{-2}(b_1)) \otimes (R^{(2)} \rightharpoondown \xi_B^{-2}(b_2)) \otimes \xi_H(R^{(1)} r^{(1)})\\
&= (R^{(2)}{}_1 \rightharpoondown \xi_B^{-2}(b_1)) \otimes (R^{(2)}{}_2 \rightharpoondown \xi_B^{-2}(b_2)) \otimes \xi_H(R^{(1)})\\
&= \varepsilon_H(R^{(2)}) \xi_B^{-1}(b_1) \otimes \xi_B^{-1}(b_2) \otimes \xi_H(R^{(1)})\\
&= \xi_B^{-1}(b_1) \otimes \xi_B^{-1}(b_2) \otimes 1_H,
\end{aligned}
$$

shows the equation (4.6).

Then, using (4.3), we can obtain:

$$
\begin{aligned}
b^{(0)}{}_1 \otimes b^{(0)}{}_2 \otimes \xi_H^{-1}(b^{(1)}) &= (R^{(2)} \rightharpoonup \xi_B^{-2}(b))_1 \otimes (R^{(2)} \rightharpoonup \xi_B^{-2}(b))_2 \otimes R^{(1)}\\
&= (\xi_H^{-1}(R^{(2)}) \rightharpoonup \xi_B^{-2}(b)) \otimes \xi_B^{-1}(b_2) \otimes R^{(1)}\\
&= (R^{(2)} \rightharpoonup \xi_B^{-2}(b)) \otimes \xi_B^{-1}(b_2) \otimes \xi_H(R^{(1)})\\
&= b_1{}^{(0)} \otimes \xi_B^{-1}(b_2) \otimes b_1{}^{(1)},
\end{aligned}
$$

and this proves (4.7), and similarly, one has (4.8). It is easy to get that $\overline{S}(b) = b^{(1)} \to S_B(b_{(0)}) = \tau(\xi_B^{-1}(b_2), R^{(2)})(\xi_H(R^{(1)}) \to S_B \xi_B^{-1}(b_1))$.

Finally, it is not hard to check that Conditions (B) hold, concluding the proof. $\square$

Let (H, ξ_H) be quasitriangular and (B, H, τ) a monoidal Hom-Hopf pairing. Similarly, we can define $h \rightharpoonup b = \tau^{-1}(b_1, \xi_H^{-1}(h))\xi_B^2(b_2)$ and $h \rightharpoondown b = \tau^{-1}(S(b_2), \xi_H^{-1}(h))\xi_B^2(b_1)$ for all $b \in (B, \xi_B)$, $h \in (H, \xi_H)$. In $\widetilde{\mathcal{H}}({}_H\mathcal{M}_1)$, it is natural that we have $\delta(b) = R^{(2)} \rightharpoonup \xi_B^{-2}(b) \otimes \xi_H(R^{(1)}) = b^{(0)} \otimes b^{(1)}$ and $\delta(b) = R^{(2)} \rightharpoonup \xi_B^{-2}(b) \otimes \xi_H(R^{(1)}) = b_{(0)} \otimes b_{(1)}$.

Thus, by (4.12)–(4.13) and Proposition 5.2 we have

$$
h \succ b = \xi_H(h_1) \rightharpoondown (h_2 \rightharpoonup \xi_B^{-1}(b)) = \tau^{-1}(b_1 S_B(b_{22}), h)\xi_B(b_{21}), \tag{6.4}
$$

$$
\begin{aligned}
a \,\overline{\star}\, b &= \xi_B(a_{(0)})(a_{(1)} \succ \xi_B^{-1}(b))\\
&= \tau^{-1}(S_B(a_2), R^{(2)}) a_1 b_{21} \tau^{-1}(b_1 S_B(b_{22}), \xi_H(R^{(1)})),
\end{aligned} \tag{6.5}
$$

$$\underline{S}(b) = b^{(1)} \succ S_B(b^{(0)}) = \tau^{-1}(b_1, R^{(2)})(R^{(1)} \succ S_B\xi_B^{-1}(b_2)) \tag{6.6}$$

for all $a, b \in (B, \xi_B)$ and $h \in (H, \xi_H)$.

