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Gauss Maps of Ruled Submanifolds and Applications II

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Abstract. The notion of pointwise 1-type Gauss map was derived from the ordinary finite type Gauss map and it gives an interesting geometric properties on surfaces of 3-dimensional Euclidean space. In particular, the helicoid and the right cone of 3-dimensional Euclidean space are characterized by pointwise 1-type Gauss map. Inspired by such a study, in this paper, we completely classify ruled submanifolds of Euclidean space with pointwise 1-type Gauss map.

1. Introduction

One of main objects in differential geometry is to study Riemannian manifolds. Due to Nash's embedding theorem, every Riemannian manifolds can be regarded as a submanifold of Euclidean space with sufficiently high codimension. Thus, it is convenient to study Riemannian manifolds by examining submanifolds of Euclidean space with the intrinsic and extrinsic properties. The theory of finite type submanifolds initiated to estimate the total mean curvature of compact submanifolds of Euclidean space in the late 1970s. Inspired by the degree of algebraic varieties, B.-Y. Chen introduced the concept of order and type on submanifolds of Euclidean space:

Let M be a submanifold of m-dimensional Euclidean space $\mathbb{E}^m$ with an isometric immersion x. We can identifying x with the position vector of $\mathbb{E}^m$. Let Δ be the Laplace operator of M in $\mathbb{E}^m$. The submanifold M is said to be of finite type if x has a spectral decomposition by $x = x_0 + x_1 + \cdots + x_k$, where x_0 is a constant vector and x_i are

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the vector fields satisfying $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$ $(i = 1, 2, \ldots, k)$. In particular, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different, it is called k-type. Many works have been done in studying finite type submanifolds of Euclidean space (see [2, 3, 5, 10]). This notion of finite type immersion was naturally extended to pseudo-Riemannian manifolds in pseudo-Euclidean space [13, 17, 19] and it can be applied to smooth functions, particularly the Gauss map defined on submanifolds of Euclidean space or pseudo-Euclidean space [2, 3].

In regarding the Gauss map of finite type, B.-Y. Chen and P. Piccini [6] studied the submanifolds in Euclidean space with finite type Gauss map so that they classified compact surfaces with 1-type Gauss map, that is, $\Delta G = \lambda(G + C)$, where C is a constant vector and $\lambda \in \mathbb{R}$. Since then, quite a few works on ruled surfaces and ruled submanifolds with finite type Gauss map in Euclidean space or pseudo-Euclidean space have been done [12, 13, 15–20].

However, some surfaces including a helicoid have an interesting property concerning the Gauss map which looks satisfying 1-type Gauss map. As a matter of fact, it is not: The helicoid in $\mathbb{E}^3$ parameterized by

$$x(u, v) = (u \cos v, u \sin v, av), \quad a \neq 0$$

has the Gauss map and its Laplacian are respectively given by

$$G = \frac{1}{\sqrt{a^2 + u^2}}(a \sin v, -a \cos v, u)$$

and

$$\Delta G = \frac{2a^2}{(a^2 + u^2)^2} G.$$

The right (or circular) cone C_a with parametrization

$$x(u, v) = (u \cos v, u \sin v, au), \quad a \geq 0$$

has the Gauss map

$$G = \frac{1}{\sqrt{1 + a^2}}(a \cos v, a \sin v, -1)$$

which satisfies

$$\Delta G = \frac{1}{u^2}\left(G + \left(0, 0, \frac{1}{\sqrt{1 + a^2}}\right)\right)$$

(cf. [7,8]). The Gauss maps of examples above are similar to 1-type, but obviously different from the usual sense of 1-type Gauss map. Based on these, we define

Definition 1.1. An oriented n-dimensional submanifold M of the Euclidean space $\mathbb{E}^m$ is said to have *pointwise* 1-*type* Gauss map if it satisfies the condition

$$\Delta G = f(G + C), \tag{1.1}$$

where f is a non-zero smooth function on M and C some constant vector. In particular, if C is zero, the Gauss map G is said to be of *the first kind*. Otherwise, it is said to be of *the second kind* [4, 7–9].

In [7, 8], M. Choi et al. proved that a ruled surface in 3-dimensional Euclidean space with pointwise 1-type Gauss map is part of a plane, a circular cylinder, a heilcoid, a cylinder over a plane curve of infinite type or a circular cone. And, in [9, 20], ruled surfaces in pseudo-Euclidean space with pointwise 1-type Gauss map were studied.

Continuing to [14], we now raise a question: *Can we completely classify ruled submanifolds in Euclidean space with pointwise* 1*-type Gauss map of the second kind?*

In this paper, we study the problem described in the question above and completely classify ruled submanifolds of Euclidean space with pointwise 1-type Gauss map.

All of geometric objects under consideration are smooth and submanifolds are assumed to be connected unless otherwise stated.

