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Shuffle Product Formulas of Two Multiples of Height-one Multiple Zeta Values

Chung-Yie Chang

Abstract. The classical Euler decomposition theorem expressed a product of two Riemann zeta values $\zeta(p)\zeta(q)$ as a sum of $\binom{p+q}{p}$ Euler double sums of weight $p+q$.

As a generalization of Euler decomposition theorem, we shall perform the shuffle product of two multiples of height-one multiple zeta values

$$\binom{j+m}{m}\zeta(\{1\}^{j+m-1}, r-\ell+2) \quad \text{and} \quad \binom{k-j+n}{n}\zeta(\{1\}^{k-j+n-1}, \ell+2)$$

with positive integers m, n and integers k, j, r, ℓ such that $0 \le j \le k$, $0 \le \ell \le r$. Then we applied the resulted shuffle relation to produce weighted sum formulas such as

$$\begin{aligned}
&(k+1) \sum_{|\boldsymbol{\alpha}|=k+r+2} \zeta(\alpha_0+1, \dots, \alpha_{k+1}+1) 2^{\alpha_{k+1}} \\
&+ 2 \sum_{|\boldsymbol{\alpha}|=k+r+1} \zeta(1, \alpha_0+1, \dots, \alpha_k+1) 2^{\alpha_k - \delta_{0k}} \\
&= \frac{1}{2} \sum_{j=0}^{k} \sum_{\substack{\ell=0 \\ \ell:\text{even}}}^{r} (-1)^j (j+1)(k-j+1) \zeta(\{1\}^j, r-\ell+2) \zeta(\{1\}^{k-j}, \ell+2)
\end{aligned}$$

when both k and r are even. Here $\delta_{mn} = 0$ unless $m = n$ and $\delta_{mm=1}$.

1. Multiple zeta values and their shuffle products

Let p and q be positive integers with $q \ge 2$. The classical Euler double sum $S_{p,q}$ is defined as [1, p. 253]

$$S_{p,q} = \sum_{m=1}^{\infty} \frac{1}{m^q} \sum_{n=1}^{m} \frac{1}{n^p}.$$

For an r-tuple of positive integers $\boldsymbol{s} = (s_1, s_2, \dots, s_r)$ with $s_r \ge 2$, the multiple zeta value or r-fold Euler sum $\zeta(\boldsymbol{s})$ is defines as [1–8, 11]

$$\zeta(\boldsymbol{s}) = \sum_{1 \le n_1 < n_2 < \cdots < n_r} n_1^{-s_1} n_2^{-s_2} \cdots n_r^{-s_r} \tag{1.1}$$

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or in free dummy variables as

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_r=1}^{\infty} n_1^{-s_1}(n_1+n_2)^{-s_2}\cdots(n_1+n_2+\cdots+n_r)^{-s_r}. \tag{1.2}$$

The numbers r and $|\boldsymbol{s}| = s_1 + s_2 + \cdots + s_r$ are called the depth and weight of $\zeta(\boldsymbol{s})$, respectively. Also the height of $\zeta(\boldsymbol{s})$ is the number of elements of the set

$$\{j \mid 1 \le j \le r, s_j > 1\}.$$

For our convenience, we let $\{1\}^k$ be k repetitions of 1, so that

$$\zeta(\{1\}^3, 2, 4) = \zeta(1,1,1,2,4) \quad \text{and} \quad \zeta(\{1\}^2, 3, \{1\}^3, 4) = \zeta(1,1,3,1,1,1,4).$$

Due to Kontsevich [4–8, 13], multiple zeta values can be represented by iterated integrals (or Drinfeld integrals) over the simplex defined by

$$E_{|\boldsymbol{s}|} : 0 < t_1 < t_2 < \cdots < t_{|\boldsymbol{s}|} < 1 \tag{1.3}$$

as

$$\int_{E_{|\boldsymbol{s}|}} \omega_1\omega_2\cdots\omega_{|\boldsymbol{s}|}, \tag{1.4}$$

where

$$\omega_j = \begin{cases} \frac{dt_j}{1-t_j} & \text{if } j = 1, s_1+1, s_1+s_2+1, \ldots, s_1+s_2+\cdots+s_{r-1}+1; \\ \frac{dt_j}{t_j} & \text{otherwise.} \end{cases}.$$

