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Existence, Localization and Multiplicity of Positive Solutions to ϕ-Laplace Equations and Systems

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Abstract. The paper presents new existence, localization and multiplicity results for positive solutions of ordinary differential equations or systems of the form $(\phi(u'))' + f(t,u) = 0$, where $\phi : (-a,a) \to (-b,b)$, $0 < a, b \leq \infty$, is some homeomorphism such that $\phi(0) = 0$. Our approach is based on Krasnosel'skiĭ type compression-expansion arguments and on a weak Harnack type inequality for positive supersolutions of the operator $(\phi(u'))'$. In the case of the systems, the localization of solutions is obtained in a component-wise manner. The theory applies in particular to equations and systems with p-Laplacian, bounded or singular homeomorphisms.

1. Introduction

We present existence, localization and multiplicity results for positive solutions of the two-point boundary value problem

$$(1.1)\qquad \begin{cases} (\phi(u'))' + f(t,u) = 0, & 0 < t < 1 \\ u'(0) = u(1) = 0. \end{cases}$$

Important motivations for this study are the cases of the equations with p-Laplacian and curvature operators in Euclidian and Minkowski spaces, for which problem (1.1) respectively becomes

$$(1.2)\qquad \begin{cases} \left(|u'|^{p-2}\, u'\right)' + f(t,u) = 0, & 0 < t < 1 \\ u'(0) = u(1) = 0, \end{cases}$$

$$(1.3)\qquad \begin{cases} \left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' + f(t,u) = 0, & 0 < t < 1 \\ u'(0) = u(1) = 0, \end{cases}$$

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$$
\begin{cases}\left(\dfrac{u'}{\sqrt{1-u'^2}}\right)' + f(t,u) = 0, \quad 0 < t < 1 \\ u'(0) = u(1) = 0. \end{cases} \tag{1.4}
$$

The problem (1.1) can be considered as a particular, as $n = 1$, of the corresponding problem for an n-dimensional system,

$$
\begin{cases}(\phi_i(u_i'))' + f_i(t,u_1,u_2,\dots,u_n) = 0, \quad 0 < t < 1 \\ u_i'(0) = u_i(1) = 0 \quad (i = 1,2,\dots,n). \end{cases} \tag{1.5}
$$

First we shall concentrate on the problem (1.1) for a single equation, and then we shall extend the results to the general case (1.5) of systems in a component-wise manner.

Such type of equations involving the ϕ-Laplacian has been investigated in a large number of papers using fixed point methods, degree theory, upper and lower solution techniques and variational methods. We refer to the papers [1–6, 9–13, 18, 19], to the survey work [11], and the bibliographies therein.

Inspired by the three typical examples (1.2), (1.3), (1.4), in the literature, the cases of homeomorphisms of $\mathbb{R}$, $\phi\colon \mathbb{R} \to \mathbb{R}$; of homeomorphisms with bounded range, $\phi\colon \mathbb{R} \to (-b,b)$; and of homeomorphisms with bounded domain, $\phi\colon (-a,a) \to \mathbb{R}$, have been discussed separately. In this paper, these three cases will be treated unitarily by assuming that ϕ is a homeomorphism from $(-a,a)$ to $(-b,b)$, and $0 < a, b \leq \infty$.

We are interested not only on the existence of positive solutions to the problems (1.1) and (1.5), but also on their localization and multiplicity. We shall succeed this by using the technique based on Krasnosel'skiĭ's fixed point theorem in cones [8].

Theorem 1.1 (Krasnosel'skiĭ)**.** *Let $(X, |\cdot|)$ be a normed linear space; $K \subset X$ a cone; $r, R \in \mathbb{R}_+$, $0 < r < R$; $K_{r,R} = \{u \in K : r \leq |u| \leq R\}$, and let $N\colon K_{r,R} \to K$ be a compact map. Assume that one of the following conditions is satisfied:*

(a) *$N(u) \not< u$ if $|u| = r$, and $N(u) \not> u$ if $|u| = R$.*

(b) *$N(u) \not> u$ if $|u| = r$, and $N(u) \not< u$ if $|u| = R$.*

Then N has a fixed point u in K with $r \leq |u| \leq R$.