2. Preliminaries

Let $x\colon M \to \mathbb{E}^m$ be an isometric immersion of an n-dimensional Riemannian manifold M into $\mathbb{E}^m$. Let $(x_1, x_2, \dots, x_n)$ be a local coordinate system of M. For the components g_{ij} of the Riemannian metric $\langle\cdot,\cdot\rangle$ on M induced from that of $\mathbb{E}^m$, we denote by (g^{ij}) (respectively, $\mathcal{G}$) the inverse matrix (respectively, the determinant) of the component matrix (g_{ij}). Then the Laplace operator Δ on M is defined by

$$\Delta = -\frac{1}{\sqrt{\mathcal{G}}}\sum_{i,j}\frac{\partial}{\partial x_i}\left(\sqrt{\mathcal{G}}g^{ij}\frac{\partial}{\partial x_j}\right).$$

We now choose an adapted local orthonormal frame $\{e_1, e_2, \dots, e_m\}$ in $\mathbb{E}^m$ such that $e_1, e_2, \dots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \dots, e_m$ normal to M. The Gauss map $G\colon M \to G(n,m) \subset \mathbb{E}^N$ $(N = {}_mC_n)$, $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$ of M is a smooth map which carries a point p in M to an oriented n-plane in $\mathbb{E}^m$ by the parallel translation of the tangent space of M at p to an n-plane passing through the origin in $\mathbb{E}^m$, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented n-planes through the origin of $\mathbb{E}^m$.

An inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on $G(n,m) \subset \mathbb{E}^N$ is defined by

$$\langle\langle e_{i_1} \wedge \cdots \wedge e_{i_n}, e_{j_1} \wedge \cdots \wedge e_{j_n}\rangle\rangle = \det(\langle e_{i_l}, e_{j_k}\rangle),$$

where l, k run over the range $\{1, 2, \dots, n\}$. Then, $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq m\}$ is an orthonormal basis of $\mathbb{E}^N$.

An $(r+1)$-dimensional submanifold M in $\mathbb{E}^m$ is called a *ruled submanifold* if M is foliated by r-dimensional totally geodesic submanifolds $E(s,r)$ of $\mathbb{E}^m$ along a regular curve $\alpha = \alpha(s)$ on M defined on an open interval I. Thus, a parametrization of a ruled submanifold M in $\mathbb{E}^m$ can be given by

$$x = x(s, t_1, t_2, \dots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \quad s \in I,\ t_i \in I_i,$$

where I_i's are some open intervals for $i = 1, 2, \dots, r$. For each s, $E(s, r)$ is open in Span $\{e_1(s), e_2(s), \dots, e_r(s)\}$, which is the linear span of linearly independent vector fields $e_1(s), e_2(s), \dots, e_r(s)$ along the curve α. We call $E(s, r)$ the *rulings* and α the *base curve* of the ruled submanifold M. In particular, the ruled submanifold M is said to be *cylindrical* if $E(s, r)$ is parallel along α, or *non-cylindrical* otherwise.

Definition 2.1. An $(r+1)$-dimensional cylindrical ruled submanifold M is called a generalized circular cylinder $\Sigma_a \times \mathbb{E}^{r-1}$ if the base curve α is a circle and the generators of rulings are orthogonal to the plane containing the circle α, where Σ_a is a circular cylinder $S^1(a) \times \mathbb{R}$ in $\mathbb{E}^3$.

For later use, we need

Lemma 2.2. [1] *Given a curve α and orthonormal vector fields $e_1, e_2, \dots, e_n$ along α in a Riemannian manifold $\overline{M}$ with the Riemannian connection $\overline{D}$, we can always choose orthonormal vector fields $f_1, \dots, f_n$ along α such that:*

(i) *The sets of vectors $\{f_j(s) : 1 \le j \le n\}$ and $\{e_j(s) : 1 \le j \le n\}$ generate the same subspace of $T_{\alpha(s)}\overline{M}$.*

(ii) *The vector fields $(\overline{D}/ds) f_i(s)$ are normal to the subspace of $T_{\alpha(s)}\overline{M}$ spanned by $\{f_j(s) : 1 \le j \le n\}$ for all $i = 1, 2, \dots, n$.*

3. Pointwise 1-type Gauss map of the second kind

In this section, we consider a ruled submanifold M in $\mathbb{E}^m$ with pointwise 1-type Gauss map of the second kind. Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{E}^m$. Then, the base curve α can be chosen to be orthogonal to the rulings by taking an integral curve of the field of normal directions to the rulings of M. Without loss of generality, we may assume that α is a unit speed curve, that is, $\langle \alpha'(s), \alpha'(s) \rangle = 1$. From now on, the prime $'$ denotes d/ds unless otherwise stated. By Lemma 2.2, we may choose orthonormal vector fields $e_1(s), \dots, e_r(s)$ along α satisfying

$$\big\langle \alpha'(s), e_i(s) \big\rangle = 0, \quad \big\langle e_i'(s), e_j(s) \big\rangle = 0, \quad \text{for } s \in I \text{ and } i, j = 1, 2, \dots, r. \tag{3.1}$$

A parametrization of M is then obtained as

$$x = x(s, t_1, t_2, \dots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \quad s \in I. \tag{3.2}$$

In this paper, we always assume that the parametrization (3.2) satisfies the condition (3.1). Then, M has the Gauss map

$$G = \frac{1}{\|x_s\|} x_s \wedge x_{t_1} \wedge \cdots \wedge x_{t_r},$$

or, equivalently

$$G = \frac{1}{q^{1/2}} \left(\Phi + \sum_{i=1}^{r} t_i \Psi_i \right), \tag{3.3}$$

where q is the function of $s, t_1, t_2, \ldots, t_r$ defined by $q = \langle x_s, x_s \rangle$, Φ and Ψ_i $(i = 1, 2, \ldots, r)$ are vector fields along α given by

$$\Phi = \alpha' \wedge e_1 \wedge \cdots \wedge e_r \quad \text{and} \quad \Psi_i = e_i' \wedge e_1 \wedge \cdots \wedge e_r.$$

Now, we separate the cases into two typical types of ruled submanifolds which are cylindrical or non-cylindrical.

