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Nonlocal Boundary Value Problems for Riemann-Liouville Fractional Differential Inclusions with Hadamard Fractional Integral Boundary Conditions

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Abstract. In this paper we study a new class of boundary value problems consisting of a fractional differential inclusion of Riemann-Liouville type and Hadamard fractional integral conditions. Some new existence results for convex as well as non-convex multivalued maps are obtained by using standard fixed point theorems. Some illustrative examples are also presented.

1. Introduction

Fractional differential equations play an important role in many research areas, such as physics, chemical technology, population dynamics, biotechnology and economics. For examples and recent development of the topic, see [1–7, 21, 25–27, 29]. It has been observed that most of the work on the topic involves either Riemann-Liouville or Caputo type fractional derivative. For background material of Hadamard fractional derivative and integral, we refer to the papers [10–12, 18, 20–22].

In this paper, we consider the following boundary value problem

$$\begin{cases} {}_{RL}D^{\alpha}x(t) \in F(t,x(t)), \quad 0 < t < T,\ 1 < \alpha \leq 2, \\ x(0) = 0, \quad \displaystyle\sum_{i=1}^{m} \mu_i\, {}_HI^{\beta_i}x(\eta_i) = \sum_{j=1}^{n} \delta_j\, {}_HI^{\gamma_j}x(\xi_j) + \lambda, \end{cases} \tag{1.1}$$

where $1 < \alpha \leq 2$, ${}_{RL}D^q$ is the standard Riemann-Liouville fractional derivative of order q, ${}_HI^{\beta_i}$, ${}_HI^{\gamma_j}$ are the Hadamard fractional integrals of order $\beta_i > 0$, $\gamma_j > 0$, $\eta_i, \xi_j \in (0,T)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $F\colon [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, and $\mu_i, \delta_j \in \mathbb{R}$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ are real constants such that

$$\Lambda := \sum_{i=1}^{m} \mu_i(\alpha-1)^{-\beta_i}\eta_i^{\alpha-1} - \sum_{j=1}^{n} \delta_j(\alpha-1)^{-\gamma_j}\xi_j^{\alpha-1} \neq 0.$$

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The present paper is motivated by a recent paper [28], where the problem (1.1) was studied for single valued maps. Here, in the present paper, we cover the multi-valued case. We establish some existence results for the problem (1.1), when the right-hand side has convex as well as non-convex values. In the case of convex values (upper semicontinuous case) we use the nonlinear alternative of Leray-Schauder type. When the right hand side is not necessarily convex valued (lower semicontinuous case) we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. Finally, in the last result (Lipschitz case) we prove the existence of solutions for the problem (1.1) with not necessary nonconvex valued right-hand side, by applying a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. The methods used are well known, however their exposition in the framework of problem (1.1) is new.

2. Preliminaries

2.1. Basic material for fractional calculus

In this subsection, we introduce some notations and definitions of fractional calculus and present preliminary results needed in the subsequent results.

Definition 2.1. The Riemann-Liouville fractional derivative of order $q > 0$ of a continuous function $f\colon (0,\infty) \to \mathbb{R}$ is defined by

$$_{RL}D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s)\, ds, \quad n-1 < q < n,$$

where $n = [q]+1$, $[q]$ denotes the integer part of a real number q, provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the Gamma function defined by $\Gamma(q) = \int_0^\infty e^{-s} s^{q-1}\, ds$.

Definition 2.2. The Riemann-Liouville fractional integral of order $q > 0$ of a continuous function $f\colon (0,\infty) \to \mathbb{R}$ is defined by

$$_{RL}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\, ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the gamma function.

Definition 2.3. The Hadamard fractional integral of order $q \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 0$, is defined as

$$_{H}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s}\right)^{q-1} f(s) \frac{ds}{s},$$

provided the integral exists.

