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Shadowable Chain Recurrence Classes for Generic Diffeomorphisms

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Abstract. Let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of a closed C^∞ manifold M ($\dim M \geq 2$) endowed with the C^1 topology. In this paper we show that for C^1 generic $f \in \mathrm{Diff}(M)$, any shadowable chain recurrence class C_f is hyperbolic if it contains a hyperbolic periodic point.

1. Introduction

Let M be a closed C^∞ manifold with $\dim M \geq 2$, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1 topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \mathrm{Diff}(M)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}$ ($-\infty \leq a < b \leq \infty$) in M is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. Let Λ be a closed f invariant set. We say that f has the *shadowing property* on Λ (or, Λ *is shadowable* for f) if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i=a}^{b} \subset \Lambda$ of $f(-\infty \leq a < b \leq \infty)$, there is a point $y \in M$ (not necessary in Λ) such that $d(f^i(y), x_i) < \epsilon$ for all $a \leq i \leq b-1$.

We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\left\|D_x f^n|_{E_x^s}\right\| \leq C\lambda^n \quad \text{and} \quad \left\|D_x f^{-n}|_{E_x^u}\right\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then f is said to be *Anosov*.

Robinson [12] and Sakai [13] proved that a diffeomorphism $f \in \mathrm{Diff}(M)$ belongs to the C^1 interior of the set of diffeomorphisms with the shadowing property if and only if f is structurally stable.

We say that Λ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$. Λ is *locally maximal* if there is a neighborhood U of Λ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. A subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is called *residual* if it contains a countable intersection of open and dense subsets of $\mathrm{Diff}(M)$. A dynamic property is called C^1 *generic* if it holds in a residual subset of $\mathrm{Diff}(M)$.

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Abdenur and Díaz [1] proved that if Λ is a locally maximal transitive set of a C^1 generic diffeomorphism, then either Λ is hyperbolic, or there are a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g is does not have the shadowing property on $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Moreover they posed an open problem:

> *"C^1 generically, a diffeomorphism has the shadowing property if and only if it is hyperbolic?"*

The above probelm is still open, but Lee and Wen [10] showed that C^1 generically, if a locally maximal chain transitive set is shadowing then it is hyperbolic. We say that a closed invariant set $\Lambda \subset M$ is *robustly transitive* if there are a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is transitive for g. Here $\Lambda_g(U)$ is called the *continuation* of Λ for g. Tian and Sun [15] showed that if a diffeomorphism f has the C^1 stable shadowing property in a robustly transitive set Λ (in the case, the shadowing points are in Λ) then it is a hyperbolic basic set. Moreover, they claimed that if f has the C^1 generic stable shadowing property in a robustly transitive set Λ then it is a hyperbolic basic set. Here, we say that f has the C^1 *stable shadowing property* in Λ if there exist a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g has the shadowing property in $\Lambda_g(U)$. We say that f has the C^1 *generic stable shadowing property* in Λ if there exist a C^1 neighborhood $\mathcal{U}(f)$ of f and a residual set $\mathcal{R} \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{R}$, g has the shadowing property in $\Lambda_g(U)$.

Using the results, we prove that C^1 generically, any shadowable chain recurrent class is hyperbolic if it contains a hyperbolic periodic point.

2. Homoclinic classses for generic diffeomorphisms

For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=0}^n$ ($n \geq 1$) of f such that $x_0 = x$ and $x_n = y$. The set $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{CR}(f)$. It is easy to see that the set is closed and $f(\mathcal{CR}(f)) = \mathcal{CR}(f)$. The relation $\leftrightsquigarrow$ induces an equivalence relation on $\mathcal{CR}(f)$ whose classes are called *chain recurrence classes* of f and is denoted by $\mathcal{C}_f$. In general, the chain recurrent class is a closed and invariant set. For any $x \in M$, the orbit of x is denoted by $\mathrm{Orb}(x) = \{f^n(x) : n \in \mathbb{Z}\}$. It is well known that if p is a hyperbolic periodic point of f with period $\pi(p)$ then the sets

$$W^s(p) = \left\{x \in M : f^{\pi(p)n}(x) \to p \text{ as } n \to \infty\right\}$$

and

$$W^u(p) = \left\{x \in M : f^{-\pi(p)n}(x) \to p \text{ as } n \to \infty\right\}$$