Sometimes, we simply write

$$\zeta(s_1, s_2, \ldots, s_r) = \int_0^1 \omega_1\omega_2\cdots\omega_{|\boldsymbol{s}|}. \tag{1.5}$$

Once multiple zeta values are expressed as iterated integrals, the shuffle product of two multiple zeta values then take the form

$$\int_0^1 \omega_1\omega_2\cdots\omega_p \int_0^1 \omega_{p+1}\omega_{p+2}\cdots\omega_{p+q} = \sum_{\sigma}\int_0^1 \omega_{\sigma(1)}\omega_{\sigma(2)}\cdots\omega_{\sigma(p+q)}, \tag{1.6}$$

where the sum is over all $\binom{p+q}{p}$ permutations σ of the set $\{1, 2, \ldots, p+q\}$ which preserve the relative orders of 1-forms $\omega_1\omega_2\cdots\omega_p$ and $\omega_{p+1}\omega_{p+2}\cdots\omega_{p+q}$. More precisely, for all $1 \le i < j \le p$ and $p+1 \le i < j \le p+q$, we have

$$\sigma^{-1}(i) < \sigma^{-1}(j).$$

The classical Euler decomposition theorem expresses the product $\zeta(p)\zeta(q)$, $p, q \geq 2$, in terms of weighted Euler double sums:

$$\zeta(p)\zeta(q) = \sum_{j=1}^{p} \binom{p+q-j-1}{p-j} \zeta(j, p+q-j) + \sum_{j=1}^{q} \binom{p+q-j-1}{q-j} \zeta(j, p+q-j)$$

or

$$\zeta(\ell+2)\zeta(r-\ell+2) = \sum_{\alpha_1+\alpha_2=r+3} \zeta(\alpha_1, \alpha_2+1) \left[\binom{\alpha_2}{\ell+1} + \binom{\alpha_2}{r-\ell+1} \right],$$

for a pair of integers r and ℓ with $0 \leq \ell \leq r$. The proof of the classical Euler decomposition theorem can be found in [2, 4, 5].

In this paper, we are going to preform the shuffle product of two multiples of multiple zeta values of height one

$$\binom{j+m}{m} \zeta(\{1\}^{j+m-1}, r-\ell+2) \quad \text{and} \quad \binom{k-j+n}{n} \zeta(\{1\}^{k-j+n-1}, \ell+2)$$

with m, n positive integers and integers k, j, r, ℓ such that $0 \leq j \leq k$, $0 \leq \ell \leq r$; through some particular integral representations developed by Eie. Indeed, we need the following propositions to express various multiple zeta values in various integrals over simplices and vice versa.

Proposition 1.1. [4, 5] *For a pair nonnegative integers p, q, we have*

$$\begin{aligned}\zeta(\{1\}^p, q+2) &= \frac{1}{(p+1)!q!} \int_0^1 \left(\log \frac{1}{1-t}\right)^{p+1} \left(\log \frac{1}{t}\right)^q \frac{dt}{t} \\ &= \frac{1}{p!q!} \iint_{0<t_1<t_2<1} \left(\log \frac{1}{1-t_1}\right)^p \left(\log \frac{1}{t_2}\right)^q \frac{dt_1 dt_2}{(1-t_1)t_2}.\end{aligned}$$

Proposition 1.2. [4, 5] *For nonnegative integers p, q, r, ℓ, we have*

$$\begin{aligned}&\sum_{|\boldsymbol{\alpha}|=q+r+1} \zeta(\{1\}^{p-1}, \alpha_0+1, \ldots, \alpha_{q-1}, \alpha_q+\ell+1) \\ &= \frac{1}{p!q!r!\ell!} \iint_{0<t_1<t_2<1} \left(\log \frac{1}{1-t_1}\right)^p \left(\log \frac{1-t_1}{1-t_2}\right)^q \left(\log \frac{t_2}{t_1}\right)^r \left(\log \frac{1}{t_2}\right)^\ell \frac{dt_1 dt_2}{t_1 t_2}.\end{aligned}$$