Here for two elements $u, v \in X$, the strict ordering $u < v$ means $v - u \in K \setminus \{0\}$.

It is a well known fact that in applications, the technique based on Krasnosel'skiĭ's theorem requires the construction of a suitable cone of positive functions, which, in the case of most boundary value problems is done using the properties of the corresponding Green functions. Alternatively, and for many other problems for which Green functions are not known, one can use weak Harnack type inequalities associated to the differential operators and the boundary conditions, as shown in [16] and [17]. In our case, such an inequality will arise as a consequence of the concavity of the positive solutions.

As concerns the systems, we shall allow the homeomorphisms ϕ_i have different domains and ranges and we shall be interested to localize each component of a solution $u = (u_1, u_2, \ldots, u_n)$. In this respect we shall use the following vectorial version of Krasnosel'skiĭ's theorem given in [14] (see also [15]).

Theorem 1.2. [14] *Let $(X, |\cdot|)$ be a normed linear space; $K_1, K_2, \ldots, K_n \subset X$ cones; $K := K_1 \times K_2 \times \cdots \times K_n$; $r, R \in \mathbb{R}_+^n$, $r = (r_1, r_2, \ldots, r_n)$, $R = (R_1, R_2, \ldots, R_n)$ with $0 < r_i < R_i$ for all i; $K_{r,R} = \{u \in K : r_i \leq |u_i| \leq R_i, i = 1, 2, \ldots, n\}$, and let $N \colon K_{r,R} \to K$, $N = (N_1, N_2, \ldots, N_n)$ be a compact map. Assume that for each $i \in \{1, 2, \ldots, n\}$, one of the following conditions is satisfied in $K_{r,R}$:*

(a) *$N_i(u) \not< u_i$ if $|u_i| = r_i$, and $N_i(u) \not> u_i$ if $|u_i| = R_i$.*

(b) *$N_i(u) \not> u_i$ if $|u_i| = r_i$, and $N_i(u) \not< u_i$ if $|u_i| = R_i$.*

Then N has a fixed point u in K with $r_i \leq |u_i| \leq R_i$ for $i = 1, 2, \ldots, n$.

Note that in the previous theorem, the same symbol $<$ is used to denote the strict ordering induced by any of the cones $K_1, K_2, \ldots, K_n$.

It deserves to be underlined the fact that asking the compression condition (a) to be satisfied by some indices i, and the expansion condition (b) by the others, it is allowed that the system nonlinearities behave differently one from another.

2. Positive solutions of ϕ-Laplace equations

This section deals with positive solutions for the problem (1.1). We make the following assumptions: $\phi \colon (-a, a) \to (-b, b)$, $0 < a, b \leq \infty$ is an increasing homeomorphism such that $\phi(0) = 0$ and $f \colon [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function.

By a positive solution of the problem (1.1) we mean a function $u \in C^1[0, 1] \cap C([0, 1]; \mathbb{R}_+)$, with $u'(0) = u(1) = 0$, such that $u'(t) \in (-a, a)$ for every $t \in [0, 1]$, $\phi \circ u'$ is continuously differentiable on $[0, 1]$ and the equation in (1.1) is satisfied on $[0, 1]$.

First we obtain the equivalent integral equation to the problem (1.1). Integration of the differential equation from (1.1) and the conditions $u'(0) = 0$ and $\phi(0) = 0$ give

$$-\phi(u'(t)) = \int_0^t f(s, u(s))\, ds.$$

Then

$$u'(t) = \phi^{-1}\left(-\int_0^t f(s, u(s))\, ds\right).$$

Integrating from t to 1 and taking into account that $u(1) = 0$, we obtain

$$u(t) = -\int_t^1 \phi^{-1}\left(-\int_0^\tau f(s,u(s))\,ds\right)d\tau. \tag{2.1}$$

Conversely, if a function $u \in C([0,1];\mathbb{R}_+)$ satisfies (2.1), which implicitly means that $\int_0^\tau f(s,u(s))\,ds < b$ for all $\tau \in [0,1]$, then u is a positive solution of the problem (1.1).