First of all, we consider the following lemma.

Lemma 3.1. *Suppose that a unit speed curve $\alpha(s)$ in an m-dimensional Euclidean space $\mathbb{E}^m$ defined on an interval I satisfies*

$$\alpha'''(s) = f(s)(\alpha'(s) + C), \tag{3.4}$$

where f is a function and C a constant vector in $\mathbb{E}^m$. Then, the curve α lies in a 3-dimensional Euclidean space. In particular, if the constant vector C is zero, we see that α is a plane curve.

Proof. We fix a point $s_0 \in I$. Let us denote by V the linear span of $\{\alpha'(s_0), \alpha''(s_0), C\}$. Then V is of at most 3-dimensional space in $\mathbb{E}^m$.

For any vector a in the orthogonal complement $V^\perp$ of V, we consider the function $h_a(s)$ defined by $h_a(s) = \langle a, \alpha'(s) \rangle$. Then, it follows from (3.4) that

$$h_a''(s) = f(s) h_a(s). \tag{3.5}$$

Hence, the function $h_a(s)$ is a solution of a second order linear differential equation with initial condition $h_a(s_0) = h_a'(s_0) = 0$. This shows that the function $h_a(s)$ vanishes identically on the interval I. Thus, we have $\alpha'(s) \in V$ for all $s \in I$, which shows that the curve α lies in a parallel displacement $\alpha(s_0) + V$ of the space V. This completes the proof. $\square$

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{E}^m$ parameterized by (3.2) satisfying Condition (3.1).

Suppose that the cylindrical ruled submanifold M has pointwise 1-type of the second kind. Equation $\Delta G = f(G + C)$ implies

$$-\Phi'' = f(\Phi + C) \tag{3.6}$$

for some non-zero function f and some non-zero constant vector C. From equation (3.6), we note that f is a function of s. We may assume that f is non-zero on some open interval I_1. Then, on the interval I_1, differentiating equation (3.6) with respect to s gives

$$\frac{f'}{f^2}\Phi'' - \frac{1}{f}\Phi''' - \Phi' = \mathbf{0}, \tag{3.7}$$

or, equivalently

$$\frac{f'}{f^2}\alpha''' - \frac{1}{f}\alpha^{(4)} - \alpha'' = \mathbf{0},$$

which means that $-\frac{1}{f}\alpha''' - \alpha'$ is a constant vector. Namely,

$$\Delta'\alpha' = -\alpha''' = f(\alpha' + D) \tag{3.8}$$

for some constant vector D on I_1. On $I - I_1$, (3.8) holds obviously. Again, using Lemma 3.1, we see that the curve α lies in a 3-dimensional Euclidean space $\mathbb{E}^3$.

Making use of Theorem 3.3 of [9] and Lemma 3.1, we have

Theorem 3.2. *An $(r+1)$-dimensional cylindrical ruled submanifold M in $\mathbb{E}^m$ has pointwise 1-type Gauss map of the second kind if and only if M is an open part of an $(r+1)$-plane or a cylinder over a curve in $\mathbb{E}^3$ of infinite type.*

We now consider the case of non-cylindrical ruled submanifold with pointwise 1-type Gauss map of the second kind. Let M be an $(r + 1)$-dimensional non-cylindrical ruled submanifold in $\mathbb{E}^m$ parameterized by (3.2).

Suppose that M has pointwise 1-type Gauss map of the second kind, that is, the Gauss map G of M satisfies the condition

$$\Delta G = f(G + C)$$

for a non-zero function f and some non-zero constant vector C. In order to make the matter simple, we introduce the following lemma.

Lemma 3.3. [14] *Let M be an $(r + 1)$-dimensional non-cylindrical ruled submanifold of $\mathbb{E}^m$ parametrized by (3.2) satisfying (3.1). Suppose some generators $e_{j_1}, e_{j_2}, \dots, e_{j_k}$ $(1 \le k < r)$ of the rulings are constant vectors along α. Then, M has pointwise 1-type Gauss map if and only if the ruled submanifold M_1 has pointwise 1-type Gauss map, where M_1 is the non-cylindrical ruled submanifold over the base curve α with the rulings generated by e_j for $j \neq j_1, j_2, \dots, j_k$.*