Lemma 2.4. [21, page 113] *Let $q > 0$ and $n > 0$. Then the following formulas hold*

$$({}_HI^q s^n)(t) = n^{-q}t^n \quad \text{and} \quad ({}_HD^q s^n)(t) = n^q t^n.$$

Lemma 2.5. *Let $1 < \alpha \leq 2$, $\beta_i, \gamma_j > 0$, $\eta_i, \xi_j \in (0,T)$, $\lambda, \mu_i, \delta_j \in \mathbb{R}$ for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, and $h \in C([0,T], \mathbb{R})$. Then the nonlocal Hadamard fractional integral problem for nonlinear Riemann-Liouville fractional differential equation*

$$(2.1) \qquad {}_{RL}D^\alpha x(t) = h(t), \quad 0 \leq t \leq T,$$

subject to the boundary conditions

$$(2.2) \qquad x(0) = 0, \quad \sum_{i=1}^{m} \mu_{i\,H} I^{\beta_i} x(\eta_i) = \sum_{j=1}^{n} \delta_{j\,H} I^{\gamma_j} x(\xi_j) + \lambda,$$

has a unique solution given by

$$(2.3) \qquad x(t) = \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n} \delta_{j\,H} I^{\gamma_j}\,{}_{RL}I^\alpha h(\xi_j) - \sum_{i=1}^{m} \mu_{i\,H} I^{\beta_i}\,{}_{RL}I^\alpha h(\eta_i) + \lambda\right) + {}_{RL}I^\alpha h(t),$$

where

$$(2.4) \qquad \Lambda := \sum_{i=1}^{m} \mu_i (\alpha-1)^{-\beta_i} \eta_i^{\alpha-1} - \sum_{j=1}^{n} \delta_j (\alpha-1)^{-\gamma_j} \xi_j^{\alpha-1} \neq 0.$$

Proof. Applying the Riemann-Liouville fractional integral of order α to both sides of (2.1), we have

$$(2.5) \qquad x(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + {}_{RL}I^\alpha h(t),$$

where $k_1, k_2 \in \mathbb{R}$. The first condition of (2.2) implies that $k_2 = 0$. Therefore (2.5) reduces to

$$(2.6) \qquad x(t) = k_1 t^{\alpha-1} + {}_{RL}I^\alpha h(t).$$

For any $p > 0$, by Lemma 2.4, it follows that

$$(2.7) \qquad {}_HI^p x(t) = k_1(\alpha-1)^{-p} t^{\alpha-1} + {}_HI^p\,{}_{RL}I^\alpha h(t).$$

The second condition of (2.2) with (2.7) leads to

$$(2.8) \qquad k_1 = \frac{1}{\Lambda}\left(\sum_{j=1}^{n} \delta_{j\,H} I^{\gamma_j}\,{}_{RL}I^\alpha h(\xi_j) - \sum_{i=1}^{m} \mu_{i\,H} I^{\beta_i}\,{}_{RL}I^\alpha h(\eta_i) + \lambda\right),$$

where Λ is defined by (2.4). Substituting the value of k_1 into (2.6), we obtain 2.3 as required. The proof is completed. $\square$

2.2. Basic material for multivalued maps

Here we outline some basic concepts of multivalued analysis [15, 19].

Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|, t \in [0,T]\}$. Also by $L^1([0,T],\mathbb{R})$ we denote the space of functions $x\colon [0,T] \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_0^T |x(t)|\, dt$.

For a normed space $(X, \|\cdot\|)$, let

$$\begin{aligned}
\mathcal{P}_{cl}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is closed}\},\\
\mathcal{P}_{b}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\},\\
\mathcal{P}_{cl,b}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\},\\
\mathcal{P}_{cp}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ is compact}\},
\end{aligned}$$

and

$$\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}.$$

A multi-valued map $G\colon X \to \mathcal{P}(X)$:

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$.

(ii) is *bounded* on bounded sets if $G(Y) = \cup_{x \in Y} G(x)$ is bounded in X for all $Y \in \mathcal{P}_b(X)$ (i.e., $\sup_{x\in Y}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

(iii) is called *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$.

(iv) is *lower semi-continuous* (*l.s.c.*) if the set $\{y \in X : G(y) \cap Y \neq \emptyset\}$ is open for any open set Y in X.

(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$; If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

(vi) is said to be *measurable* if for every $y \in X$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable.

(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$.

3. Existence results

3.1. The Carathéodory case

In this subsection we consider the case when F has convex values and prove an existence result based on nonlinear alternative of Leray-Schauder type, assuming that F is Carathéodory.