Next, assuming in addition that $f(t,x) < b$ for all $t \in [0,1]$ and $x \in \mathbb{R}_+$, we may define the integral operator $N\colon C([0,1];\mathbb{R}_+) \to C([0,1];\mathbb{R}_+)$ by

$$N(u)(t) = -\int_t^1 \phi^{-1}\left(-\int_0^\tau f(s,u(s))\,ds\right)d\tau, \tag{2.2}$$

and thus, finding positive solutions to (1.1) is equivalent to the fixed point problem for the operator N on $C([0,1];\mathbb{R}_+)$. Note that by standard arguments based on Ascoli-Arzela's theorem, N is completely continuous. Let $|\cdot|_\infty$ denote the max norm on $C[0,1]$.

In order to apply Krasnosel'skiĭ's fixed point theorem in cones we need a weak Harnack type inequality for the differential operator $Lu := -(\phi(u'))'$ and the boundary conditions $u'(0) = u(1) = 0$.

Lemma 2.1. *For each $c \in (0,1)$, and any $u \in C^1[0,1] \cap C([0,1];\mathbb{R}_+)$ with $u'(0) = u(1) = 0$, $u'(t) \in (-a,a)$ for every $t \in [0,1]$, $\phi \circ u' \in W^{1,1}(0,1)$ and $(\phi(u'))' \le 0$ on $[0,1]$, one has*

$$u(t) \ge (1-c)\,|u|_\infty\,, \quad \textit{for all } t \in [0,c]. \tag{2.3}$$

Proof. From $(\phi(u))' \le 0$ on $[0,1]$, one has that the function $\phi\circ u'$ is nonincreasing on $[0,1]$. Then, from $u' = \phi^{-1}(\phi \circ u')$, and ϕ^{-1} increasing, we deduce that u' is nonincreasing on $[0,1]$. Thus u is concave on $[0,1]$. On the other hand, since the function $\phi \circ u'$ vanishes at $t = 0$, $\phi(u'(t)) \le 0$ for every $t \in [0,1]$. Then $u' \le 0$ on $[0,1]$, which shows that u is nonincreasing on $[0,1]$. Hence u is nonnegative, nonincreasing, concave and $|u|_\infty = u(0)$. If $\min_{t\in[0,c]} u(t) = 0$, then the concavity of u implies $u = 0$ on $[0,1]$, and so (2.3) holds. If $\min_{t\in[0,c]} u(t) > 0$, then we may assume without loss of generality that $\min_{t\in[0,c]} u(t) = 1$ (otherwise, multiply (2.3) by a suitable positive constant). Then $u(c) = 1$. The function u being concave, its graph on $[0,c]$ is under the line containing the points $(1,0)$ and $(c,1)$ and so we have $u(0) \le \frac{1}{1-c}$. Hence $(1-c)\,|u|_\infty \le 1$. Finally, since $1 \le u(t)$ for $t \in [0,c]$, we obtain (2.3). □

For our first result we make the following assumptions:

(A1) $\phi\colon (-a,a) \to (-b,b)$, $0 < a,b \le \infty$ is an increasing homeomorphism such that $\phi(0) = 0$;

(A2) $f\colon [0,1]\times\mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $f(t,\cdot)$ is nondecreasing on $\mathbb{R}_+$ for each $t \in [0,1]$, and $f(t,x) < b$ for all $t \in [0,1]$ and $x \in \mathbb{R}_+$.

Theorem 2.2. *Let the conditions* (A1) *and* (A2) *hold and assume that there exist* $c, \alpha, \beta > 0$ *with* $c < 1$ *and* $\alpha \neq \beta$ *such that*

$$\Phi(\alpha) := -\int_0^c \phi^{-1}\left(-\int_0^\tau f(s,(1-c)\alpha)\,ds\right)d\tau \geq \alpha, \tag{2.4}$$

$$\Psi(\beta) := -\int_0^1 \phi^{-1}\left(-\int_0^\tau f(s,\beta)\,ds\right)d\tau < \beta. \tag{2.5}$$

Then (1.1) *has at least one positive solution* u *with* $r \leq |u|_\infty \leq R$*, where* $r = \min\{\alpha,\beta\}$, $R = \max\{\alpha,\beta\}$.

