TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 1, pp. 1–11, February 2016
DOI: 10.11650/tjm.20.2016.5062
This paper is available online at http://journal.tms.org.tw

b-coloring of Cartesian Product of Trees

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Abstract. A b-coloring of a graph G with k colors is a proper coloring of G using k colors in which each color class contains a color dominating vertex, that is, a vertex which has a neighbor in each of the other color classes. The largest positive integer k for which G has a b-coloring using k colors is the b-chromatic number $b(G)$ of G. The b-spectrum $S_b(G)$ of a graph G is the set of positive integers k, $\chi(G) \leq k \leq b(G)$, for which G has a b-coloring using k colors. A graph G is b-continuous if $S_b(G) = \{\chi(G), \ldots, b(G)\}$. It is known that for any two graphs G and H, $b(G \square H) \geq \max\{b(G), b(H)\}$, where $\square$ stands for the Cartesian product. In this paper, we determine some families of graphs G and H for which $b(G \square H) \geq b(G) + b(H) - 1$. Further if T_i, $i = 1, 2, \ldots, n$, are trees with $b(T_i) \geq 3$, then $b(T_1 \square \cdots \square T_n) \geq \sum_{i=1}^{n} b(T_i) - (n-1)$ and $S_b(T_1 \square \cdots \square T_n) \supseteq \{2, \ldots, \sum_{i=1}^{n} b(T_i) - (n-1)\}$. Also if $b(T_i) = \Delta(T_i) + 1$ for each i, then $b(T_1 \square \cdots \square T_n) = \Delta(T_1 \square \cdots \square T_n) + 1$, and $T_1 \square \cdots \square T_n$ is b-continuous.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. A b-coloring of a graph G is a proper coloring of G in which each color class has a color dominating vertex (c.d.v.), that is, a vertex that has a neighbor in each of the other color classes. The b-chromatic number $b(G)$ of G is the largest k such that G has a b-coloring using k colors. For a given b-coloring of a graph, a set of c.d.v.'s, one from each class, is known as a color dominating system (c.d.s.) of that b-coloring. A k-stable dominating system denotes a b-coloring using k colors containing a color dominating system which is independent. Recently, there has been an increasing interest in the study of b-coloring. See, for instance, [7,10–15]. The concept of b-coloring was introduced by Irving and Manlove [9] in analogy to the achromatic number of a graph G (which gives the maximum number of color classes in a complete coloring of G [8]). They have shown that the determination of $b(G)$ is NP-hard for general graphs, but polynomial for trees. From the very definition of $b(G)$, the chromatic number $\chi(G)$ of G is the least k for which G admits a b-coloring using k colors. Thus $\chi(G) \leq b(G) \leq 1 + \Delta(G)$, where $\Delta(G)$ is the maximum degree of G.

Received July 17, 2014, accepted July 29, 2015.
Communicated by Xuding Zhu.
2010 *Mathematics Subject Classification*. 05C15.
Key words and phrases. b-Chromatic number, b-Continuity, b-Spectrum, Cartesian product, Trees.
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While considering the hypercube Q_3, it is easy to note that Q_3 has a b-coloring using 2 colors and 4 colors but none with 3 colors. Thus a statement similar to the interpolation theorem for complete coloring [8] is not true for b-coloring. Graphs G for which there exists a b-coloring using k colors for every $k \in \{\chi(G), \ldots, b(G)\}$ are known as b-continuous graphs. From the time of its introduction, there had been several papers on b-continuity of graphs [4–6]. Some of the known families of graphs which are b-continuous are chordal graphs (which include trees), cographs and P_4-sparse graphs [4, 5]. The b-spectrum of a graph G, denoted by $S_b(G)$, is defined by:

$$S_b(G) = \{k : G \text{ has a } b\text{-coloring using } k \text{ colors}\}.$$

Clearly $S_b(G) \subseteq \{\chi(G), \ldots, b(G)\}$ and G is b-continuous iff $S_b(G) = \{\chi(G), \ldots, b(G)\}$.