are C^1 injectively immersed submanifolds of M. A point $x \in W^s(p) \cap W^u(p)$ is called a *homoclinic point* of f associated to p, and it is said to be a *transversal homoclinic point* of f if the above intersection is transversal at x; i.e., $x \in W^s(p) \pitchfork W^u(p)$. The closure of the transversal homoclinic points of f associated to p is called the *homoclinic class* of f associated to p, and it is denoted by $H_f(p)$. It is clear that $H_f(p)$ is a compact, invariant and transitive set. Note that any chain recurrence class contains a homoclinic class. Bonatti and Crovisier [2] proved that every chain recurrence class containing a hyperbolic periodic point p is the homoclinic class $H_f(p)$ in the C^1 generic sense. For any hyperbolic periodic point p, we say that f has a *homoclinic tangency* if $W^s(p)$ and $W^u(p)$ intersect nontransversaly. Denote by $\overline{\mathcal{HT}}$ the closure of the set of diffeomorphisms exhibiting a homoclinic tangency. We say that a compact invariant set Λ admits a *dominated splitting* for f if the tangent bundle $T_\Lambda M$ has a continuous Df invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that for all $x \in \Lambda$ and $n \geq 0$, we have

$$\left\|Df^n|_{E(x)}\right\| \cdot \left\|Df^{-n}|_{F(f^n(x))}\right\| \leq C\lambda^n.$$

If p is a hyperbolic periodic point then there are a C^1-neighborhood $\mathcal{U}(f)$ and a neighborhood U of p such that for any $g \in \mathcal{U}(f)$, there is a hyperbolic periodic point $p_g \in P(g)$, where $p_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is called *the continuation* of p. The following was proved by Gourmelon in [8, Theorem 1.1].

Theorem 2.1. *For generic $f \in \mathrm{Diff}(M)$, if f has a homoclinic class $H_f(p)$ which contains hyperbolic saddles of indices i and j $(i \leq j)$, then either*

(a) *$H_f(p)$ admits a dominated splitting of the form*

$$T_{H_f(p)}M = E \oplus E_1 \oplus \cdots \oplus E_{j-i} \oplus F$$

with $\dim E = i$, $\dim F = \dim M - j$ and $\dim E_i = 1$ for all $i = 1, 2, \ldots, j - i$. Moreover, all subbundles E_i are non-hyperbolic, or

(b) *for any C^1-neighborhood $\mathcal{U}(f)$ of f there is a $g \in \mathcal{U}(f)$ having a homoclinic tangency associated with a saddle of the homoclinic class $H_g(p_g)$, where p_g is the continuation of p.*

For any two hyperbolic periodic points p and q, we say that f has a *heterodimensional cycles* if

(i) $\dim W^s(p) \neq \dim W^s(q)$,

(ii) $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$.

Denote by $\overline{\mathcal{HC}}$ the closure of the set of diffeomorphisms exhibiting a heterodimensional cycle. In [4], Crovisier proved the following.

Theorem 2.2. *For generic $f \in \mathrm{Diff}(M) \setminus \overline{\mathcal{HT} \cup \mathcal{HC}}$, any homoclinic class $H_f(p)$ admits a partially hyperbolic spitting: $T_{H_f(p)}M = E^s \oplus E^c_1 \oplus E^c_2 \oplus E^u$, where $\dim E^c_i = 0$ or 1, for $i = 1, 2$ and $\dim(E^s \oplus E^c_1)$ coincides with the stable dimension of p.*

Let p and q be hyperbolic periodic points. We say that q is *homoclinically related to* p (denoted by $p \sim q$) if

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \quad \text{and} \quad W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

Let μ be an invariant measure of f. Then for μ-almost every $x \in M$, there exist real numbers

$$\xi^1_f(\mu, x) \leq \cdots \leq \xi^n_f(\mu, x)$$

with $n = \dim M$ such that for every non-zero vector $v \in T_xM$, one has

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| = \xi^i_f(\mu, x)$$

for some $i \in \{1, 2, \ldots, n\}$. The number $\xi^i_f(\mu, x)$ is called the *i-th Lyapunov exponent* of the invariant measure μ at x. Moreover, if μ is taken to be ergodic, $\xi^i_f(\mu, x)$ is independent of x in a μ-full measure set. An ergodic measure μ is *hyperbolic* if all of its Lyapunov exponents are non-zero, or else, it is said to be non-hyperbolic.