Proof. We shall apply Krasnosel'skiĭ's fixed point theorem in cones. In our case, $X = C[0,1]$, the cone K is the following one

$$K = \{u \in C([0,1];\mathbb{R}_+) : u(t) \geq (1-c)\,|u|_\infty \text{ for all } t \in [0,c]\},$$

and N is the operator given by (2.2). Notice that if $u, v \in C([0,1];\mathbb{R}_+)$ and $v < u$, that is $u - v \in K \setminus \{0\}$, then $(u-v)(0) \geq (1-c)\,|u-v|_\infty > 0$. Hence

$$|u|_\infty \geq u(0) > v(0). \tag{2.6}$$

First we remark that $N(K) \subset K$. Indeed, if $u \in K$ and $v := N(u)$, then $-(\phi(v'))' = f(t,u)$. We have $f(t,u(t)) \geq 0$ for every $t \in [0,1]$, so $(\phi(v'))' \leq 0$ on $[0,1]$. Then Lemma 2.1 guarantees that $v(t) \geq (1-c)\,|v|_\infty$ for $t \in [0,c]$, that is $v \in K$ as desired. Next we prove that

$$u \not> N(u) \text{ for every } u \in K \text{ with } |u|_\infty = \alpha. \tag{2.7}$$

To this end, assume the contrary, i.e., $u > N(u)$ for some $u \in K$ with $|u|_\infty = \alpha$. Then using (2.6), the definition of K, and the monotonicity of f and ϕ, we deduce

$$\begin{aligned}\alpha = |u|_\infty > N(u)(0) &= -\int_0^1 \phi^{-1}\left(-\int_0^\tau f(s,u(s))\,ds\right)d\tau \\ &\geq -\int_0^c \phi^{-1}\left(-\int_0^\tau f(s,u(s))\,ds\right)d\tau \geq -\int_0^c \phi^{-1}\left(-\int_0^\tau f(s,(1-c)\alpha)\,ds\right)d\tau,\end{aligned}$$

which contradicts (2.4). Thus (2.7) holds. The next step is to prove that

$$u \not< N(u) \text{ for every } u \in K \text{ with } |u|_\infty = \beta. \tag{2.8}$$

Assume the contrary, i.e., $u < N(u)$ for some $u \in K$ with $|u|_\infty = \beta$. Then we would obtain

$$\begin{aligned}\beta = |u|_\infty \leq |N(u)|_\infty = N(u)(0) &= -\int_0^1 \phi^{-1}\left(-\int_0^\tau f(s,u(s))\,ds\right)d\tau \\ &\leq -\int_0^1 \phi^{-1}\left(-\int_0^\tau f(s,\beta)\,ds\right)d\tau,\end{aligned}$$

which contradicts (2.5). Thus (2.8) holds. Therefore, Krasnosel'skiĭ's theorem applies and yields the result. □

Remark 2.3. The existence and localization result, Theorem 2.2, immediately yields multiplicity results for the problem (1.1), in case that several (finitely many or infinitely many) couples of distinct numbers α, β satisfying (2.4), (2.5) exist such any two of the corresponding intervals (α, β) are disjoint.

Remark 2.4. (a) If we do not assume that $f(t,\cdot)$ is nondecreasing on $\mathbb{R}_+$ for each $t \in [0,1]$, then the conclusion of Theorem 2.2 remains true under the following conditions replacing (2.4) and (2.5):

$$-\int_0^c \phi^{-1}\left(-\int_0^\tau \min_{x\in[(1-c)\alpha,\alpha]} f(s,x)\,ds\right) d\tau \geq \alpha,$$
$$-\int_0^1 \phi^{-1}\left(-\int_0^\tau \max_{x\in[0,\beta]} f(s,x)\,ds\right) d\tau < \beta.$$

(b) The conclusion of Theorem 2.2 remains true if we replace the hypothesis that $f(t,x) < b$ for all $t \in [0,1]$ and $x \in \mathbb{R}_+$, by the assumption

$$f(t,x) < b \text{ for all } t \in [0,1] \text{ and } x \in [0,R]. \tag{2.9}$$

Indeed, the operator N does not need to be defined on the whole cone K. It suffices to be defined on the conical annulus $K_{r,R}$ and for this the condition (2.9) is enough.