Definition 3.1. A multivalued map $F\colon [0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t\longmapsto F(t,x)$ is measurable for each $x\in\mathbb{R}$;

(ii) $x\longmapsto F(t,x)$ is upper semicontinuous for almost all $t\in[0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho>0$, there exists $\varphi_\rho\in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\|=\sup\{|v|:v\in F(t,x)\}\le\varphi_\rho(t)$$

for all $\|x\|\le\rho$ and for a.e. $t\in[0,T]$.

For each $y\in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y}:=\left\{v\in L^1([0,T],\mathbb{R}):v(t)\in F(t,y(t))\text{ on }[0,T]\right\}.$$

We define the graph of G to be the set $\mathrm{Gr}(G)=\{(x,y)\in X\times Y, y\in G(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

Lemma 3.2. [15, Proposition 1.2] *If $G\colon X\to\mathcal{P}_{cl}(Y)$ is u.s.c., then $\mathrm{Gr}(G)$ is a closed subset of $X\times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$ and $\{y_n\}_{n\in\mathbb{N}}\subset Y$, if when $n\to\infty$, $x_n\to x_*$, $y_n\to y_*$ and $y_n\in G(x_n)$, then $y_*\in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

The following lemma will be used in the sequel.

Lemma 3.3. [24] *Let X be a Banach space. Let $F\colon J\times\mathbb{R}\to\mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta\circ S_F\colon C(J,X)\to\mathcal{P}_{cp,c}(C(J,X)),\quad x\mapsto(\Theta\circ S_F)(x)=\Theta(S_{F,x})$$

is a closed graph operator in $C(J,X)\times C(J,X)$.

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps.

Lemma 3.4 (Nonlinear alternative for Kakutani maps)**.** [17] *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F\colon \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Theorem 3.5. *Assume that:*

(H_1) *$F\colon [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*

(H_2) *there exists a continuous nondecreasing function $\psi\colon [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \text{ for each } (t,x) \in [0,T] \times \mathbb{R};$$

(H_3) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M)\,\|p\|\,\Lambda_1 + \frac{T^{\alpha-1}|\lambda|}{|\Lambda|}} > 1,$$

where

$$\Lambda_1 = \frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda|}\sum_{i=1}^{m}|\mu_i|\,\alpha^{-\beta_i}\eta_i^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda|}\sum_{j=1}^{n}|\delta_j|\,\alpha^{-\gamma_j}\xi_j^{\alpha}\right). \tag{3.1}$$

Then the boundary value problem (1.1) *has at least one solution on* $[0,T]$.

Proof. Define the operator $\mathcal{F}\colon C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ by

$$\mathcal{F}(x) = \left\{\begin{array}{l} h \in C([0,T],\mathbb{R}) : \\ h(t) = {}_{RL}I^{\alpha}v(t) + \dfrac{t^{\alpha-1}}{\Lambda}\left(\displaystyle\sum_{j=1}^{n}\delta_j{}_{H}I^{\gamma_j}{}_{RL}I^{\alpha}v(\xi_j) - \sum_{i=1}^{m}\mu_i{}_{H}I^{\beta_i}{}_{RL}I^{\alpha}v(\eta_i) + \lambda\right)\end{array}\right\} \tag{3.2}$$

for $v \in S_{F,x}$. It is obvious that the fixed points of $\mathcal{F}$ are solutions of the boundary value problem (1.1).

We will show that $\mathcal{F}$ satisfies the assumptions of Leray-Schauder nonlinear alternative (Lemma 3.4). The proof consists of several steps.

Step 1. $\mathcal{F}(x)$ is convex for each $x \in C([0,T],\mathbb{R})$.
This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

Step 2. $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $C([0,T],\mathbb{R})$.
For a positive number ρ, let $B_\rho = \{x \in C([0,T],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathcal{F}(x), x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$h(t) = {}_{RL}I^\alpha v(t) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n} \delta_j {}_H I^{\gamma_j} {}_{RL}I^\alpha v(\xi_j) - \sum_{i=1}^{m} \mu_i {}_H I^{\beta_i} {}_{RL}I^\alpha v(\eta_i) + \lambda\right).$$