For a homoclinic class $H_f(p)$ if for any $\delta > 0$, there is a hyperbolic periodic point q such that (1) q is homoclinically related with p, and (2) q has one Lyapunov exponent in $(-\delta, \delta)$, then we say that $H_f(p)$ contains *weak periodic orbits related to* p. In [5], Corivisier *et. al* proved the following.

Theorem 2.3. *For generic $f \in \mathrm{Diff}(M) \setminus \overline{\mathcal{HT}}$, any homoclinic class $H_f(p)$ satisfy: either*

(a) *$H_f(p)$ is hyperbolic, or*

(b) *$H_f(p)$ contains weak periodic orbits related to p.*

Let Λ be a closed f invariant set. We say that Λ is *expansive* if there is $e > 0$ such that for any $x, y \in \Lambda$ if $d(f^i(x), f^i(y)) < e$ for all $i \in \mathbb{Z}$ then $x = y$. It is well known that if Λ is hyperbolic then Λ is expansive. Yang and Gan [17] proved that C^1 generically, every expansive homoclinic class $H_f(p)$ is hyperbolic.

For the result, we study that for a C^1 generic diffeomorphism f, if any chain recurrence class $\mathcal{C}_f$ which contains a hyperbolic periodic point p ($\mathcal{C}_f(p)$) is shadowable then it is hyperbolic. The following is our main result.

Main Theorem. *For generic $f \in \mathrm{Diff}(M)$, any shadowable chain recurrence class $\mathcal{C}_f$ of f is hyperbolic if it contains a hyperbolic periodic point.*

3. Proof of main theorem

Let Λ be a closed f invariant set. For $x \in \Lambda$ and any $\epsilon > 0$, let

$$W^s_\epsilon(x) = \{y \in M : d(f^n(x), f^n(y)) \le \epsilon, \text{ for all } n \ge 0\}$$

and

$$W^u_\epsilon(x) = \left\{y \in M : d(f^{-n}(x), f^{-n}(y)) \le \epsilon, \text{ for all } n \ge 0\right\}$$

be the *local stable set* and the *local unstable set* of x, respectively. If Λ is hyperbolic then there is $\eta > 0$ such that for any $0 < \epsilon \le \eta$, and $x \in \Lambda$, the set $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$ are embedded manifolds.

Lemma 3.1. *If f has the shadowing property on $H_f(p)$ then for any hyperbolic saddle $q \in H_f(p)$ we have $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$.*

Proof. Let q be a hyperbolic saddle. Since $H_f(p)$ is transitive there is a point $x \in H_f(p)$ such that $\omega(x) = H_f(p)$. Thus for any $\eta > 0$ there are $j_1 > 0$ and $j_2 > 0$ such that $d(f^{j_1}(x), p) < \eta$ and $d(f^{j_2}(x), q) < \eta$. For simplicity, we assume that $f(p) = p$, $f(q) = q$. Since p, q are hyperbolic there exist $\epsilon(p) > 0$ and $\epsilon(q) > 0$ such that $W^\sigma_{\epsilon(p)}(p)$ and $W^\sigma_{\epsilon(q)}(q)$ are submanifolds of M, as above, where $\sigma = s, u$. Take $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$. Let $0 < \delta(\epsilon) = \delta \le \eta/4$ be the number obtained by the shadowing property. Without loss of generality, $j_2 = j_1 + k$ for some $k > 0$. Then we construct a δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ as follows:

(i) $x_{-i} = f^{-i}(p)$ for $i \ge 0$,

(ii) $f^{j_1+i}(x) = x_{1+i}$ for $i = 0, 1, \ldots, k-1$, and

(iii) $x_i = f^i(q)$ for $i \ge k+1$.

By (i), (ii) and (iii), $\{x_i\}_{i\in\mathbb{Z}}$ is the δ-pseudo orbit

$$\begin{aligned}\{x_i\}_{i\in\mathbb{Z}} &= \left\{\ldots, p, f^{j_1}(x), f^{j_1+1}(x), \ldots, f^{j_1+k-1}(x), q, \ldots\right\} \\ &= \{\ldots, x_0 = p, x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1} = q, q, \ldots\}.\end{aligned}$$

Then it is clear $\{x_i\}_{i\in\mathbb{Z}} \subset H_f(p)$. By the shadowing property, there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Then $y \in W^u(p)$ and $f^k(y) \in W^s(q)$. Since $y \in f^{-k}(W^s(q)) \subset W^s(q)$, we have $W^u(p) \cap W^s(q) \neq \emptyset$. Similarly, we have $W^s(p) \cap W^u(q) \neq \emptyset$. □

Let $P(f)$ be the set of all periodic points of f. We say that f is *Kupka-Smale* if every periodic point is hyperbolic, and all their invariant manifolds are transverse. Denote by $\mathcal{KS}(M)$ the set of Kupka-Smale diffeomorphisms. It is well-known that $\mathcal{KS}(M)$ is a residual set in $\mathrm{Diff}(M)$.