According to Lemma 3.3, we may assume that $e_i' \neq 0$ for all $i = 1, 2, \ldots, r$. Let us define functions:

$$u_i = \langle \alpha', e_i' \rangle, \quad w_{ij} = \langle e_i', e_j' \rangle, \quad \phi = \langle\langle \Phi, \Phi'' \rangle\rangle \quad \text{and} \quad \varphi_i = \langle\langle \Phi, \Psi'' \rangle\rangle$$

on M for $i, j = 1, 2, \ldots, r$. Note that $u_i \neq 0$ for all $i = 1, 2, \ldots, r$. Then, $\Delta G = f(G + C)$ can be written as

$$\begin{aligned}
&\left(\frac{\partial q}{\partial s}\right)^2 \left(\Phi + \sum_{j=1}^r \Psi_j t_j\right) - \frac{3}{2} q \frac{\partial q}{\partial s} \left(\Phi' + \sum_{j=1}^r \Psi_j' t_j\right) - \frac{1}{2} q \frac{\partial^2 q}{\partial s^2} \left(\Phi + \sum_{j=1}^r \Psi_j t_j\right) \\
&+ q^2 \left(\Phi'' + \sum_{j=1}^r \Psi_j'' t_j\right) + \frac{1}{2} q \sum_{i=1}^r \left(\frac{\partial q}{\partial t_i}\right)^2 \left(\Phi + \sum_{j=1}^r \Psi_j t_j\right) - \frac{1}{2} q^2 \sum_{i=1}^r \frac{\partial q}{\partial t_i} \Psi_i \\
&- \frac{1}{2} q^2 \sum_{i=1}^r \frac{\partial^2 q}{\partial t_i^2} \left(\Phi + \sum_{j=1}^r \Psi_j t_j\right) + f \left(q^3 \left(\Phi + \sum_{j=1}^r \Psi_j t_j\right) + q^{\frac{7}{2}} C\right) \\
&= \mathbf{0}.
\end{aligned} \tag{3.9}$$

By putting

$$\begin{aligned}
P(t) = &\left(\frac{\partial q}{\partial s}\right)^2 \left(1 + \sum_{j=1}^r u_j t_j\right) - \frac{3}{2} q \frac{\partial q}{\partial s} \sum_{j=1}^r x_j t_j - \frac{1}{2} q \frac{\partial^2 q}{\partial s^2} \left(1 + \sum_{j=1}^r u_j t_j\right) \\
&+ q^2 \left(\phi + \sum_{j=1}^r \varphi_j t_j\right) + \frac{1}{2} q \sum_{i=1}^r \left(\frac{\partial q}{\partial t_i}\right)^2 \left(1 + \sum_{j=1}^r u_j t_j\right) - \frac{1}{2} q^2 \sum_{i=1}^r \frac{\partial q}{\partial t_i} u_i \\
&- \frac{1}{2} q^2 \sum_{i=1}^r \frac{\partial^2 q}{\partial t_i^2} \left(1 + \sum_{j=1}^r u_j t_j\right),
\end{aligned} \tag{3.10}$$

we may put

$$f = -\frac{P(t)}{q^3 \left(1 + \sum_{j=1}^r u_j t_j\right) + q^{\frac{7}{2}} \gamma(s)}, \tag{3.11}$$

where $\gamma(s) = \langle\langle C, \Phi(s) \rangle\rangle$. Substituting equation (3.11) into (3.9), we get

$$\begin{aligned}
&\left(1 + \sum_{j=1}^r u_j t_j\right) \left\{ -\frac{3}{2} q \frac{\partial q}{\partial s} \left(\Phi' + \sum_{j=1}^r \Psi_j' t_j\right) + q^2 \left(\Phi'' + \sum_{j=1}^r \Psi_j'' t_j\right) - \frac{1}{2} q^2 \sum_{i=1}^r \frac{\partial q}{\partial t_i} \Psi_i \right\} \\
&- \left(\Phi + \sum_{j=1}^r \Psi_j t_j\right) \left\{ -\frac{3}{2} q \frac{\partial q}{\partial s} \sum_{j=1}^r x_j t_j + q^2 \left(\phi + \sum_{j=1}^r \varphi_j t_j\right) - \frac{1}{2} q^2 \sum_{i=1}^r \frac{\partial q}{\partial t_i} u_i \right\} \\
&= -q^{\frac{1}{2}} \gamma(s) \left\{ \left(\frac{\partial q}{\partial s}\right)^2 \left(\Phi + \sum_{j=1}^r \Psi_j t_j\right) - \frac{3}{2} q \frac{\partial q}{\partial s} \left(\Phi' + \sum_{j=1}^r \Psi_j' t_j\right) \right.
\end{aligned} \tag{3.12}$$

$$- \frac{1}{2} q \frac{\partial^2 q}{\partial s^2} \left(\Phi + \sum_{j=1}^{r} \Psi_j t_j \right) + q^2 \left(\Phi'' + \sum_{j=1}^{r} \Psi_j'' t_j \right) + \frac{1}{2} q \sum_{i=1}^{r} \left(\frac{\partial q}{\partial t_i} \right)^2 \left(\Phi + \sum_{j=1}^{r} \Psi_j t_j \right)$$
$$- \frac{1}{2} q^2 \sum_{i=1}^{r} \frac{\partial q}{\partial t_i} \Psi_i - \frac{1}{2} q^2 \sum_{i=1}^{r} \frac{\partial^2 q}{\partial t_i^2} \left(\Phi + \sum_{j=1}^{r} \Psi_j t_j \right) \Bigg\} + q^{\frac{1}{2}} C P(t).$$

Regarding equation (3.12), we have two possible cases whether $q^{\frac{1}{2}}$ is a polynomial or not.