Then we have

$$\begin{aligned}
|h(x)| &\le {}_{RL}I^\alpha |v(t)| + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n} \delta_j {}_H I^{\gamma_j} {}_{RL}I^\alpha |v(\xi_j)| - \sum_{i=1}^{m} \mu_i {}_H I^{\beta_i} {}_{RL}I^\alpha |v(\eta_i)| + |\lambda|\right) \\
&\le \frac{\|p\|\,\psi(\|x\|)}{\Gamma(\alpha+1)}\left(T^\alpha + \frac{T^{\alpha-1}}{|\Lambda|}\sum_{i=1}^{m} |\mu_i|\,\alpha^{-\beta_i}\eta_i^\alpha + \frac{T^{\alpha-1}}{|\Lambda|}\sum_{j=1}^{n} |\delta_j|\,\alpha^{-\gamma_j}\xi_j^\alpha\right) \\
&\quad + \frac{T^{\alpha-1}|\lambda|}{|\Lambda|} \\
&= \Lambda_1 \|p\|\,\psi(\|x\|) + \frac{T^{\alpha-1}|\lambda|}{|\Lambda|}.
\end{aligned}$$

Thus

$$\|h\| \le \Lambda_1 \|p\|\,\psi(\rho) + \frac{T^{\alpha-1}|\lambda|}{|\Lambda|}.$$

Step 3. $\mathcal{F}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.
Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_\rho$. For each $h \in \mathcal{F}(x)$, we obtain

$$\begin{aligned}
&|h(\tau_2) - h(\tau_1)| \\
&\le \frac{1}{\Gamma(q)}\left|\int_0^{\tau_1}\left[(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}\right] v(s)\,ds + \int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1} v(s)\,ds\right| \\
&\quad + \frac{(\tau_2^{\alpha-1} - \tau_1^{\alpha-1})}{|\lambda|}\left(\sum_{j=1}^{n} \delta_j {}_H I^{\gamma_j} {}_{RL}I^\alpha |v(\xi_j)| - \sum_{i=1}^{m} \mu_i {}_H I^{\beta_i} {}_{RL}I^\alpha |v(\eta_i)|\right) \\
&\le \frac{\|p\|\,\psi(\rho)}{\Gamma(q)}\left|\int_0^{\tau_1}\left[(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}\right] ds + \int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1}\,ds\right| \\
&\quad + \frac{(\tau_2^{q-1} - \tau_1^{\alpha-1})\,\|p\|\,\psi(\rho)}{|\lambda|}\left(\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{m} |\mu_i|\,\alpha^{-\beta_i}\eta_i^\alpha + \frac{1}{\Gamma(\alpha+1)}\sum_{j=1}^{n} |\delta_j|\,\alpha^{-\gamma_j}\xi_j^\alpha\right).
\end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $\tau_2 - \tau_1 \to 0$. As $\mathcal{F}$ satisfies the above three assumptions, therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{F}\colon C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

Since $\mathcal{F}$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph. Thus, in our next step, we show that

Step 4. $\mathcal{F}$ has a closed graph.

Let $x_n \to x_*$, $h_n \in \mathcal{F}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{F}(x_*)$. Associated with $h_n \in \mathcal{F}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$h_n(t) = {}_{RL}I^{\alpha} v_n(t) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n} \delta_j {}_H I^{\gamma_j} {}_{RL}I^{\alpha} v_n(\xi_j) - \sum_{i=1}^{m} \mu_i {}_H I^{\beta_i} {}_{RL}I^{\alpha} v_n(\eta_i) + \lambda\right).$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$h_*(t) = {}_{RL}I^{\alpha} v_*(t) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n} \delta_j {}_H I^{\gamma_j} {}_{RL}I^{\alpha} v_*(\xi_j) - \sum_{i=1}^{m} \mu_i {}_H I^{\beta_i} {}_{RL}I^{\alpha} v_*(\eta_i) + \lambda\right).$$

Let us consider the linear operator $\Theta\colon L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = {}_{RL}I^{\alpha} v(t) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n} \delta_j {}_H I^{\gamma_j} {}_{RL}I^{\alpha} v(\xi_j) - \sum_{i=1}^{m} \mu_i {}_H I^{\beta_i} {}_{RL}I^{\alpha} v(\eta_i) + \lambda\right).$$