We now have the following:

Theorem 6.2. *Let (H, ξ_H) be quasitriangular. With the notations (6.4)–(6.6) above, there exists a braided monoidal Hom-Hopf algebra $(\underline{B}, \xi_B)$ in $\widetilde{\mathcal{H}}({}_H\mathcal{M}_1)$, where $\underline{B} = B$ is a linear space with module structure given by (6.4). The coalgebra structure and unit in $(\underline{B}, \xi_B)$ coincide with that of (B, ξ_B). The multiplication is given by (6.5) and the antipode is given by (6.6).*

Proof. Similar to Theorem 6.1. □

Let $(H, \xi_H, \langle \cdot \mid \cdot \rangle)$ be coquasitriangular and (B, ξ_B) a monoidal Hom-Hopf algebra. Assume that $f\colon B \to H$ is a monoidal Hom-Hopf algebra map. Define $\delta(b) = b_2 \otimes S_H^{-1} f(b_1) \overset{\text{def}}{=} b^{(0)} \otimes b^{(1)}$ and $\rho(b) = b_1 \otimes f(b_2) \overset{\text{def}}{=} b_{(0)} \otimes b_{(1)}$ for $b \in (B, \xi_B)$. Then we have $h \rightharpoonup b = \langle \xi_H^{-1}(h) \mid S_H^{-1} f(b_1) \rangle \xi_B^2(b_2)$ and $h \rightharpoondown b = \langle \xi_H^{-1}(h) \mid f(b_2) \rangle \xi_B^2(b_1)$ for $h \in (H, \xi_H)$, $b \in (B, \xi_B)$. It is easy to check that $(B^{\mathrm{op}}, \xi_B, \rightharpoonup, \delta)$ is an algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$, and $(B^{\mathrm{op}}, \xi_B, \rightharpoondown, \rho)$ an algebra in $\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ such that $(B^{\mathrm{op}}, \xi_B, \rightharpoonup, \rho)$ is in $\widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$, and $(B^{\mathrm{op}}, \xi_B, \rightharpoondown, \delta)$ is in $\widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$.

Then by (4.10)–(4.11) and Proposition 5.1, one has

$$\chi_B(b) = \xi_B({b_{(0)}}^{(0)}) \otimes \xi_H^{-1}(b_{(1)}){b_{(0)}}^{(1)} = \xi_B(b_{12}) \otimes f\xi_B^{-1}(b_2) S_H^{-1} f(b_{11}), \tag{6.7}$$

$$\begin{aligned} a \star b &= (b^{(1)} \rightharpoondown \xi_B^{-1}(a))\xi_B(b^{(0)}) \\ &= \langle S_H^{-1} f(b_1) \mid f\xi_B^{-1}(a_2) S_H^{-1} f(a_{11}) \rangle \xi_B^2(a_{12})\xi_B(b_2), \end{aligned} \tag{6.8}$$

$$\overline{S}(b) = b_{(1)} \rightharpoondown S_B(b_{(0)}) = \langle f(b_2) \mid f S_B(b_{11}) f\xi_B(b_{122}) \rangle S_B\xi_B^3(b_{121}) \tag{6.9}$$

for all $a, b \in (B, \xi_B)$ and $h \in (H, \xi_H)$. It is easy to show that Conditions (A) and (B) are satisfied, and so by Theorem 4.4 and Proposition 5.1, we have:

Theorem 6.3. *Let $(H, \xi_H, \langle \cdot \mid \cdot \rangle)$ be coquasitriangular and (B, ξ_B) a monoiodal Hom-Hopf algebra. Let $f\colon B \to H$ be a monoidal Hom-Hopf algebra map. Then there is a braided monoidal Hom-Hopf algebra $(\overline{B}, \xi_B)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_2^H)$, where $\overline{B} = B$ is a linear space with H-comodule structure given by (6.7). The coalgebra structure and counit in $(\overline{B}, \xi_B)$ coincide with that of (B, ξ_B). The multiplication is given by (6.8) and the antipode is given by (6.9).*

Let (B, ξ_B) be any monoidal Hom-bialgebra and $f\colon H \to B$ be a monoidal Hom-bialgebra map. If f is a convolution invertible map with an inverse f^{-1}, then $f^{-1}\colon H \to B$ is an anti-Hom-bialgebra map, i.e., $f^{-1}(hl) = f^{-1}(l)f^{-1}(h)$ and $\Delta_B f^{-1}(h) = f^{-1}(h_2) \otimes f^{-1}(h_1)$.

Example 6.4. If (H,ξ_H) is a monoidal Hom-Hopf algebra, then $f^{-1}=fS_H$ is convolution invertible.