The next theorems answer the question how can the numbers α, β satisfying the conditions (2.4), (2.5) be guaranteed. The first result is about the existence of at least one pair of such numbers.

Theorem 2.5. *Let* (A1) *and* (A2) *hold and assume that one of the following conditions is satisfied:*

(i) $\limsup_{\lambda\to\infty} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \dfrac{\Psi(\lambda)}{\lambda} < 1$;

(ii) $\limsup_{\lambda\to 0} \dfrac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \dfrac{\Psi(\lambda)}{\lambda} < 1$.

Then (1.1) *has at least one positive solution.*

Proof. In order to apply Theorem 2.2, we look for two numbers α, $\beta > 0$, $\alpha \neq \beta$ with

$$\Phi(\alpha) \geq \alpha \quad \text{and} \quad \Psi(\beta) < \beta.$$

In case (i), one can chose α large enough and β small enough; while in case (ii), α is chosen small enough and β is chosen large enough. □

The next result is about a sequence of positive solutions of the problem (1.1), whose existence is guaranteed by the oscillations of f towards infinity or zero.

Theorem 2.6. *Let* (A1) *and* (A2) *hold. If the condition*

(iii) $\limsup_{\lambda\to\infty} \frac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to\infty} \frac{\Psi(\lambda)}{\lambda} < 1$ *holds, then* (1.1) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $|u_n|_\infty \to \infty$ *as* $n\to\infty$.

If the condition

(iv) $\limsup_{\lambda\to 0} \frac{\Phi(\lambda)}{\lambda} > 1$ *and* $\liminf_{\lambda\to 0} \frac{\Psi(\lambda)}{\lambda} < 1$ *holds, then* (1.1) *has a sequence of positive solutions* $(u_n)_{n\geq 1}$ *such that* $|u_n|_\infty \to 0$ *as* $n\to\infty$.

Proof. Clearly (iii) guarantees the existence of two sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ such that

$$0<\alpha_n<\beta_n<\alpha_{n+1}<\beta_{n+1} \quad \text{for every } n\geq 1, \text{ and } \alpha_n\to\infty \text{ as } n\to\infty. \tag{2.10}$$

For each n, Theorem 2.2 yields a positive solution u_n with $\alpha_n \leq |u_n|_\infty \leq \beta_n$. The condition (2.10) implies that these solutions are distinct and that $|u_n|_\infty \to\infty$ as $n\to\infty$. A similar reasoning can be done in case (iv). □

Remark 2.7. Since $\Phi(\lambda)\in[0,a)$ for every $\lambda>0$, the conditions (i) from Theorem 2.5 and (iii) from Theorem 2.6 can not occur if $a<\infty$.

For the remaining part of this section, we shall take into consideration some particular cases, including the problems (1.2), (1.3) and (1.4).

First note that if ϕ is odd, then the conditions (2.4), (2.5) become

$$\int_0^c \phi^{-1}\left(\int_0^\tau f(s,(1-c)\alpha)\,ds\right) d\tau \geq \alpha,$$
$$\int_0^1 \phi^{-1}\left(\int_0^\tau f(s,\beta)\,ds\right) d\tau < \beta.$$

In particular, for $\phi(t)=t$, which was the case considered in [7], these conditions reduce to

$$\int_0^c\int_0^\tau f(s,(1-c)\alpha)\,ds\,d\tau \geq \alpha, \qquad \int_0^1\int_0^\tau f(s,\beta)\,ds\,d\tau < \beta.$$

If in addition $f(t,x)$ does not depend on t, they read as

$$\frac{f((1-c)\alpha)}{\alpha} \geq \frac{2}{c^2}, \qquad \frac{f(\beta)}{\beta} < 2.$$

Theorem 2.2 gives the following results for the problems (1.2), (1.3) and (1.4):