Observe that

$$\begin{aligned}\|h_n(t) - h_*(t)\| = \Bigg\| & {}_{RL}I^{\alpha}(v_n(t) - v_*(t)) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n} \delta_j {}_H I^{\gamma_j} {}_{RL}I^{\alpha}(v_n(\xi_j) - v_*(\xi_j))\right.\\ & \left. - \sum_{i=1}^{m} \mu_i {}_H I^{\beta_i} {}_{RL}I^{\alpha}(v_n(\eta_i) - v_*(\eta_i))\right)\Bigg\| \to 0,\end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 3.3 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = {}_{RL}I^{\alpha} v_*(t) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n} \delta_j {}_H I^{\gamma_j} {}_{RL}I^{\alpha} v_*(\xi_j) - \sum_{i=1}^{m} \mu_i {}_H I^{\beta_i} {}_{RL}I^{\alpha} v_*(\eta_i) + \lambda\right),$$

for some $v_* \in S_{F,x_*}$.

Step 5. We show there exists an open set $U \subseteq C([0,T],\mathbb{R})$ with $x \notin \theta\mathcal{F}(x)$ for any $\theta \in (0,1)$ and all $x \in \partial U$.

Let $\theta \in (0,1)$ and $x \in \theta\mathcal{F}(x)$. Then there exists $v \in L^1([0,T],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,T]$, we have

$$x(t) = \theta {}_{RL}I^{\alpha} v(t) + \theta\frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n} \delta_j {}_H I^{\gamma_j} {}_{RL}I^{\alpha} v(\xi_j) - \sum_{i=1}^{m} \mu_i {}_H I^{\beta_i} {}_{RL}I^{\alpha} v(\eta_i) + \lambda\right).$$

Using the computations of the second step above we have

$$\|x\| \le \psi(\|x\|)\,\|p\| \left\{ \frac{1}{\Gamma(\alpha+1)} \left(T^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda|} \sum_{i=1}^{m} |\mu_i|\, \alpha^{-\beta_i} \eta_i^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda|} \sum_{j=1}^{n} |\delta_j|\, \alpha^{-\gamma_j} \xi_j^{\alpha} \right) \right\}$$
$$+ \frac{T^{\alpha-1}\,|\lambda|}{|\Lambda|}$$
$$= \psi(\|x\|)\,\|p\|\,\Lambda_1 + \frac{T^{\alpha-1}\,|\lambda|}{|\Lambda|},$$

which implies that

$$\frac{\|x\|}{\psi(\|x\|)\,\|p\|\,\Lambda_1 + \frac{T^{\alpha-1}|\lambda|}{|\Lambda|}} \le 1.$$

In view of (H_3), there exists M such that $\|x\| \ne M$. Let us set

$$U = \{x \in C(I, \mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\mathcal{F}\colon \overline{U} \to \mathcal{P}(C(I,\mathbb{R}))$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $x \in \partial U$ such that $x \in \theta \mathcal{F}(x)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that $\mathcal{F}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

3.2. The lower semicontinuous case

In the next result, F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [9] for lower semi-continuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and $G\colon X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \ne \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,T] = J$, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.6. Let Y be a separable metric space and let $N\colon Y \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F\colon [0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F}\colon C([0,T]\times\mathbb{R})\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x)=\left\{w\in L^1([0,T],\mathbb{R}) : w(t)\in F(t,x(t)) \text{ for a.e. } t\in[0,T]\right\},$$

which is called the Nemytskii operator associated with F.

Definition 3.7. Let $F\colon [0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 3.8. [16] *Let Y be a separable metric space and let $N\colon Y\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g\colon Y\to L^1([0,T],\mathbb{R})$ such that $g(x)\in N(x)$ for every $x\in Y$.*

Theorem 3.9. *Assume that* (H_2), (H_3) *and the following condition holds:*

(H_4) $F\colon [0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(a) $(t,x)\longmapsto F(t,x)$ *is* $\mathcal{L}\otimes\mathcal{B}$ *measurable,*

(b) $x\longmapsto F(t,x)$ *is lower semicontinuous for each* $t\in[0,T]$.