Similarly, let $(H,\xi_H,\langle\cdot\mid\cdot\rangle)$ be coquasitriangular and (B,ξ_B) a monoidal Hom-Hopf algebra. Let $f\colon B\to H$ be a monoidal Hom-Hopf algebra map. Define $\delta(b)=b_2\otimes S_Hf(b_1)\stackrel{\text{def}}{=}b^{(0)}\otimes b^{(1)}$ and $\rho(b)=b_1\otimes f(b_2)\stackrel{\text{def}}{=}b_{(0)}\otimes b_{(1)}$ for $b\in(B,\xi_B)$. Naturally, we get: $h\rightharpoonup b=\left\langle\xi_H^{-1}(h)\mid S_Hf(b_1)\right\rangle\xi_B^2(b_2)$ and $h\rightharpoondown b=\left\langle\xi_H^{-1}(h)\mid f(b_2)\right\rangle\xi_B^2(b_1)$ for $h\in(H,\xi_H)$, $b\in(B,\xi_B)$. It is easy to check that $(B^{\mathrm{op}},\xi_B,\rightharpoonup,\delta)\in\widetilde{\mathcal{H}}({}_H\mathcal{YD}_1^H)$ is a monoidal Hom-algebra and $(B^{\mathrm{op}},\xi_B,\rightharpoondown,\rho)\in\widetilde{\mathcal{H}}({}_H\mathcal{YD}_2^H)$ is a monoidal Hom-algebra such that $(B^{\mathrm{op}},\xi_B,\rightharpoonup,\rho)$ is in $\widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$ and $(B^{\mathrm{op}},\xi_B,\rightharpoondown,\delta)$ is in $\widetilde{\mathcal{H}}({}_H\mathcal{L}^H)$.

Thus by (4.13), (4.14) and Proposition 5.2, one has

$$\zeta_B(b)=\xi_B({b_{(0)}}^{(0)})\otimes {b_{(0)}}^{(1)}\xi_H^{-1}(b_{(1)})=\xi_B(b_{12})\otimes S_Hf(b_{11})f\xi_B(b_2),\tag{6.10}$$

$$\begin{aligned}a\,\overline{\star}\,b&=\xi_B(a_{(0)})(a_{(1)}\succ\xi_B^{-1}(b))\\&=\left\langle f(a_2)\mid S_Hf\xi_B^{-1}(b_1)f(b_{22})\right\rangle\xi_B(a_1)\xi_B^2(b_21),\end{aligned}\tag{6.11}$$

$$\underline{S}(b)=b^{(1)}\succ S_B(b^{(0)})=\left\langle S_Hf(b_1)\mid f\xi_B^{-1}(b_{22})fS_B(b_{211})\right\rangle S_B\xi_B^2(b_{212})\tag{6.12}$$

for all $a,b\in(B,\xi_B)$ and $h\in(H,\xi_H)$.

Finally, it is not hard to see that (4.1), (4.3)–(4.5), (4.7), (4.8), (4.15), (4.16) and Conditions (C) are satisfied. By Theorem 4.5 and Proposition 5.2, we have

Theorem 6.5. *Let $(H,\xi_H,\langle\cdot\mid\cdot\rangle)$ be coquasitriangular and (B,ξ_B) a monoidal Hom-Hopf algebra. Let $f\colon B\to H$ be a monoidal Hom-Hopf algebra map. Then there is a braided monoidal Hom-Hopf algebra $(\underline{B},\xi_B)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_1^H)$, where $\underline{B}=B$ is a linear space with H-comodule structure given by* (6.10). *The coalgebra structure coincides with that of B. The multiplication is given by* (6.11) *and the antipode is given by* (6.12).

By Theorem 6.1, we give an example explicitly as follows.

Remark 6.6. If (H,ξ_H,R) is a quasitriangular monoidal Hom-Hopf algebra, so is (H^{cop},ξ_H) with the quasitriangular structure $R'=R^{(2)}\otimes R^{(1)}$. The braided category $\widetilde{\mathcal{H}}({}_H\mathcal{M}_1)$ is identified with $\widetilde{\mathcal{H}}({}_{H^{\mathrm{cop}}}\mathcal{M}_2)$, and hence with $\widetilde{\mathcal{H}}(\mathcal{M}_{H^{\mathrm{bop}}2})$, the second kind braided category of right modules over $H^{\mathrm{bop}}:=(H^{\mathrm{op}})^{\mathrm{cop}}$. In addition, if $\tau\colon B\otimes H\to k$ is a monoidal Hom-Hopf pairing, so is $\tau^{-1}\colon B\otimes H^{\mathrm{bop}}\to k$, as shown by (DP1)$'$ and (DP2)$'$. Therefore, it is not hard to check that Theorem 6.2 follows from a variation of Theorem 6.1 which gives a construction of monoidal Hom-Hopf algebras in $\widetilde{\mathcal{H}}(\mathcal{M}_{H2})$.