The Cartesian product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, denoted by $G \Box H$, has vertex set $V_1 \times V_2$, and two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G \Box H$ iff either $x_1 = x_2$ and y_1 is adjacent to y_2 in H, or $y_1 = y_2$ and x_1 is adjacent to x_2 in G.

This paper deals with the b-chromatic number of Cartesian products of graphs. The study of the b-chromatic number of Cartesian product of graphs was initiated by Kouider and Mahéo in [13] wherein they have proved the following results.

Theorem 1.1. (M. Kouider and M. Mahéo [13]) *For any two graphs G and H, $b(G \Box H) \geq \max\{b(G), b(H)\}$.*

Theorem 1.2. (M. Kouider and M. Mahéo [13]) *Let G and H be two graphs such that G has a $b(G)$-stable dominating system, and H has a $b(H)$-stable dominating system. Then $b(G \Box H) \geq b(G) + b(H) - 1$, and the graph $G \Box H$ has a $(b(G) + b(H) - 1)$-stable dominating system.*

The above result can be generalized as follows (with the same proof).

Observation 1.3. Let G and H be two graphs such that G has a k-stable dominating system, and H has an ℓ-stable dominating system. Then $G \Box H$ has a $(k + \ell - 1)$-stable dominating system.

One of the main problems concerning b-colorings is to completely characterize those graphs G and H for which $b(G \Box H) = \max\{b(G), b(H)\}$. Equivalently, one has to characterize those graphs G and H for which $b(G \Box H) > \max\{b(G), b(H)\}$. Theorem 1.2 gives one such family. In [1, 2], we found a few more classes of graphs G and H for which $b(G \Box H) \geq b(G) + b(H) - 1$. These include odd graphs. In particular, we have proved for odd graphs O_{k_i}, $1 \leq i \leq n$ and $k_i \geq 4$ for each i, $O_{k_1} \Box O_{k_2} \Box \cdots \Box O_{k_n}$ is b-continuous and $b(O_{k_1} \Box O_{k_2} \Box \cdots \Box O_{k_n}) = 1 + \sum_{i=1}^{n} k_i$.

In this paper, we prove that if T_i is a tree with $b(T_i) \geq 3$, for $1 \leq i \leq n$, then $b(T_1 \Box \cdots \Box T_n) \geq \sum_{i=1}^{n} b(T_i) - (n-1)$ and $S_b(T_1 \Box \cdots \Box T_n) \supseteq \{2, \ldots, \sum_{i=1}^{n} b(T_i) - (n-1)\}$.

Also if $b(T_i) = \Delta(T_i) + 1$ for each i, then $b(T_1\Box\cdots\Box T_n) = \Delta(T_1\Box\cdots\Box T_n) + 1$, and $T_1\Box\cdots\Box T_n$ is b-continuous.

2. b-coloring of Cartesian product of trees

We start with the following observation from [2].

Observation 2.1. (i) If G has a b-coloring using k colors and H has a b-coloring using ℓ colors with $k \leq \ell$, then $G\Box H$ has a b-coloring using ℓ colors (and hence $b(G\Box H) \geq \ell$).

(ii) If G and H are b-continuous graphs, then

$$S_b(G\Box H) \supseteq \{\chi(G\Box H) = \max\{\chi(G), \chi(H)\}, \ldots, \max\{b(G), b(H)\}\}.$$

In particular, if G and H are b-continuous and $b(G\Box H) = \max\{b(G), b(H)\}$, then $G\Box H$ is b-continuous.

We now give a lower bound for the b-chromatic number of the Cartesian product of trees. First we recall a lemma given by Kratochvíl, Tuza and Voigt [12] on connected graphs G with $b(G) = 2$. Let G be a bipartite graph with bipartition X and Y. A vertex $x \in X$ ($y \in Y$) is called a full vertex (or a charismatic vertex) of X (Y) if it is adjacent to all the vertices of Y (X).

Lemma 2.2. [12] *Let G be a non-trivial connected graph. Then $b(G) = 2$ iff G is bipartite and has a full vertex in each part of the bipartition.*

Observation 2.3. For trees T with $b(T) \geq 3$, P_5 is an induced subgraph. Any P_5 can be given a b-coloring using 3 colors in which the three middle vertices are c.d.v.'s of distinct color classes. Moreover this b-coloring of P_5 can be extended to a b-coloring of T using the same three colors. Thus for trees with $b(T) \geq 3$, there exists a b-coloring using 3 colors for which we have a c.d.s. forming a star.