The shuffle products of two multiple zeta values we are going to perform are quite different from those according to the definition of shuffle product (1.6). Instead, we express our candidates in integrals of just one variable as

$$\int_0^1 f(t) \frac{dt}{t} \quad \text{and} \quad \int_0^1 g(u) \frac{du}{u},$$

so that the resulted shuffle relation is given by

$$\int_0^1 f(t)\, dt \cdot \int_0^1 g(u)\, du = \iint_{0<t<u<1} f(t)g(u)\, dt du + \iint_{0<u<t<1} f(t)g(u)\, dt du.$$

However, the main difficulty is to evaluate the two double integrals over two dimensional simplices in terms of multiple zeta values.

2. The special case $m = n = 1$

A multiple zeta values of height one appears to be the form

$$\zeta(\{1\}^p, q+2)$$

with p, q nonnegative integers. It has the iterated integral representation

$$\int_{E_{m+n+2}} \prod_{j=1}^{p+1} \frac{dt_j}{1-t_j} \prod_{k=p+2}^{p+q+2} \frac{dt_k}{t_k}$$

or by Proposition 1.1, as a double integral

$$\frac{1}{p!q!} \iint_{0<t_1<t_2<1} \left(\log \frac{1}{1-t_1}\right)^p \left(\log \frac{1}{t_2}\right)^q \frac{dt_1 dt_2}{(1-t_1)t_2}$$

or just an integral

$$\frac{1}{(p+1)!q!} \int_0^1 \left(\log \frac{1}{1-t}\right)^{p+1} \left(\log \frac{1}{t}\right)^q \frac{dt}{t}.$$

Therefore the integral

$$\frac{1}{p!q!} \int_0^1 \left(\log \frac{1}{1-t}\right)^{p+1} \left(\log \frac{1}{t}\right)^q \frac{dt}{t}$$

represents the multiple zeta value $(p+1)\zeta(\{1\}^p, q+2)$. More general, for positive integer m, we obtain

$$\binom{m+p}{m}\zeta(\{1\}^{p+m-1}, q+2) = \frac{1}{p!m!q!} \int_0^1 \left(\log \frac{1}{1-t}\right)^{p+m} \left(\log \frac{1}{t}\right)^q \frac{dt}{t}. \tag{2.1}$$

In the following, we are going to derive the shuffle product formula of

$$\binom{j+m}{m}\zeta(\{1\}^{j+m}, r-\ell+2) \quad \text{and} \quad \binom{k-j+n}{n}\zeta(\{1\}^{k-j+n}, \ell+2)$$

through their integral representations. Here m and n are positive integers and k, r, j, ℓ are nonnegative integers such that

$$0 \le j \le k \quad \text{and} \quad 0 \le \ell \le r.$$

First we consider the case $m = n = 1$.

Theorem 2.1. *For nonnegative integers k, r, j, ℓ such that*

$$0 \le j \le k \quad \textit{and} \quad 0 \le \ell \le r,$$

the shuffle product formula of

$$(j+1)\zeta(\{1\}^j, r-\ell+2) \quad \textit{and} \quad (k-j+1)\zeta(\{1\}^{k-j}, \ell+2)$$