Then the boundary value problem (1.1) *has at least one solution on* $[0,T]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. Then from Lemma 3.8, there exists a continuous function $f\colon C^2([0,T],\mathbb{R})\to L^1([0,T],\mathbb{R})$ such that $f(x)\in\mathcal{F}(x)$ for all $x\in C([0,T],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}_{RL}D^q x(t)=f(x(t)), \quad 0<t<T,\ 1<\alpha\le 2,\\ x(0)=0,\quad \displaystyle\sum_{i=1}^{m}\mu_{i\,H}I^{\beta_i}x(\eta_i)=\sum_{j=1}^{n}\delta_{j\,H}I^{\gamma_j}x(\xi_j)+\lambda. \end{cases} \tag{3.3}$$

Observe that if $x\in C^2([0,T],\mathbb{R})$ is a solution of (3.3), then x is a solution to the problem (1.1). In order to transform the problem (3.3) into a fixed point problem, we define the operator $\overline{\mathcal{F}}$ as

$$\overline{\mathcal{F}}x(t)={}_{RL}I^{\alpha}f(x(t))+\frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n}\delta_{j\,H}I^{\gamma_j}{}_{RL}I^{\alpha}f(x(\xi_j))-\sum_{i=1}^{m}\mu_{i\,H}I^{\beta_i}{}_{RL}I^{\alpha}f(x((\eta_i))+\lambda\right).$$

It can easily be shown that $\overline{\mathcal{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.5. So we omit it. This completes the proof. $\square$

3.3. The Lipschitz case

In this subsection we prove the existence of solutions for the problem (1.1) with a not necessary nonconvex valued right-hand side, by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [14].

Let (X,d) be a metric space induced from the normed space $(X;\|\cdot\|)$. Consider $H_d\colon \mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B)=\max\left\{\sup_{a\in A} d(a,B),\sup_{b\in B} d(A,b)\right\},$$

where $d(A,b)=\inf_{a\in A} d(a;b)$ and $d(a,B)=\inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{cl,b}(X),H_d)$ is a metric space (see [23]).

Definition 3.10. A multivalued operator $N\colon X\to\mathcal{P}_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma>0$ such that

$$H_d(N(x),N(y))\le \gamma d(x,y)\quad \text{for each } x,y\in X;$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma<1$.

Lemma 3.11. [14] *Let (X,d) be a complete metric space. If $N\colon X\to\mathcal{P}_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N\neq\emptyset$.

Theorem 3.12. *Assume that:*

(A_1) *$F\colon[0,T]\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,x)\colon[0,T]\to\mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x\in\mathbb{R}$;*

(A_2) *$H_d(F(t,x),F(t,\overline{x}))\le m(t)\,|x-\overline{x}|$ for almost all $t\in[0,T]$ and $x,\overline{x}\in\mathbb{R}$ with $m\in C([0,T],\mathbb{R}^+)$ and $d(0,F(t,0))\le m(t)$ for almost all $t\in[0,T]$.*

Then the boundary value problem (1.1) *has at least one solution on* $[0,T]$ *if* $\|m\|\,\Lambda_1<1$, *i.e.,*

$$\|m\|\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^\alpha+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{i=1}^{m}|\mu_i|\,\alpha^{-\beta_i}\eta_i^\alpha+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{j=1}^{n}|\delta_j|\,\alpha^{-\gamma_j}\xi_j^\alpha\right)\right\}<1.$$

Proof. Consider the operator $\mathcal{F}$ defined by (3.2). Observe that the set $S_{F,x}$ is nonempty for each $x\in C([0,T],\mathbb{R})$ by the assumption (A_1), so F has a measurable selection (see [13, Theorem III.6]). Now we show that the operator $\mathcal{F}$ satisfies the assumptions of Lemma 3.11. We show that $\mathcal{F}(x)\in\mathcal{P}_{cl}((C[0,T],\mathbb{R}))$ for each $x\in C([0,T],\mathbb{R})$. Let

$\{u_n\}_{n\geq 0} \in \mathcal{F}(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,T],\mathbb{R})$. Then $u \in C([0,T],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,T]$,

$$u_n(t) = {}_{RL}I^{\alpha}v_n(t) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n}\delta_j {}_H I^{\gamma_j}{}_{RL}I^{\alpha}v_n(\xi_j) - \sum_{i=1}^{m}\mu_i {}_H I^{\beta_i}{}_{RL}I^{\alpha}v_n(\eta_i) + \lambda\right).$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$, we have

$$u_n(t) \to v(t) = {}_{RL}I^{\alpha}v(t) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n}\delta_j {}_H I^{\gamma_j}{}_{RL}I^{\alpha}v(\xi_j) - \sum_{i=1}^{m}\mu_i {}_H I^{\beta_i}{}_{RL}I^{\alpha}v(\eta_i) + \lambda\right).$$

Hence, $u \in \mathcal{F}(x)$.