Suppose that $q^{\frac{1}{2}}$ is not a polynomial in t. We will show that it is impossible. In this case, the polynomial in the left side must vanish and we can rewrite it as follows:

$$\begin{aligned} & - \frac{3}{2} q \frac{\partial q}{\partial s} \left(\Phi' + \sum_{j=1}^{r} t_j \Psi_j' \right) \left(1 + \sum_{k=1}^{r} u_k t_k \right) + \frac{3}{2} q \frac{\partial q}{\partial s} \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j \right) \sum_{k=1}^{r} x_k t_k \\ & + q^2 \left(\Phi'' + \sum_{j=1}^{r} t_j \Psi_j'' \right) \left(1 + \sum_{k=1}^{r} u_k t_k \right) - q^2 \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j \right) \left(\phi + \sum_{k=1}^{r} \varphi_k t_k \right) \\ & - \frac{1}{2} q^2 \sum_{i=1}^{r} \left(\frac{\partial q}{\partial t_i} \right) \Psi_i \left(1 + \sum_{k=1}^{r} u_k t_k \right) + \frac{1}{2} q^2 \sum_{i=1}^{r} \left(\frac{\partial q}{\partial t_i} \right) u_i \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j \right) \\ & = \mathbf{0}. \end{aligned} \tag{3.13}$$

We put (3.13) in the following form

$$- \frac{3}{2} \left(\frac{\partial q}{\partial s} \right) R(t) = q Q(t), \tag{3.14}$$

where

$$R(t) = \left(\Phi' + \sum_{j=1}^{r} t_j \Psi_j' \right) \left(1 + \sum_{k=1}^{r} u_k t_k \right) - \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j \right) \sum_{k=1}^{r} x_k t_k \tag{3.15}$$

and

$$\begin{aligned} Q(t) = & - \left(\Phi'' + \sum_{j=1}^{r} t_j \Psi_j'' \right) \left(1 + \sum_{k=1}^{r} u_k t_k \right) + \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j \right) \left(\phi + \sum_{k=1}^{r} \varphi_k t_k \right) \\ & + \frac{1}{2} \sum_{i=1}^{r} \left(\frac{\partial q}{\partial t_i} \right) \Psi_i \left(1 + \sum_{k=1}^{r} u_k t_k \right) - \frac{1}{2} \sum_{i=1}^{r} \left(\frac{\partial q}{\partial t_i} \right) u_i \left(\Phi + \sum_{j=1}^{r} t_j \Psi_j \right). \end{aligned}$$

It yields

$$R(t) = B(s) q(t)$$

for some vector field $B(s)$ along α since the degree of the polynomial of (3.15) is 2 and $\deg q = 2$. We can show that u_i and w_{ij} are constant along α for all $i, j = 1, 2, \ldots, r$. It means $\frac{\partial q}{\partial s} = 0$. Thus, we have

$$\frac{\partial q}{\partial s} = 0. \tag{3.16}$$

For the details, see [14]. Then, the right side of equation (3.12) implies

$$\frac{1}{2}\sum_{i=1}^{r}\left(\frac{\partial q}{\partial t_i}\right)^2\left\{-\gamma(s)\left(\Phi+\sum_{j=1}^{r}\Psi_j t_j\right)+C\left(1+\sum_{j=1}^{r}u_j t_j\right)\right\}$$

$$\begin{aligned}(3.17)\qquad &= q\left\{\gamma(s)\left(\Phi''+\sum_{j=1}^{r}\Psi_j'' t_j-\frac{1}{2}\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}\Psi_i-\frac{1}{2}\sum_{i=1}^{r}\frac{\partial^2 q}{\partial t_i^2}\left(\Phi+\sum_{j=1}^{r}\Psi_j t_j\right)\right)\right.\\ &\left.\quad - C\left(\phi+\sum_{j=1}^{r}\varphi_j t_j-\frac{1}{2}\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}u_i-\frac{1}{2}\sum_{i=1}^{r}\frac{\partial^2 q}{\partial t_i^2}\left(1+\sum_{j=1}^{r}u_j t_j\right)\right)\right\}.\end{aligned}$$

By (3.17), we have

$$(3.18)\qquad \sum_{i=1}^{r}\left(\frac{\partial q}{\partial t_i}\right)^2 = aq(t)$$

for a constant a because of (3.16).

We now put

$$(3.19)\qquad \left(\frac{\partial q}{\partial t_j}\right)^2 = a_j q(t) + r_j(t)$$

for some non-zero constants a_j and polynomials $r_j(t) = \sum_{k=1}^{r} c_{jk}t_k + b_j$ in t for $j = 1, 2, \ldots, r$. According to (3.18), we must have

$$(3.20)\qquad \sum_{j=1}^{r} r_j(t) = 0.$$

On the other hand, (3.19) with $w_{jj} \neq 0$ gives

$$(3.21)\qquad 4w_{jj} = a_j.$$

(3.21) implies

$$r_j(t) = \sum_{k=1}^{r} c_{jk}t_k + b_j = \sum_{k=1}^{r} 8(u_j w_{jk} - w_{jj}u_k)t_k + 4(u_j^2 - w_{jj})$$

for $j = 1, 2, \ldots, r$. By the definitions of u_i and w_{ii}, we can see that $b_j = 4(u_j^2 - w_{jj})$ are non-positive constants of the form

$$b_j = -4\sum_{a=r+1}^{m-1}\left\langle e_j', e_a\right\rangle^2$$

for $j = 1, 2, \ldots, r$. By (3.20), we see that $\sum_{j=1}^{r} b_j = 0$ and hence we have $e_j' = u_j\alpha'$, that is, $w_{jk} = u_j u_k$ for $j = 1, 2, \ldots, r$ and for $k = 1, 2, \ldots, r$. Using this, we can see that