Proposition 3.2. *There is a residual set $\mathcal{G}_1 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_1$, if f has the shadowing property on $H_f(p)$ then for any hyperbolic periodic point $q \in H_f(p)$,*

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \quad \textit{and} \quad W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

Proof. Let $\mathcal{G}_1 = \mathcal{KS}(M)$ and let $f \in \mathcal{G}_1$ have the shadowing property on $H_f(p)$. By Lemma 3.1, for any $q \in H_f(p) \cap P_h(f)$, we have

$$W^s(p) \cap W^u(q) \neq \emptyset \quad \text{and} \quad W^u(p) \cap W^s(q) \neq \emptyset,$$

where $P_h(f)$ is the set of all hyperbolic periodic points of f. Since $f \in \mathcal{KS}(M)$, we know

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \quad \text{and} \quad W^u(p) \pitchfork W^s(q) \neq \emptyset. \qquad \square$$

Lemma 3.3. [9, Lemma 2.2] *There is a residual set $\mathcal{G}_2 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_2$, if for any C^1 neighborhood $\mathcal{U}(f)$ of f there is $g \in \mathcal{U}(f)$ such that g has two distinct hyperbolic periodic points p_g and q_g with $\mathrm{index}(p_g) \neq \mathrm{index}(q_g)$, then f has two distinct hyperbolic periodic points p and q with $\mathrm{index}(p) \neq \mathrm{index}(q)$, where $\mathrm{index}(p) = \dim W^s(p)$.*

For $\eta > 0$, a C^1-curve $\mathcal{J} \subset M$ is called η *simply periodic curve* (see [17]) of f if

(a) $\mathcal{J}$ is diffeomorphic to $[0,1]$, and its end points are hyperbolic periodic points of f;

(b) $\mathcal{J}$ is periodic with period $\pi(\mathcal{J})$, that is, $f^{\pi(\mathcal{J})}(\mathcal{J}) = \mathcal{J}$, and $L(f^i(\mathcal{J})) < \eta$ for all $i = 0, 1, \ldots, \pi(\mathcal{J}) - 1$, where $L(\mathcal{J})$ denotes the length of $\mathcal{J}$;

(c) $\mathcal{J}$ is normally hyperbolic.

Lemma 3.4. [17, Lemma 2.1] *There is a residual set $\mathcal{G}_3 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_3$, and $p \in P(f)$ if for any $\eta > 0$ and a C^1 neighborhood $\mathcal{U}(f)$ of f there is $g \in \mathcal{U}(f)$ such that g has an η simply periodic curve $\mathcal{J}$ which two endpoints are homoclinically related with p_g, then f has an η-simply periodic curve $\mathcal{I}$ such that the two endpoints of $\mathcal{I}$ are homoclinically related to p.*

For $p, q \in P_h(f)$, if $p \sim q$ then it is clear that $\dim W^s(p) = \dim W^s(q)$, that is, $\mathrm{index}(p) = \mathrm{index}(q)$. For $p \in P(f)$, the set of *normalized eigenvalues of $D_p f^{\pi(p)}$* is the set $\{\lambda^{1/\pi(p)} : \lambda \text{ eigenvalues of } D_p f^{\pi(p)}\}$. The following Franks' lemma [7] will play essential roles in our proofs.

Lemma 3.5. *Let $\mathcal{U}(f)$ be any given C^1 neighborhood of f. Then there exist $\epsilon > 0$ and a C^1 neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \ldots, x_N\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_N\}$ and linear maps $L_i \colon T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i} g\| \leq \epsilon$ for all $1 \leq i \leq N$, there exists $\widehat{g} \in \mathcal{U}(f)$ such that $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \ldots, x_N\} \cup (M \setminus U)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \leq i \leq N$.*

Lemma 3.6. *There is a residual set $\mathcal{G}_4 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_4$, if f has the shadowing property on $H_f(p)$, then for any $q \in H_f(p) \cap P_h(f)$ with $q \sim p$, the moduli of the normalized eigenvalues of q are uniformly bounded away from* 1.