Next we show that there exists $\delta < 1$ such that

$$H_d(\mathcal{F}(x), \mathcal{F}(\overline{x})) \leq \delta \|x - \overline{x}\| \quad \text{for each } x, \overline{x} \in C^2([0,T],\mathbb{R}).$$

Let $x, \overline{x} \in C^2([0,T],\mathbb{R})$ and $h_1 \in \mathcal{F}(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,T]$,

$$h_1(t) = {}_{RL}I^{\alpha}v_1(t) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n}\delta_j {}_H I^{\gamma_j}{}_{RL}I^{\alpha}v_1(\xi_j) - \sum_{i=1}^{m}\mu_i {}_H I^{\beta_i}{}_{RL}I^{\alpha}v_1(\eta_i) + \lambda\right).$$

By (A_2), we have

$$H_d(F(t,x), F(t,\overline{x})) \leq m(t)\,|x(t) - \overline{x}(t)|\,.$$

So, there exists $w \in F(t, \overline{x}(t))$ such that

$$|v_1(t) - w(t)| \leq m(t)\,|x(t) - \overline{x}(t)|\,,\ t \in [0,T].$$

Define $U\colon [0,T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)\,|x(t) - \overline{x}(t)|\}\,.$$

Since the multivalued operator $U(t) \cap F(t, \overline{x}(t))$ is measurable [13, Proposition III.4], there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \overline{x}(t))$ and for each $t \in [0,T]$, we have $|v_1(t) - v_2(t)| \leq m(t)\,|x(t) - \overline{x}(t)|$.

For each $t \in [0,T]$, let us define

$$h_2(t) = {}_{RL}I^{\alpha}v_2(t) + \frac{t^{\alpha-1}}{\Lambda}\left(\sum_{j=1}^{n}\delta_j {}_H I^{\gamma_j}{}_{RL}I^{\alpha}v_2(\xi_j) - \sum_{i=1}^{m}\mu_i {}_H I^{\beta_i}{}_{RL}I^{\alpha}v_2(\eta_i) + \lambda\right).$$

Thus,

$$
\begin{aligned}
&|h_1(t)-h_2(t)|\\
&\le {}_{RL}I^{\alpha}\left|v_1(t)-v_2(t)\right|\\
&\quad+\frac{t^{\alpha-1}}{|\Lambda|}\left(\sum_{j=1}^{n}\delta_j{}_HI^{\gamma_j}{}_{RL}I^{\alpha}\left|v_1(\xi_j)-v_2(\xi_j)\right|+\sum_{i=1}^{m}\mu_i{}_HI^{\beta_i}{}_{RL}I^{\alpha}\left|v_1(\eta_i)-v_2(\eta_i)\right|\right)\\
&\le \|m\|\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{i=1}^{m}|\mu_i|\,\alpha^{-\beta_i}\eta_i^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{j=1}^{n}|\delta_j|\,\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\}\|x-\overline{x}\|\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|h_1-h_2\|\\
&\le \|m\|\left\{\frac{1}{\Gamma(\alpha+)}\left(T^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{i=1}^{m}|\mu_i|\,\alpha^{-\beta_i}\eta_i^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{j=1}^{n}|\delta_j|\,\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\}\|x-\overline{x}\|\,.
\end{aligned}
$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\mathcal{F}(x),\mathcal{F}(\overline{x}))\le\delta\,\|x-\overline{x}\|$$

where

$$\delta=\|m\|\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{i=1}^{m}|\mu_i|\,\alpha^{-\beta_i}\eta_i^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{j=1}^{n}|\delta_j|\,\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\}\|x-\overline{x}\|<1.$$

So $\mathcal{F}$ is a contraction. Therefore, it follows by Lemma 3.11 that $\mathcal{F}$ has a fixed point x which is a solution of (1.1). This completes the proof. □