We use this fact in the proof of the next theorem.

Theorem 2.4. *Let T_1 and T_2 be any two trees with $b(T_1), b(T_2) \geq 3$, then $b(T_1\Box T_2) \geq b(T_1) + b(T_2) - 1$ and $\{2, \ldots, b(T_1) + b(T_2) - 1\} \subseteq S_b(T_1\Box T_2)$. In particular, if $b(T_1) = 1 + \Delta(T_1)$ and $b(T_2) = 1 + \Delta(T_2)$, then $T_1\Box T_2$ is b-continuous.*

Proof. By Observation 2.1, $T_1\Box T_2$ has a b-coloring using s colors, for every $s \in \{2, \ldots, \max\{b(T_1), b(T_2)\}\}$. Hence all that remains is to show that $T_1\Box T_2$ has a b-coloring using s colors for $s \in \{\max\{b(T_1), b(T_2)\} + 1, \ldots, b(T_1) + b(T_2) - 1\}$, where $\max\{b(T_1), b(T_2)\} + 1 \geq 4$. As already mentioned in the introduction, trees are b-continuous and hence it suffices to show that if T_1 has a b-coloring using k colors and T_2 has a b-coloring using ℓ

colors and if $b(T_1) \geq k \geq 2$ and $b(T_2) \geq \ell \geq 3$, then $T_1 \Box T_2$ has a b-coloring using $k+\ell-1$ colors.

Let g be a b-coloring of T_1 using k colors with $S = \{x_0, x_1, \ldots, x_{k-1}\}$ as a c.d.s. Also let h be a b-coloring of T_2 using ℓ colors with $S^* = \{y_0, y_1, \ldots, y_{\ell-1}\}$ as a c.d.s. Clearly, $\langle S \rangle$ and $\langle S^* \rangle$ are forests. Let U_i denote the color class of g containing x_i, $0 \leq i \leq k-1$ and V_j denote the color class of h containing y_j, $0 \leq j \leq \ell - 1$. Set $X = V(T_1) \setminus S$ and $Y = V(T_2) \setminus S^*$. Let us first consider $k, \ell \geq 4$.

If both S and S^* are stable, then by Observation 1.3, $T_1 \Box T_2$ has a b-coloring using $k+\ell-1$ colors. If not, at least one of S or S^* is not stable. Without loss of generality, let S^* be the set that is not stable. As $\langle S \rangle$ is a forest, there exists at least one vertex, say x_0, such that $d_S(x_0) \leq 1$. In what follows, we assume that whenever $d_S(x_0) = 1$, then the neighbor of x_0 is x_1 in $\langle S \rangle$. While considering S^*, we have the following two cases.

Case 1. $\langle S^* \rangle$ is a star with center at y_0.
As T_1 is a tree, it is a bipartite graph with bipartition, say, S_0 and S_1. Without loss of generality, let $x_0 \in S_0$ and $x_1 \in S_1$. We shall construct a b-coloring, say, c of $T_1 \Box T_2$ using $k+\ell-1$ colors by means of g and h as follows:

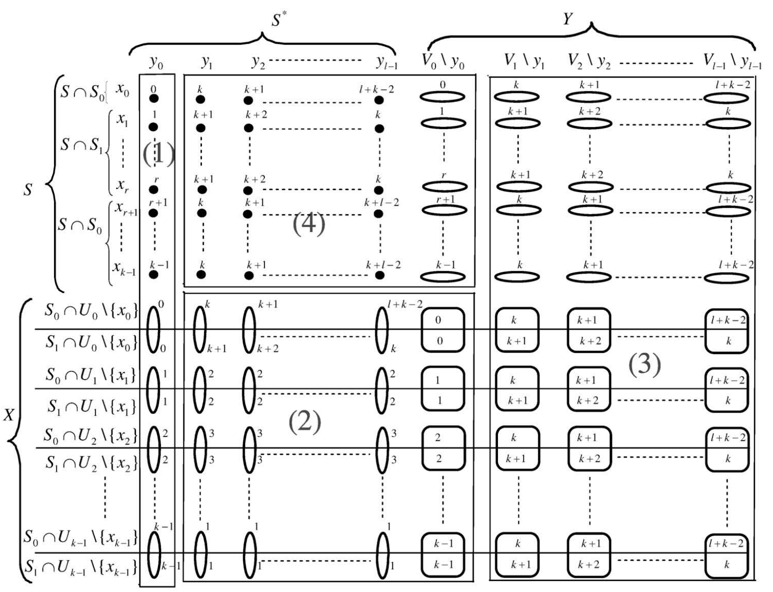

Figure 1: Coloring c in Case 1 of the proof of Theorem 2.4

(1) For $x \in U_i$, $i = 0, 1, \ldots, k-1$ (See box (1) of Figure 1), set

$$c(x, y_0) = i.$$

(2) Consider the vertices in $X \times ((S^* \cup V_0) - \{y_0\})$. (See box (2) of Figure 1).

(i) For $x \in U_0 - \{x_0\}$ and $y \in ((S^* \cup V_0) - \{y_0\})$, set

$$c(x,y) = \begin{cases} k + [(i+j-1) \text{ mod } (\ell-1)] & \text{if } x \in (U_0 \cap S_i) - \{x_0\}, i = 0,1 \text{ and} \\ & y = y_j, 1 \le j \le \ell - 1, \\ 0 & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

(ii) For $x \in X \setminus U_0$, $y \in (S^* \cup V_0) - \{y_0\}$, set

$$c(x,y) = \begin{cases} 1 + [i \text{ mod } (k-1)] & \text{if } x \in U_i,\ 1 \le i \le k-1 \text{ and } y \in S^* - \{y_0\}, \\ c(x, y_0) & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

(3) Consider the vertices in $V(T_1) \times (Y \setminus V_0)$. (See box (3) of Figure 1). For $x \in S_i$, $i = 0,1$, and $y \in V_j - \{y_j\}$, $1 \le j \le \ell - 1$, set

$$c(x,y) = k + [(i+j-1) \text{ mod } (\ell - 1)].$$

(4) Finally we consider the vertices in $S \times (S^* \cup V_0 - \{y_0\})$. (See box (4) of Figure 1), set

$$c(x,y) = \begin{cases} k+[(i+j-1) \text{ mod } (\ell-1)] & \text{if } x \in S \cap S_i,\ i = 0,1,\ y = y_j,\ 1 \le j \le \ell - 1 \\ c(x, y_0) & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

Clearly, this coloring is proper. Consider the vertices in $(S \times \{y_0\}) \cup (\{x_0\} \times S^*)$. We shall show that these vertices are c.d.v.'s of distinct color classes. It is quite evident that the vertices in $S \times \{y_0\}$ are c.d.v.'s of their corresponding color classes.

When $d_S(x_0) = 0$, the vertices in $\{x_0\} \times S^*$ are c.d.v.'s for c and hence c is a b-coloring using $k+\ell-1$ colors. Recall that $d_S(x_0) \le 1$. Thus the only other possibility is $d_S(x_0) = 1$ and in this case as assumed earlier, let $N_S(x_0) = x_1$. Here suppose x_0 has a neighbor in $U_1 \setminus \{x_1\}$, then again the vertices in $\{x_0\} \times S^*$ are c.d.v.'s for c and hence c is a b-coloring using $k + \ell - 1$ colors, or else, x_0 has no neighbor in $U_1 \setminus \{x_1\}$ in which case the vertices in $\{x_0\} \times S^*$ have no neighbors with color 2 in $T_1 \square T_2$.