Corollary 2.8. *Let $\phi(\lambda) = |\lambda|^{p-2}\lambda$ for $\lambda \in \mathbb{R}$, where $p > 1$, and let* (A2) *holds with $b = \infty$. If there exist $c, \alpha, \beta > 0$ with $c < 1$ and $\alpha \neq \beta$ such that*

$$\int_0^c \left(\int_0^\tau f(s,(1-c)\alpha)\,ds\right)^{\frac{1}{p-1}} d\tau \geq \alpha,$$
$$\int_0^1 \left(\int_0^\tau f(s,\beta)\,ds\right)^{\frac{1}{p-1}} d\tau < \beta,$$

then (1.2) *has at least one positive solution u with $r \leq |u|_\infty \leq R$, where $r = \min\{\alpha,\beta\}$, $R = \max\{\alpha,\beta\}$.*

Proof. In this case $\phi^{-1}(\lambda) = |\lambda|^{1/(p-1)} \operatorname{sign} \lambda$ for $\lambda \in \mathbb{R}$. □

Corollary 2.9. *Let $\phi(\lambda) = \lambda/\sqrt{1+\lambda^2}$ for $\lambda \in \mathbb{R}$, and let* (A2) *hold with $b = 1$. If there exist $c, \alpha, \beta > 0$ with $c < 1$ and $\alpha \neq \beta$ such that*

$$\int_0^c \left(\frac{\int_0^\tau f(s,(1-c)\alpha)\,ds}{\sqrt{1-\left(\int_0^\tau f(s,(1-c)\alpha)\,ds\right)^2}}\right) d\tau \geq \alpha,$$
$$\int_0^1 \left(\frac{\int_0^\tau f(s,\beta)\,ds}{\sqrt{1-\left(\int_0^\tau f(s,\beta)\,ds\right)^2}}\right) d\tau < \beta,$$

then (1.3) *has at least one positive solution u with $r \leq |u|_\infty \leq R$, where $r = \min\{\alpha,\beta\}$, $R = \max\{\alpha,\beta\}$.*

Proof. Here $\phi^{-1}(\lambda) = \lambda/\sqrt{1-\lambda^2}$ for $\lambda \in (-1,1)$. □

Corollary 2.10. *Let $\phi(\lambda) = \lambda/\sqrt{1-\lambda^2}$ for $\lambda \in (-1,1)$, and let* (A2) *hold with $b = \infty$. If there exist $c, \alpha, \beta > 0$ with $c < 1$ and $\alpha \neq \beta$ such that*

$$\int_0^c \left(\frac{\int_0^\tau f(s,(1-c)\alpha)\,ds}{\sqrt{1+\left(\int_0^\tau f(s,(1-c)\alpha)\,ds\right)^2}}\right) d\tau \geq \alpha,$$
$$\int_0^1 \left(\frac{\int_0^\tau f(s,\beta)\,ds}{\sqrt{1+\left(\int_0^\tau f(s,\beta)\,ds\right)^2}}\right) d\tau < \beta,$$

then (1.4) *has at least one positive solution u with $r \leq |u|_\infty \leq R$, where $r = \min\{\alpha,\beta\}$, $R = \max\{\alpha,\beta\}$.*

Proof. This time $\phi^{-1}(\lambda) = \lambda/\sqrt{1+\lambda^2}$ for $\lambda \in \mathbb{R}$. □

As in the general case, one can discuss the existence of the numbers α, β with the required properties, and the multiplicity of solutions, for each one of the problems (1.2), (1.3) and (1.4), by taking into consideration the asymptotic behavior of the nonlinearity towards infinity and zero.

We conclude this section by two examples illustrating Corollaries 2.9 and 2.10.