Example 6.7. In Example 3.2, when $c^2=1$, (H_4,ξ) is also a quasitriangular monoidal Hom-Hopf algebra with

$$R_\alpha=\frac{1}{2}(1\otimes 1+1\otimes g+g\otimes 1-g\otimes g)+\frac{\alpha}{2}(x\otimes x-x\otimes y+y\otimes x+y\otimes y).$$

Then two actions (H_4,ξ) on (H_4^{cop},ξ) are respectively defined by

$\rightharpoonup$	1	g	x	y
1	1	g	cx	cy
g	1	$-g$	cx	$-cy$
x	0	0	αg	$\alpha 1$
y	0	0	$-\alpha g$	$\alpha 1$

and

$\rightharpoondown$	1	g	x	y
1	1	g	cx	cy
g	1	$-g$	$-cx$	cy
x	0	0	$\alpha 1$	$-\alpha g$
y	0	0	$\alpha 1$	αg

Thus, by the formula (6.1), (H_4^{cop},ξ) is a left H_4-module where the H_4-module structure is given by

$\rightarrow$	1	g	x	y
1	1	g	cx	cy
g	1	g	$-cx$	$-cy$
x	0	0	$\alpha(1+cg)$	$\alpha(1+cg)$
y	0	0	$\alpha(1+cg)$	$\alpha(1+cg)$

By the equation (6.2), the multiplication on (H_4^{cop},ξ) is obtained by the following table

$\star$	1	g	x	y
1	1	g	cx	cy
g	g	1	y	x
x	x	cy	$-\alpha^3(c^{-1}1+g)$	$\alpha^3(1+c^{-1}g)$
y	y	$-cx$	$-\alpha^3(c^{-1}1+g)$	$\alpha^3(1+c^{-1}g)$

Therefore, by Theorem 6.1, $(H_4,\xi,\Delta_{H_4}^{\mathrm{cop}},\star)$ is a braided monoidal Hom-Hopf algebra in $\widetilde{\mathcal{H}}({}_{H_4}\mathcal{M}_2)$. Its antipode is defined by

$$\overline{S}(1)=1,\quad \overline{S}(g)=g,\quad \overline{S}(x)=y,\quad \overline{S}(y)=x.$$

Similarly, applying Theorem 6.2, we have

Example 6.8. Let (H_4,ξ) be the Sweedler's 4-dimensional monoidal Hom-Hopf algebra. Then $(H_4,\xi,\Delta_{H_4}^{\mathrm{cop}},\overline{\star})$ is a braided monoidal Hom-Hopf algebra in $\widetilde{\mathcal{H}}({}_{H_4}\mathcal{M}_1)$. Its antipode is defined by

$$\underline{S}(1)=1,\quad \underline{S}(g)=g,\quad \underline{S}(x)=-c^{-1}(1+y),\quad \underline{S}(y)=-c^{-1}x.$$

The H_4-module structure and the multiplication on (H_4^{cop}) is given respectively by the following tables

$\succ$	1	g	x	y
1	1	g	$c^{-1}x$	$c^{-1}(1+y)$
g	1	g	$-c^{-1}x$	$-c^{-1}(1+y)$
x	0	0	$c^{-1}\alpha(1+g)$	$-\alpha g$
y	0	0	$-c^{-1}\alpha(1+g)$	αg

and

$\rightharpoondown$	1	g	x	y
1	1	g	$c^{-1}x$	$c^{-1}(1+y)$
g	g	1	$-y$	$-(c^{-1}g+x)$
x	x	$-c^{-1}y$	$c^{-1}\alpha^3(1+g)$	$-\alpha^3(1+g)$
y	y	$-cx$	$c^{-1}\alpha^3(1+g)$	$-\alpha^3(1+g)$

Remark 6.9. In Example 6.8, our two H_4-module structures associated to the H_4-module structures $\succ$ are given by respectively

$\rightharpoonup$	1	g	x	y
1	1	g	cx	cy
g	1	$-g$	cx	$-cy$
x	0	0	αg	$-\alpha 1$
y	0	0	αg	$\alpha 1$

and

$\rightharpoondown$	1	g	x	y
1	1	g	cx	cy
g	1	$-g$	$-cx$	cy
x	0	0	$\alpha 1$	αg
y	0	0	$-\alpha 1$	αg

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