In order to overcome this case we shall recolor some of the vertices in $\{x_0\} \times Y$ by using the fact that these colors are also present in box (4) of Figure 1. Recall that S^* is a star having center y_0 and with $y_1, \ldots, y_{\ell-1}$ forming an independent set in T_2. As the y_j's are c.d.v.'s in T_2 for $1 \le j \le \ell - 1$, each y_j should have a neighbor in $V_s \setminus \{y_s\}$, for each $s = 1, \ldots, j-1, j+1, \ldots, \ell - 1$. Call such a neighbor in $V_s \setminus \{y_s\}$ as y_{j_s}. As x_0 is adjacent to x_1, the vertex (x_0, y_j) is adjacent to the vertices (x_1, y_j), receiving the colors $k + [j \ (\text{mod}\,(\ell-1))]$. Now recolor the vertex (x_0, y_{j_s}) by color 2, where $s = 1 + [j(\text{mod}(\ell - 1))]$. After this recoloring, it can be seen that the set of vertices

$\{(x_0, y_j) : 1 \le j \le \ell - 1\}$ forms c.d.v.'s of their corresponding color classes and hence in this case also we have found a b-coloring using $k + \ell - 1$ colors.

Case 2. $\langle S^* \rangle$ is not a star.
If $\langle S \rangle$ is a star, then we can interchange T_2 by T_1 in Case 1 and get the result. Therefore we assume that $\langle S \rangle$ also is not a star.

As T_1 is a tree, it is a bipartite graph with bipartition, say, S_0 and S_1. Without loss of generality, let $x_0 \in S_0$. As $\langle S^* \rangle$ is a forest but not stable, S^* has at least one vertex y_0 such that $d_{S^*}(y_0) = 1$. Let $y_1 \in S^*$ be the neighbor of y_0 in $\langle S^* \rangle$. As $\langle S^* \rangle$ is not a star, there exists a vertex, say y_2, in S^* such that $y_1 y_2 \notin E(T_2)$.

As y_1 is a c.d.v., y_1 should have a neighbor in $V_2 \setminus \{y_2\}$, say, y_{1_2} (see Figure 2). Consider the neighbors of y_{1_2} in S^*, say, S_1^*. Note that y_0 is not a neighbor of y_{1_2} (Otherwise, we get a K_3). Without loss of generality let $S_1^* = \{y_1, y_3, y_4, \ldots, y_r\}$, $r \le \ell - 1$. As $(S^* \setminus S_1^*) \cup V_0$ is bipartite (because T_2 is a tree), $(S^* \setminus S_1^*) \cup V_0$ has a bipartition, say, S_0^*, S_2^*, where S_0^* contains y_0. That is $S^* \cup V_0 = S_0^* \cup S_1^* \cup S_2^*$. Now we shall construct a b-coloring, say c, using $k + \ell - 1$ colors by means of g and h as follows:

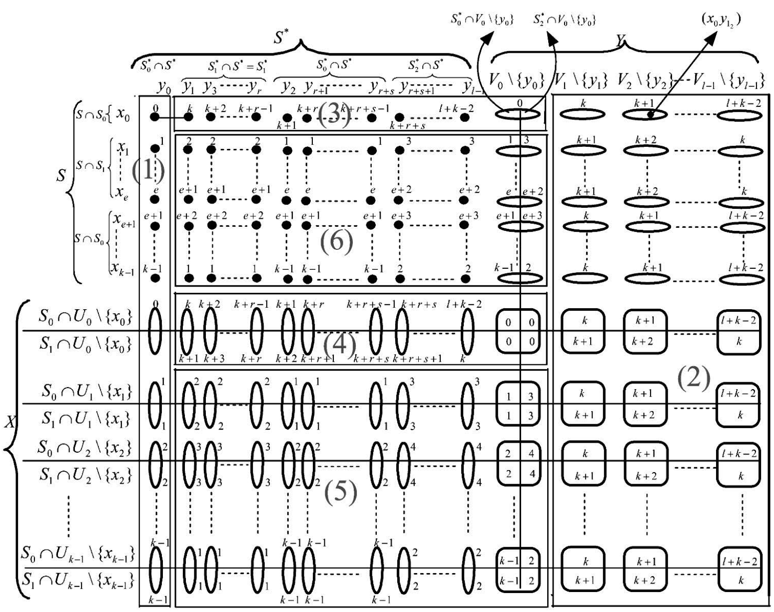

Figure 2: Coloring c in Case 2 of the proof of Theorem 2.4

(1) For $x \in U_i$, $0 \le i \le k - 1$ (See box (1) of Figure 2), set

$$c(x, y_0) = i.$$

(2) Now we color the vertices in $V(T_1) \times Y \setminus V_0$ (See box (2) of Figure 2): For $x \in S_i$,