$r_j(t) = 0$, which means that $\left(\frac{\partial q}{\partial t_j}\right)^2$ are the multiples of $q(t)$ for $j = 1, 2, \ldots, r$. Thus we see that

$$\left(\frac{\partial q}{\partial t_i}\right)^2 = 4u_i^2 q(t)$$

for $i = 1, 2, \ldots, r$.

Comparing the both sides of the above equation, we have

$$w_{ij} = u_i u_j$$

for $i, j = 1, 2, \ldots, r$, which yields

$$q = \left(1 + \sum_{j=1}^{r} u_j t_j\right)^2,$$

which contradicts that $q^{1/2}$ is not a polynomial in t. Thus, we have

$$q = \left(1 + \sum_{j=1}^{r} u_j t_j\right)^2. \tag{3.22}$$

Together with (3.22) and the definitions of u_i and w_{ii}, we see that

$$e_i' = u_i \alpha' \tag{3.23}$$

for $i = 1, 2, \ldots, r$. Hence, we have

$$G = \Phi,$$

from which,

$$\phi(s) = \langle\langle \Phi(s), \Phi''(s) \rangle\rangle = -\langle\langle \Phi'(s), \Phi'(s) \rangle\rangle. \tag{3.24}$$

Therefore, $\Delta G = f(G + C)$ yields

$$\frac{1}{2q^2}\frac{\partial q}{\partial s}\Phi' - \frac{1}{q}\Phi'' = f(\Phi + C). \tag{3.25}$$

Taking the inner product to (3.25) with Φ, we obtain

$$-\frac{1}{q}\phi(s) = f(1 + \gamma(s)). \tag{3.26}$$

On the other hand, the function $\phi(s)$ is reduced to

$$\phi(s) = -\sum_{a=r+1}^{m-1} u_a^2(s), \tag{3.27}$$

where $u_a(s) = \langle \alpha'(s), e_a'(s)\rangle$ and $e_{r+1}, e_{r+2}, \ldots, e_{m-1}$ are unit normal vector fields to M along α. Note that $\phi(s) = 0$ iff $\Phi'(s) = \mathbf{0}$.

Suppose $\phi(s) \equiv 0$ on I. Then, $\Phi'(s) \equiv \mathbf{0}$. This means the Gauss map G is a constant vector field. Hence, M is an open part of an $(r+1)$-plane $\mathbb{E}^{r+1}$.

We now suppose that the open subset $J = \{s \in I \mid \phi(s) \neq 0\}$ is not empty. We may put

$$f = -\frac{\phi(s)}{q(1+\gamma(s))}. \tag{3.28}$$

Putting $q = \left(1 + \sum_{j=1}^{r} u_j t_j\right)^2$ and substituting (3.28) into (3.25), we have

$$\left(\sum_{j=1}^{r} u_j' t_j\right)\Phi' - \left(1 + \sum_{j=1}^{r} u_j t_j\right)\Phi'' = -\frac{\phi}{1+\gamma}\left(1 + \sum_{j=1}^{r} u_j t_j\right)(\Phi + C). \tag{3.29}$$

Considering the constant terms with respect to t and coefficients of terms containing t_j in (3.29), we get

$$\Phi'' = \frac{\phi}{1+\gamma}(\Phi + C) \quad \text{and} \quad u_j'\Phi' = \mathbf{0} \tag{3.30}$$

for all $j = 1, 2, \ldots, r$.

Since $\Phi'(s) \neq \mathbf{0}$, $u_j'(s) = 0$ for all $j = 1, 2, \ldots, r$ and $s \in J$. Thus, u_j and w_{ij} are constant on J for all $i, j = 1, 2, \ldots, r$. Hence, we have

Lemma 3.4. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by* (3.2) *in $\mathbb{E}^m$. If M has pointwise 1-type Gauss map of the second kind, then the functions*

$$u_i(s) = \left\langle \alpha'(s), e_i'(s)\right\rangle \quad \textit{and} \quad w_{ij}(s) = \left\langle e_i'(s), e_j'(s)\right\rangle$$

are constant functions on the open interval $J = \{s \in I \mid \phi(s) \neq 0\}$ for all $i, j = 1, 2, \ldots, r$.

The following lemma tells us how to choose some suitable normal vector fields along the base curve α for non-cylindrical ruled submanifold of $\mathbb{E}^m$ with pointwise 1-type Gauss map of the second kind.