Proof. Let $\mathcal{G}_4 = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3$. Suppose that $f \in \mathcal{G}_4$ has the shadowing property on $H_f(p)$. Assume that there is $q \in H_f(p) \cap P_h(f)$ with $q \sim p$ such that the normalized eigenvalue λ of $D_q f^{\pi(q)}$ is close to 1.

Note that if $D_q f^{\pi(q)}$ has the normalized eigenvalue λ which is closed to 1 then by Lemma 3.5, there is g C^1 close to f such that $D_{q_g} g^{\pi(q)}$ has an eigenvalue μ with $\mu = 1$. Then clearly we have $\mathrm{index}(q_g) < \mathrm{index}(q)$ (for more detail see [14, Proposition 3]).

By Lemma 3.5, there is g C^1 close to f such that g has an η simply periodic curve $\mathcal{J}_{q_g}$ which two endpoints are homoclinically related with p_g, and $\mathrm{index}(q_g) \neq \mathrm{index}(p_g)$. By Lemmas 3.3 and 3.4, f has an η simply periodic curve $\mathcal{I}_q$ which two endpoints are homoclinically related with p, and $\mathrm{index}(q) \neq \mathrm{index}(p)$. This is a contradiction since $q \sim p$. □

The family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$ generated by Df along the hyperbolic periodic points $q \in H_f(p)$ with $q \sim p$ is *uniformly hyperbolic* means that there is $\epsilon > 0$ such that for any g (C^1 close to f), the hyperbolic periodic point $q_g \in P(g)$ with $q_g \sim p_g$ (see [14, p. 475]).

Proposition 3.7. *If $f \in \mathcal{G}_4$ has the shadowing property on $H_f(p)$, then there is $\lambda \in (0,1)$ such that for any $q \in H_f(p) \cap P_h(f)$ and $x \in \mathrm{Orb}(q)$, we have*

$$\prod_{i=0}^{\pi(q)-1} \left\| Df^i|_{E^s(f^i(x))} \right\| \le \lambda^{\pi(q)} \quad \text{and} \quad \prod_{i=0}^{\pi(q)-1} \left\| Df^{-i}|_{E^u(f^{-i}(x))} \right\| \le \lambda^{\pi(q)},$$

where $\pi(q)$ is the period of q.

Proof. Let $f \in \mathcal{G}_4$ has the shadowing property on $H_f(p)$. For any $q \in H_f(p) \cap P_h(f)$, we have $q \sim p$ by Proposition 3.2. Since f has the shadowing property on $H_f(p)$, the moduli of the normalized eigenvalues of $D_q f^{\pi(q)}$ are uniformly bounded away from 1. Thus by [11, Lemma II.3], we obtain that for any $x \in \mathrm{Orb}(q)$, we have

$$\prod_{i=0}^{\pi(q)-1} \left\| Df^i|_{E^s(f^i(x))} \right\| \le \lambda^{\pi(q)} \quad \text{and} \quad \prod_{i=0}^{\pi(q)-1} \left\| Df^{-i}|_{E^u(f^{-i}(x))} \right\| \le \lambda^{\pi(q)},$$

where $\pi(q)$ is the period of q. □

For any $p \in P(f)$ and $\delta \in (0,1)$, we say p has a δ *weak eigenvalue* of f if $D_p f^{\pi(p)}$ has an eigenvalue λ such that $(1-\delta)^{\pi(p)} < |\lambda| < (1+\delta)^{\pi(p)}$.

Lemma 3.8. *For any $f \in \mathcal{G}_4$, if f has the shadowing property on $H_f(p)$ then all the Lyapunov exponents of all periodic points homoclinically related to p are uniformly away from* 0.