$0 \le i \le 1$, and $y \in V_j - \{y_j\}$, $1 \le j \le \ell - 1$, set

$$c(x,y) = k + [(i+j-1) \bmod (\ell-1)].$$

(3) For the vertices in $U_0 \times (S^* \cup V_0 - \{y_0\})$ (See boxes (3) and (4) of Figure 2), set

$$c(x,y) = \begin{cases} k + [(i+j-1) \bmod (\ell-1)] & \text{if } x \in U_0 \cap S_i,\ 0 \le i \le 1 \text{ and} \\ & y = y_j, 1 \le j \le \ell - 1, \\ 0 & \text{if } y \in V_0 - \{y_0\}. \end{cases}$$

(4) Finally, we consider the vertices in $(V(T_1) \setminus U_0) \times (\bigcup_{j=0}^{2} S_j^* \setminus \{y_0\})$ (See boxes (5) and (6) of Figure 2). For $x \in U_i$, $1 \le i \le k-1$ and $y \in S_j^*$, $0 \le j \le 2$, set

$$c(x,y) = 1 + [(i+j-1) \bmod (k-1)].$$

In a routine way, one can check that c is a proper coloring using $k + \ell - 1$ colors. As usual, we try to make $(\{x_0\} \times S^*) \cup (S \times \{y_0\})$ as a c.d.s. for c. Obviously $\{x_0\} \times S^*$ are c.d.v.'s for their respective colors.

As y_0 is adjacent to y_1, y_0 may have no neighbors in $V_1 \setminus \{y_1\}$. So we recolor the vertices in $(S \setminus \{x_0\}) \times \{y_1\}$ by setting

$$c(x, y_1) = c(x,y) = k + i,\ x \in (S \cap S_i) \setminus \{x_0\},\ i = 0, 1, \text{ and } y \in V_1 \setminus \{y_1\},\ 1 \le i \le k-1$$

(see box (1) of Figure 3).

Clearly this recoloring does not disturb the proper coloring and this recoloring guarantees that the vertices in $S \times \{y_0\}$ are c.d.v.'s of distinct color classes. But note that there is a possibility for (x_0, y_1) to loss its color dominating property.

If $d_S(x_0) = 0$, then all vertices in $\{x_0\} \times S^*$ are c.d.v.'s of their corresponding color classes and therefore this becomes a b-coloring using $k+\ell-1$ colors. Otherwise $d_S(x_0) = 1$. Recall that x_1 is adjacent to x_0 in S. If x_0 has a neighbor in $U_1 \setminus \{x_1\}$, then we are done. If not, (x_0, y_1) has no neighbor in the color class 2 in $T_1 \square T_2$, so recolor the vertex (x_0, y_{1_2}) by 2 (see box (2) of Figure 3).

This may lead to the vertices in $\{x_0\} \times (S_1^* \setminus \{y_1\})$ having no neighbors with color $k+1$. In order to overcome this problem we do the following recoloring in $\{x_1\} \times (S_1^* \setminus \{y_1\})$:

$$c(x_1, y) = k + 1, \quad y \in S_1^* \setminus \{y_1\}$$

(see box (3) of Figure 3). Thus, $\{x_0\} \times S_1^*$ are c.d.v.'s.