Lemma 3.5. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parametrized by* (3.2) *satisfying* (3.1) *in $\mathbb{E}^m$. If M has pointwise 1-type Gauss map of the second kind, we can choose orthonormal frame $\{e_a\}_{a=r+1}^{m-1}$ of the normal space $T_{\alpha(s)}N$ of M along $\alpha(s)$ satisfying*

$$e_a'(s) \wedge \alpha'(s) = \mathbf{0} \tag{3.31}$$

for all $a = r+1, \ldots, m-1$.

Proof. Let $\{\overline{e}_a\}_{a=r+1}^{m-1}$ denote an orthonormal frame of the normal space $T_{\alpha(s)}N$ of M along α. If we apply Lemma 2.2 to the normal space $T_{\alpha(s)}N$, then there exists an orthonormal frame $\{e_a\}_{a=r+1}^{m-1}$ of the normal space $T_{\alpha(s)}N$ satisfying

$$\left\langle e_a'(s), e_b(s)\right\rangle = 0 \tag{3.32}$$

for all $a, b = r+1, \dots, m-1$.

It follows from (3.23) that for all $i = 1, \dots, r$ and $a = r+1, \dots, m-1$,

$$\left\langle e_a'(s), e_i(s)\right\rangle = -\left\langle e_i'(s), e_a(s)\right\rangle = 0. \tag{3.33}$$

Together with (3.32), (3.33) completes the proof. □

We now give the following definition of a generalized right cone.

Definition 3.6. Suppose $\beta = \beta(s)$ is a circle on the unit sphere centered at the origin. Let $\boldsymbol{a}_2, \boldsymbol{a}_3, \dots, \boldsymbol{a}_r$ be orthonormal constant vectors satisfying $\langle \beta'(s), \boldsymbol{a}_i\rangle = \langle \beta(s), \boldsymbol{a}_i\rangle = 0$ for all $i = 2, 3, \dots, r$ and s. A ruled submanifold M parametrized by

$$x(s, t_1, t_2, \dots, t_r) = t_1\beta(s) + \sum_{i=2}^{r} t_i \boldsymbol{a}_i + D \tag{3.34}$$

is called a generalized right cone $C_a \times \mathbb{E}^{r-1}$, where C_a is a right cone in $\mathbb{E}^3$, D a constant vector and $t_i \in I_i$ for some open intervals I_i and $i = 2, 3, \dots, r$.

Remark 3.7. If $\beta = \beta(s)$ is a great circle on the unit sphere in Definition 3.6, a generalized right cone is an $(r+1)$-plane $\mathbb{E}^{r+1}$.

Theorem 3.8. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold in $\mathbb{E}^m$. Then, M has pointwise 1-type Gauss map of the second kind if and only if M is an open portion of a generalized right cone.*

Proof. Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold with pointwise 1-type Gauss map of the second kind.

Suppose $\phi \equiv 0$ on the whole domain I of α. In this case, we see that M is part of an $(r+1)$-plane $\mathbb{E}^{r+1}$. It is a special case of a generalized right cone.

Now, we suppose that M is not part of $(r+1)$-plane, that is, $J = \{s \in I \mid \phi(s) \neq 0\}$ is non-empty. Note that

$$\left\langle\left\langle \Phi', \Phi''\right\rangle\right\rangle = -\frac{\phi'}{2}, \quad \left\langle\left\langle \Phi', \Phi\right\rangle\right\rangle = 0 \quad \text{and} \quad \phi = -\sum_{a=r+1}^{m-1} u_a^2. \tag{3.35}$$

Then, the first equation in (3.30) implies

$$-\frac{\phi'}{2} = \frac{\phi}{1+\gamma}\left\langle\left\langle \Phi', C\right\rangle\right\rangle = \frac{\phi}{1+\gamma}\gamma',$$

or,

$$-\frac{1}{2}\frac{\phi'}{\phi} = \frac{(1+\gamma)'}{(1+\gamma)} \tag{3.36}$$

on J. Equation (3.36) yields

$$\frac{1}{\sqrt{|\phi|}} = \widetilde{\lambda}\,|1+\gamma|$$

for some positive constant $\widetilde{\lambda}$. Therefore, we get

$$\Phi'' = \lambda\sqrt{|\phi|^3}(\Phi + C)$$

for some non-zero constant λ.

On the other hand, according to Lemma 3.5, we have

$$\alpha'' = -\sum_{i=1}^{r} u_i e_i - \sum_{a=r+1}^{m-1} u_a e_a \quad \text{and} \quad e_a' = u_a \alpha' \tag{3.37}$$

for all $a = r+1, \ldots, m-1$. Since $\frac{1}{\lambda\sqrt{|\phi|^3}}\Phi'' - \Phi$ is a constant vector, we have

$$\begin{aligned}
\mathbf{0} = &\sum_{a=r+1}^{m-1} \left\{ \left(\frac{\mu}{\lambda\sqrt{|\phi|^3}} + 1 \right) u_a - \left(\frac{u_a'}{\lambda\sqrt{|\phi|^3}} \right)' \right\} e_a \wedge e_1 \wedge \cdots \wedge e_r \\
&+ \sum_{k=1}^{r} \sum_{a=r+1}^{m-1} u_k \left\{ \left(\frac{2}{\lambda\sqrt{|\phi|^3}} \right) u_a' + \left(\frac{1}{\lambda\sqrt{|\phi|^3}} \right)' u_a \right\} \\
&\times \alpha' \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_a \wedge e_{k+1} \wedge \cdots \wedge e_r.
\end{aligned} \tag{3.38}$$

Considering the orthogonality of the vectors in (3.38), we get

$$\left(\frac{2}{\lambda\sqrt{|\phi|^3}} \right) u_a' + \left(\frac{1}{\lambda\sqrt{|\phi|^3}} \right)' u_a = 0 \tag{3.39}$$

for all $a = r+1, \ldots, m-1$.