3.4. Examples

In this section, we will illustrate our main results with the help of some examples. Let us consider the following nonlocal boundary value problem for Riemann-Liouville fractional differential inclusions with Hadamrd fractional integral boundary conditions

$$
\begin{cases}
{}_{RL}D^{\frac{5}{3}}x(t)\in F(t,x(t)), \quad t\in(0,\frac{3}{2}),\\
x(0)=0,\\
4{}_HI^{\frac{2}{5}}x(\frac{1}{3})+3{}_HI^{\frac{1}{2}}x(\frac{2}{3})=\frac{2}{3}{}_HI^{\frac{1}{2}}x(\frac{1}{4})+\frac{3}{5}{}_HI^{\frac{1}{3}}x(\frac{4}{3})+\frac{1}{7}.
\end{cases}
\tag{3.4}
$$

Here we have $\alpha=5/3$, $T=3/2$, $m=2$, $n=2$, $\mu_1=4$, $\mu_2=3$, $\beta_1=2/5$, $\beta_2=1/2$, $\eta_1=1/3$, $\eta_2=2/3$, $\delta_1=2/3$, $\delta_2=3/5$, $\gamma_1=1/2$, $\gamma_2=1/3$, $\xi_1=1/4$, $\xi_2=4/3$, $\lambda=1/7$. By using the MAPLE program, we can find

$$\Lambda:=\sum_{i=1}^{m}\mu_i(\alpha-1)^{-\beta_i}\eta_i^{\alpha-1}-\sum_{j=1}^{n}\delta_j(\alpha-1)^{-\gamma_j}\xi_j^{\alpha-1}\approx 6.221625494\neq 0.$$

(a) Let $F\colon [0,3/2]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$x\to F(t,x)=\left[\frac{1+\cos^2 x}{2+\sin^2 x},\frac{e^x}{2e^x+3}+\frac{2t^3}{9}+1\right]. \tag{3.5}$$

For $f\in F$, we have

$$|f|\le \max\left(\frac{1+\cos^2 x}{2+\sin^2 x},\frac{e^x}{2e^x+3}+\frac{2t^3}{9}+1\right)\le\frac{9}{4},\ x\in\mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathcal{P}}:=\sup\{|y|:y\in F(t,x)\}\le\frac{9}{4}=p(t)\psi(\|x\|),\ x\in\mathbb{R},$$

with $p(t)=9$, $\psi(\|x\|)=1/4$. Further, using the condition (H_6),

$$\frac{M}{\psi(M)\,\|p\|\,\Lambda_1+\frac{T^{\alpha-1}|\lambda|}{|\Lambda|}}>1,$$

we find that $M>3.779988106$. Therefore, all the conditions of Theorem 3.5 are satisfied. So, the problem (3.4) with $F(t,x)$ given by (3.5) has at least one solution on $[0,3/2]$.

(b) Let $F\colon [0,3/2]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$x\to F(t,x)=\left[0,\frac{3\sin^2 x}{(\sqrt{8}+2t)^2}+\frac{3}{128}\right]. \tag{3.6}$$

Then we have

$$\sup\{|x|:x\in F(t,x)\}\le\frac{3}{(\sqrt{8}+2t)^2}+\frac{3}{128},$$

and

$$H_d(F(t,x),F(t,\overline{x}))\le\frac{3}{(\sqrt{8}+2t)^2}\,|x-\overline{x}|\,.$$

Let $m(t)=3/(\sqrt{8}+2t)^2$. Then $H_d(F(t,x),F(t,\overline{x}))\le m(t)\,|x-\overline{x}|$ with $d(0,F(t,0)\le m(t)$ and $\|m\|=3/8$. We can show that

$$\|m\|\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^\alpha+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{i=1}^m|\mu_i|\,\alpha^{-\beta_i}\eta_i^\alpha+\frac{T^{\alpha-1}}{|\Lambda|}\sum_{j=1}^n|\delta_j|\,\alpha^{-\gamma_j}\xi_j^\alpha\right)\right\}$$
$$\approx 0.624983363<1.$$

Thus all the conditions of Theorem 3.12 are satisfied. Therefore, by the conclusion of Theorem 3.12, the problem (3.4) with $F(t,x)$ given by (3.6) has at least one solution on $[0,3/2]$.

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