Note that the vertices in $\{x_1\} \times (V_1 \setminus \{y_1\})$ received color $k+1$ and these vertices may have a neighbor in $\{x_1\} \times (S_1^* \setminus \{y_1\})$ and this might make c improper. We get over this

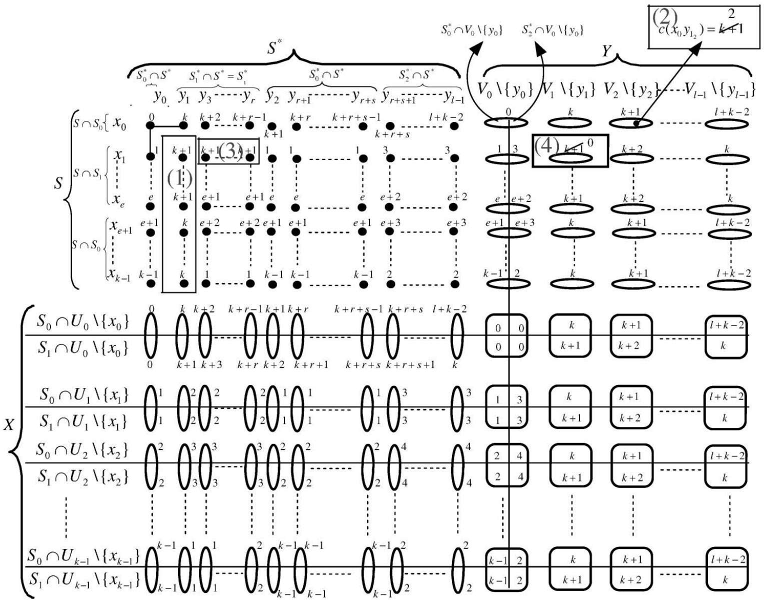

Figure 3: Recoloring of c in Case 2 of the proof of Theorem 2.4

by recoloring the vertices in $\{x_1\} \times (V_1 \setminus \{y_1\})$ by 0 (see box (4) of Figure 3). Checking this recolored c for $G \square H$ to be proper is routine. Thus c is a b-coloring of $T_1 \square T_2$ using $k + \ell - 1$ colors, and hence $\{7, 8, \ldots, b(T_1) + b(T_2) - 1\} \subseteq S_b(T_1 \square T_2)$.

Next, we consider the case when $k \geq 3$ and $\ell = 3$. By Observation 2.3, we can always find a b-coloring using 3 colors for T_2 with a c.d.s. which is a star. Thus by using arguments similar to those used in Case 1, we can show that there exists a b-coloring using $k + 3 - 1$ colors for $T_1 \square T_2$. When $k = 3$ and $\ell \geq 3$, we can find, in a similar way, a b-coloring using $\ell + 3 - 1$ colors for $T_1 \square T_2$. This shows that $\{5, 6\} \in S_b(T_1 \square T_2)$ when $b(T_1) = b(T_2) = 3$.

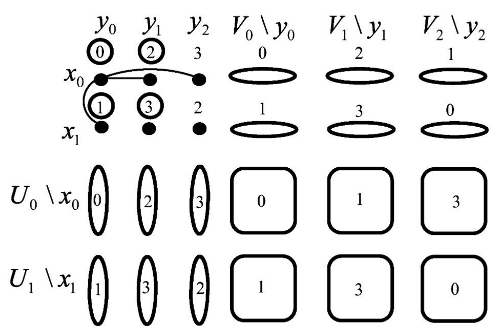

Figure 4: Coloring when $k = 2$ and $\ell = 3$ in the proof of Theorem 2.4

The only case left out is when either k or ℓ is 2 and the other is 3. Without loss of generality, assume that $k = 2$ and $\ell = 3$. In this case, we can give a b-coloring using $2 + 3 - 1 = 4$ colors as shown in Figure 4. This proves that $4 \in S_b(T_1 \Box T_2)$ when $b(T_1) = b(T_2) = 3$.

Thus $T_1 \Box T_2$ has a b-coloring using s colors, for each $s \in \{2, 3, \ldots, b(T_1) + b(T_2) - 1\}$ and hence $b(T_1 \Box T_2) \geq b(T_1) + b(T_2) - 1$. □

Corollary 2.5. *Let T_i, $i = 1, 2, \ldots, n$, be trees with $b(T_i) \geq 3$. Then $b(T_1 \Box \cdots \Box T_n) \geq \sum_{i=1}^{n} b(T_i) - (n-1)$ and $S_b(T_1 \Box \cdots \Box T_n) \supseteq \{2, \ldots, \sum_{i=1}^{n} b(T_i) - (n-1)\}$. In particular, if $b(T_i) = \Delta(T_i) + 1$ for each i, then $b(T_1 \Box \cdots \Box T_n) = \Delta(T_1 \Box \cdots \Box T_n) + 1$, and $T_1 \Box \cdots \Box T_n$ is b-continuous.*