Since $\phi \neq 0$ on J, there exists $u_b \neq 0$ for some $b = r+1, \ldots, m-1$. Then, equation (3.39) implies

$$\frac{3}{4}\frac{|\phi|'}{|\phi|} = \frac{u_b'}{u_b}.$$

So we see that

$$|\phi|^{\frac{3}{4}} = \varepsilon_b u_b \quad \text{or,} \quad u_b^2 = \frac{1}{\varepsilon_b^2}\,|\phi|^{\frac{3}{2}}$$

for some non-zero real number ε_b. Together with (3.35), we can see

$$\phi = c\,|\phi|^{\frac{3}{2}}$$

for some negative constant c, which means that the function ϕ is constant and hence the function u_b is also constant. By continuity, the interval J is the whole domain I of α.

It follows from (3.39) that u_a are constant for all $a = r+1, \dots, m-1$. Thus, (3.37) implies

$$\alpha''' = -\mu\alpha'$$

for the constant $\mu = \sum u_i^2 + \sum u_a^2$. According to Lemma 3.1, α is a space curve in $\mathbb{E}^3$. By considering the Frenet formula for α, we see that the base curve α is a plane curve with non-zero constant curvature and thus it is part of a circle.

Since $e_i'(s) \neq 0$ for all $i = 1, 2, \dots, r$ as indicated in Section 3, we have

$$\alpha(s) = \frac{1}{u_1}(e_1(s) - \boldsymbol{a}_1) \quad \text{and} \quad e_i(s) = \frac{u_i}{u_1}e_1(s) + \boldsymbol{b}_i \tag{3.40}$$

for some constant vectors $\boldsymbol{a}_1$ and $\boldsymbol{b}_i$ for $i = 2, 3, \dots, r$ such that $e_1(s), \boldsymbol{b}_2, \boldsymbol{b}_3, \dots, \boldsymbol{b}_r$ are linearly independent for each s. By applying the Gram-Schmidt's method for orthogonalization, we have orthonormal constant vectors $\boldsymbol{a}_2, \dots, \boldsymbol{a}_r$ from $\boldsymbol{b}_2, \boldsymbol{b}_3, \dots, \boldsymbol{b}_r$. In this case, $\langle e_1(s), \boldsymbol{b}_i\rangle$ are constant and thus $\langle e_1(s), \boldsymbol{a}_i\rangle$ are constant for all $i = 2, \dots, r$.

We put $v_i = \langle e_1, \boldsymbol{a}_i\rangle$ for all $i = 2, \dots, r$. Define $\beta_1(s) = e_1(s) - \sum_{i=2}^r v_i\boldsymbol{a}_i$. Then the length $\|\beta_1(s)\| = \sqrt{1 - \sum_{i=2}^r v_i^2}$ of the vector field β_1 is a non-zero constant since $e_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_r$ are linearly independent. Take $\beta(s) = \frac{\beta_1(s)}{\|\beta_1(s)\|}$. After appropriate change of parameters $t_1, t_2, \dots, t_r$, the parametrization (3.2) for M can be reduced to

$$x(s, \bar{t}_1, \bar{t}_2, \dots, \bar{t}_r) = \bar{t}_1\beta(s) + \sum_{i=2}^r \bar{t}_i\boldsymbol{a}_i + D \tag{3.41}$$

for some constant vector D.

Since α is a circle, the first equation of (3.40) indicates that the trace of position vectors of $\beta(s)$ is a circle on the unit sphere. Thus, the parametrization given by (3.41) representing the ruled submanifold M is an open part of a generalized right cone.

Conversely, suppose that M is an open part of a generalized right cone $C_a \times \mathbb{E}^{r-1}$ parametrized by (3.34). In Introduction, we see that C_a has pointwise 1-type Gauss map of the second kind. According to Lemma 3.3, we see that M has pointwise 1-type Gauss map of the second kind. □

From above theorem, we immediately have

Corollary 3.9. [7] *Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$. Then, M has pointwise 1-type Gauss map of the second kind if and only if M is an open portion of a right cone C_a.*

Combining Theorem 3.7 of [14], Theorem 3.2 and Theorem 3.8, we have

Theorem 3.10 (Classification)**.** *An $(r+1)$-dimensional ruled submanifold M of Euclidean space $\mathbb{E}^m$ with pointwise* 1*-type Gauss map is an open part of one of an $(r+1)$-plane, a generalized circular cylinder $\Sigma_a \times \mathbb{E}^{r-1}$, a cylinder over a curve in $\mathbb{E}^3$ of infinite type, a generalized helicoid or a generalized right cone $C_a \times \mathbb{E}^{r-1}$.*

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