Example 2.11. Consider the problem (1.3) where

$$f\colon [0,1]\times\mathbb{R}_+\to\mathbb{R}_+,\quad f(t,x)=\frac{\gamma x}{x+\delta} \tag{2.11}$$

and $\gamma,\delta>0$. In this case $a=\infty$, $b=1$ and one can easily check that the condition (A2), particularly, the inequality $f(t,x)<1$, holds if and only if $\gamma\le 1$. Direct computation shows that

$$\Phi(\alpha)=\frac{1-\sqrt{1-A^2c^2}}{A},\quad \Psi(\beta)=\frac{1-\sqrt{1-B^2}}{B},$$

where

$$A=\frac{\gamma(1-c)\alpha}{(1-c)\alpha+\delta},\quad B=\frac{\gamma\beta}{\beta+\delta}. \tag{2.12}$$

Now it is easy to see that

$$\lim_{\lambda\to 0}\frac{\Phi(\lambda)}{\lambda}=\frac{\gamma(1-c)c^2}{2\delta}\quad\text{and}\quad \lim_{\lambda\to\infty}\frac{\Psi(\lambda)}{\lambda}=0. \tag{2.13}$$

Hence the condition (ii) from Theorem 2.5 is satisfied if $\frac{\gamma(1-c)c^2}{2\delta}>1$. Thus, if $\delta<\gamma\frac{(1-c)c^2}{2}$ and $\gamma\le 1$, then the problem (1.3) has at least one positive solution.

Example 2.12. Consider the problem (1.4) for the same function (2.11). In this case $a=1$, $b=\infty$ and the condition (A2) holds for any $\gamma,\delta>0$. We have

$$\Phi(\alpha)=\frac{\sqrt{1+A^2c^2}-1}{A},\quad \Psi(\beta)=\frac{\sqrt{1+B^2}-1}{B},$$

where A,B are given by (2.12), and the limits (2.13) also hold. Thus, if $\delta<\gamma\frac{(1-c)c^2}{2}$ and $\gamma>0$, then the problem (1.4) has at least one positive solution.

3. Positive solutions of ϕ-Laplace systems

In this section we extend the above results to the general case (1.5). We shall allow the homeomorphisms ϕ_i have different domains and ranges, namely $\phi_i\colon(-a_i,a_i)\to(-b_i,b_i)$, $0<a_i,b_i\le\infty$, and we shall assume that ϕ_i are increasing with $\phi_i(0)=0$, and that $f_i\colon[0,1]\times\mathbb{R}_+^n\to\mathbb{R}_+$ are continuous functions ($i=1,2,\dots,n$. Under these assumptions problem (1.5) is equivalent to the integral system

$$u_i(t)=-\int_t^1\phi_i^{-1}\left[-\int_0^\tau f_i(s,u(s))\,ds\right]d\tau,\quad i=1,2,\dots,n,$$

where $u=(u_1,u_2,\dots,u_n)$.

According to Lemma 2.1, for each i and any constant $c_i\in(0,1)$, a weak Harnack type inequality holds for the differential operator $L_i v := -(\phi_i(v'))'$ and the boundary conditions $v'(0)=v(1)=0$. Based on this we define the cones

$$K_i=\{u_i\in C([0,1];\mathbb{R}_+) : u_i(t)\geq(1-c_i)\left|u_i\right|_\infty \text{ for all } t\in[0,c_i]\}\,,\tag{3.1}$$

for $i=1,2,\dots,n$, and take the product cone

$$K:=K_1\times K_2\times\cdots\times K_n$$

in $C([0,1],\mathbb{R}^n)$.

Let $N\colon C([0,1];\mathbb{R}^n_+)\to C([0,1];\mathbb{R}^n_+)$, $N=(N_1,N_2,\dots,N_n)$ be defined by

$$N_i(u)(t)=-\int_t^1\phi_i^{-1}\left[-\int_0^\tau f_i(s,u(s))\,ds\right]d\tau\quad(i=1,2,\dots,n).$$

If $u_j\in K_j$ for each j, then $f_i(s,u(s))\geq 0$ and from Lemma 2.1, one has $N_i(u)\in K_i$. Thus the cone K is invariant by N. Moreover, the operator N is completely continuous since, by standard arguments, the components N_i are completely continuous.

The following result is a generalization of Theorem 2.2 and guarantees the existence of positive solutions to the problem (1.5) and their component-wise localization. For any index $i\in\{1,2,\dots,n\}$, we shall say that the homeomorphism $\phi_i\colon(-a_i,a_i)\to(-b_i,b_i)$ satisfies (A1) if ϕ_i is increasing and $\phi_i(0)=0$, and that the continuous function $f_i\colon[0,1]\times\mathbb{R}^n_+\to\mathbb{R}_+$ satisfies (A2) if for each $t\in[0,1]$, $f_i(t,x_1,\dots,x_n)$ is nondecreasing on $\mathbb{R}_+$ with respect to any variable x_j, $j=1,2,\dots,n$, and $f_i(t,x)<b_i$ for all $t\in[0,1]$ and $x\in\mathbb{R}^n_+$.