Proof. First let us prove the first part. Proof is by induction on n. By Theorem 2.4, the result is true for $n = 2$. So assume that the result is true for $j \leq n - 1$. We shall show that the result is true for n. Consider $T_1 \Box T_2 \Box \cdots \Box T_n = (T_1 \Box T_2 \Box \cdots \Box T_{n-1}) \Box T_n$. By induction hypothesis $b(T_1 \Box T_2 \Box \cdots \Box T_{n-1}) \geq \sum_{i=1}^{n-1} b(T_i) - (n-2)$ and $S_b(T_1 \Box T_2 \Box \cdots \Box T_{n-1}) \supseteq \left\{2, 3, \ldots, \sum_{i=1}^{n-1} b(T_i) - (n-2)\right\}$. Note that by applying the technique used in Theorem 2.4 step by step to $T_1 \Box T_2 \Box \cdots \Box T_{n-1}$, we can find a b-coloring using k colors (where $2 \leq k \leq \sum_{i=1}^{n-1} b(T_i) - (n-2)$) for which there is a c.d.s. S of $T_1 \Box T_2 \Box \cdots \Box T_{n-1}$ which has a vertex of degree one in $\langle S \rangle$. We know that $\chi(T_1 \Box T_2 \Box \cdots \Box T_{n-1}) = 2$. Thus by using arguments similar to Theorem 2.4 to $[T_1 \Box T_2 \Box \cdots \Box T_{n-1}] \Box T_n$, we can prove that $b(T_1 \Box \cdots \Box T_n) \geq \sum_{i=1}^{n} b(T_i) - (n-1)$ and $S_b(T_1 \Box \cdots \Box T_n) \supseteq \{2, \ldots, \sum_{i=1}^{n} b(T_i) - (n-1)\}$.

Next we prove the second part. Suppose $b(T_i) = \Delta(T_i) + 1$, $1 \leq i \leq n$, then

$$\begin{aligned} b(T_1 \Box \cdots \Box T_n) \geq \sum_{i=1}^{n} b(T_i) - (n-1) &= \sum_{i=1}^{n} (\Delta(T_i) + 1) - (n-1) \\ &= \sum_{i=1}^{n} \Delta(T_i) + 1 = \Delta(T_1 \Box \cdots \Box T_n) + 1. \end{aligned}$$

Since for any graph G, $b(G) \leq \Delta(G) + 1$, $b(T_1 \Box \cdots \Box T_n) = \Delta(T_1 \Box \cdots \Box T_n) + 1$. Since $S_b(T_1 \Box \cdots \Box T_n) \supseteq \{2, \ldots, \sum_{i=1}^{n} b(T_i) - (n-1) = \Delta(T_1 \Box \cdots \Box T_n) + 1\}$, $T_1 \Box \cdots \Box T_n$ is b-continuous. □

One can observe that the technique used in Theorem 2.4 can be extended to a more general setup as given below.

Theorem 2.6. *Let G be a graph having a b-coloring using k colors with a c.d.s. S containing a vertex x whose degree is at most one in $\langle S \rangle$. Let H be a bipartite graph having a b-coloring using ℓ colors with a c.d.s. S^* such that $\langle S^* \rangle$ is a forest other than a star. If $4 \leq k < \ell$ and $b(G) < b(H)$, then $G \Box H$ has a b-coloring using $k + \ell - 1$ colors and $b(G \Box H) \geq b(G) + b(H) - 1$.*

Acknowledgments

The authors thank the referees for their critical study of the paper and useful comments. For the first author, this research was supported by the Department of Science and Technology, Government of India grant DST SR / S4 / MS: 497 / 2009 while for the third author, it was supported by Dr.D.S.Kothari Post Doctoral Fellowship, University Grants Commission, Government of India at the Department of Mathematics, Bharathidasan University, Tiruchirappalli, India.

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