Proof. Let $f \in \mathcal{G}_4$ has the shadowing property on $H_f(p)$, and let $q \in H_f(p) \cap P_h(f)$. Note that if there is a periodic point q of f that is homoclinically related to p and has a Lyapunov exponent arbitrarily close to 0 then there is g C^1 close to f such that

$$\chi(q_g, v^c) = \lim_{n\to\infty} \frac{1}{n} \log \left\| D_{q_g} g^n(v^c) \right\| = 0,$$

for $v^c \in E^c$, where E^c is associated to an eigenvalue $\lambda\,(|\lambda| = 1)$ of $D_{q_g} g^{\pi(q_g)}$. Here, the periodic point q has a Lyapunov exponent arbitrarily close to 0 means that for any $\delta > 0$, q has a δ weak eigenvalue. Now, we know that for any periodic point $q \in H_f(p)$, q is homoclinically related to p by applying Proposition 3.2. To prove the lemma, we may assume that for any $\delta > 0$, q has a δ weak eigenvalue. Since f has the shadowing property on $H_f(p)$ and $q \in H_f(p)$, we know $q \sim p$. By Proposition 3.7, q has no δ-weak eigenvalue. This is a contradiction and so completes the proof. $\square$

Note that if Lemma 3.8 holds, then $H_f(p)$ admits a dominated splitting $T_{H_f(p)}M = E \oplus F$ with $\dim E = \text{index}(p)$.

Theorem 3.9. [16] *There is a residual set $\mathcal{G}_5 \subset \text{Diff}(M)$ such that for any $f \in \mathcal{G}_5$, a homoclinic class $H_f(p)$ either is hyperbolic, or contains periodic orbits with arbitrarily long periods that are homoclinically related to p and has a Lyapunov exponent arbitrarily close to* 0.

Proof of main theorem. Let $\mathcal{G} = \mathcal{G}_4 \cap \mathcal{G}_5$, and assume that $f \in \mathcal{G}$ has the shadowing property on $H_f(p)$. By Proposition 3.2, for any hyperbolic periodic point $q \in H_f(p)$, we know $q \sim p$, and by Lemma 3.8, all Lyapunov exponents of q are uniformly away from 0. Consequently, by Theorem 3.9, $H_f(p)$ is hyperbolic. $\square$

A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is said to be a *limit pseudo orbit* of f if $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm\infty$. We say that f has the *limit shadowing property* in Λ (or Λ is limit shadowable for f) if for any $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ there is $y \in \Lambda$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Let $f\colon M \to M$ be a diffeomorphism which has the limit shadowing property. Then by [3, Theorem A], f has the shadowing property. Thus we have the following corollary.

Corollary 3.10. *For generic $f \in \text{Diff}(M)$, any limit shadowable chain recurrence class $\mathcal{C}_f$ of f is hyperbolic if it contains a hyperbolic periodic point.*

For any $\delta > 0$, a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ is said to be a *δ-ergodic pseudo orbit* of f if for

$$Np_n^+(\xi, f, \delta) = \{i : d(f(x_i), x_{i+1}) \geq \delta\} \cap \{0, 1, \dots, n-1\}$$

and

$$Np_n^-(\xi, f, \delta) = \left\{-i : d(f^{-1}(x_{-i}), x_{-i-1}) \geq \delta\right\} \cap \{-n+1, \ldots, -1, 0\},$$

we have

$$\lim_{n\to\infty} \frac{\#Np_n^+(\xi, f, \delta)}{n} = 0 \quad \text{and} \quad \lim_{n\to-\infty} \frac{\#Np_n^-(\xi, f, \delta)}{n} = 0.$$

We say that f has the *ergodic shadowing property* in Λ (or Λ is ergodic shadwoable for f) if for any $\epsilon > 0$, there is a $\delta > 0$ such that every δ-ergodic pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ of f there is a point $z \in \Lambda$ such that for

$$Ns_n^+(\xi, f, z, \epsilon) = \left\{i : d(f^i(z), x_i) \geq \epsilon\right\} \cap \{0, 1, \ldots, n-1\}$$

and

$$Ns_n^-(\xi, f, z, \epsilon) = \left\{-i : d(f^{-i}(z), x_{-i}) \geq \epsilon\right\} \cap \{-n+1, \ldots, -1, 0\},$$

we have

$$\lim_{n\to\infty} \frac{\#Ns_n^+(\xi, f, z, \epsilon)}{n} = 0 \quad \text{and} \quad \lim_{n\to-\infty} \frac{\#Ns_n^-(\xi, f, z, \epsilon)}{n} = 0.$$

Corollary 3.11. *For generic $f \in \mathrm{Diff}(M)$, any ergodic shadwoable chain recurrence class $\mathcal{C}_f$ of f is hyperbolic if it contains a hyperbolic periodic point.*

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