Theorem 3.1. *Let ϕ_i, f_i satisfy* (A1) *and* (A2) *for $i=1,2,\dots,n$. Assume that there exist $c_i,\alpha_i,\beta_i>0$ with $c_i<1$ and $\alpha_i\neq\beta_i$ such that*

$$\Phi_i(\alpha):=-\int_0^{c_i}\phi_i^{-1}\left(-\int_0^\tau f_i(s,(1-c_1)\alpha_1,\dots,(1-c_n)\alpha_n)\,ds\right)d\tau\geq\alpha_i,\tag{3.2}$$

$$\Psi_i(\beta):=-\int_0^1\phi_i^{-1}\left(-\int_0^\tau f_i(s,\beta)\,ds\right)d\tau<\beta_i,\tag{3.3}$$

for $i=1,2,\dots,n$, where $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\dots,\beta_n)$. Then (1.5) *has at least one positive solution $u=(u_1,u_2,\dots,u_n)$ with $r_i\leq\left|u_i\right|_\infty\leq R_i$, where $r_i=\min\{\alpha_i,\beta_i\}$, $R_i=\max\{\alpha_i,\beta_i\}$, $i=1,2,\dots,n$.*

Proof. The result is a consequence of the vectorial version of Krasnosel'skiĭ's fixed point theorem in cones. □

We conclude this paper by the following generalization of Theorem 2.6, on the existence of a sequence of positive solutions to the problem (1.5). We shall say that for a given index i, the condition (i) holds if for every $\lambda_1, \lambda_2, \ldots, \lambda_{i-1} > 0$,

$$\limsup_{\lambda_i \to \infty} \frac{\Phi_i(\lambda)}{\lambda_i} > 1 \quad \text{and} \quad \liminf_{\lambda_i \to 0} \frac{\Psi_i(\lambda)}{\lambda_i} < 1,$$

uniformly with respect to $\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_n \in (0, \infty)$. We shall understand the condition (ii) in a similar manner. Analogously, we say that (iii) holds for some index i, if for every $\lambda_1, \lambda_2, \ldots, \lambda_{i-1} > 0$,

$$\limsup_{\lambda_i \to \infty} \frac{\Phi_i(\lambda)}{\lambda_i} > 1 \quad \text{and} \quad \liminf_{\lambda_i \to \infty} \frac{\Psi_i(\lambda)}{\lambda_i} < 1,$$

uniformly with respect to $\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_n \in (0, \infty)$. The condition (iv) is understood in a similar manner.

Theorem 3.2. *Let ϕ_i, f_i satisfy* (A1) *and* (A2) *for every $i = 1, 2, \ldots, n$. Assume that the set of indices $I = \{1, 2, \ldots, n\}$ admits the partition $I = I_1 \cup I_2 \cup I_3 \cup I_4$, $I_j \cap I_k = \emptyset$ for $j \neq k$, such that condition* (i) *holds for every $i \in I_1$, condition* (ii) *holds for every $i \in I_2$, condition* (iii) *holds for every $i \in I_3$, and condition* (iv) *holds for every $i \in I_4$. If $I_3 \neq \emptyset$ or $I_4 \neq \emptyset$, then the problem* (1.5) *has a sequence of positive solutions.*

Proof. We apply Theorem 3.1. To this aim, the pairs of positive numbers α_i, β_i are successively obtained for $i = 1, 2, \ldots, n$. At step i, in case that $i \in I_1 \cup I_2$, such a pair of numbers is obtained as explained in the proof of Theorem 2.5; in case that $i \in I_3 \cup I_4$, as shown in the proof of Theorem 2.6, an entire sequence of distinct pairs of such numbers can be obtained, which finally guarantees the existence of a sequence of positive solutions for the problem (1.5). □

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