is given by

$$
\begin{aligned}
&(j+1)\zeta(\{1\}^{j},r-\ell+2)\cdot(k-j+1)\zeta(\{1\}^{k-j},\ell+2)\\
&=\sum_{p+q=k}(p+1)(p+2)\binom{p}{j}\sum_{|\boldsymbol{\alpha}|=q+r+1}\zeta(\{1\}^{p+1},\alpha_0+1,\ldots,\alpha_q+1)\binom{\alpha_q-\delta_{0q}}{\ell+1}\\
&\quad+\sum_{p+q=k}(p+1)(q+1)\binom{p}{j}\sum_{|\boldsymbol{\alpha}|=q+r+2}\zeta(\{1\}^{p},\alpha_0+1,\ldots,\alpha_q,\alpha_{q+1}+1)\binom{\alpha_{q+1}}{\ell+1}\\
&\quad+\sum_{p+q=k}(p+1)(p+2)\binom{p}{k-j}\sum_{|\boldsymbol{\alpha}|=q+r+1}\zeta(\{1\}^{p+1},\alpha_0+1,\ldots,\alpha_q+1)\binom{\alpha_q-\delta_{0q}}{r-\ell+1}\\
&\quad+\sum_{p+q=k}(p+1)(q+1)\binom{p}{k-j}\sum_{|\boldsymbol{\alpha}|=q+r+2}\zeta(\{1\}^{p},\alpha_0+1,\ldots,\alpha_q,\alpha_{q+1}+1)\binom{\alpha_{q+1}}{r-\ell+1}.
\end{aligned}
$$

Here $\delta_{0q}=0$ *unless* $q=0$ *and* $\delta_{00}=1$.

Proof. First we express the product $(j+1)\zeta(\{1\}^{j},r-\ell+2)\cdot(k-j+1)\zeta(\{1\}^{k-j},\ell+2)$ as a separable double integral

$$
\frac{1}{j!(k-j)!\ell!(r-\ell)!}\int_0^1\int_0^1\left(\log\frac{1}{1-t}\right)^{j+1}\left(\log\frac{1}{1-u}\right)^{k-j+1}\left(\log\frac{1}{t}\right)^{r-\ell}\left(\log\frac{1}{u}\right)^{\ell}\frac{dt}{t}\frac{du}{u}.
$$

Then decompose the region of integration $I^2=[0,1]\times[0,1]$ into two simplices of dimension two as

$$
D_1: 0<t<u<1 \quad\text{and}\quad D_2: 0<u<t<1.
$$

In the following, we want to evaluate the integrations over D_1 and D_2 in terms of sums of multiple zeta values.

On the first simplex $D_1: 0<t<u<1$, we substitute the factors

$$
\left(\log\frac{1}{1-u}\right)^{k-j},\quad \log\frac{1}{1-u}\quad\text{and}\quad\left(\log\frac{1}{t}\right)^{r-\ell}
$$

by their binomial expansions

$$
\sum_{a+b=k-j}\frac{(k-j)!}{a!b!}\left(\log\frac{1}{1-t}\right)^{a}\left(\log\frac{1-t}{1-u}\right)^{b},\quad \log\frac{1}{1-t}+\log\frac{1-t}{1-u}
$$

and

$$
\sum_{c+d=r-\ell}\frac{(r-\ell)!}{c!d!}\left(\log\frac{u}{t}\right)^{c}\left(\log\frac{1}{u}\right)^{d}.
$$

So by Proposition 1.2, the value of the integration over D_1, in terms of multiple zeta values, is

$$\sum_{a+b=k-j}\binom{j+a+1+1}{j,a,1,1}\sum_{c+d=r-\ell}\binom{d+\ell}{\ell}$$
$$\times\sum_{|\boldsymbol{\alpha}|=b+c+1}\zeta(\{1\}^{j+a+1},\alpha_0+1,\ldots,\alpha_b+d+\ell+1)$$
$$+\sum_{a+b=k-j}\binom{j+a+1}{j,a,1}(b+1)\sum_{c+d=r-\ell}\binom{d+\ell}{\ell}$$
$$\times\sum_{|\boldsymbol{\alpha}|=b+c+2}\zeta(\{1\}^{j+a},\alpha_0+1,\ldots,\alpha_b,\alpha_{b+1}+d+\ell+1).$$

Here

$$\binom{k_1+k_2+\cdots+k_n}{k_1,k_2,\ldots,k_n}$$

is the multinomial coefficient

$$\frac{(k_1+k_2+\cdots+k_n)!}{k_1!k_2!\cdots k_n!}.$$

With $p=j+a$ and $q=b$ as new dummy variables in place of a,b, the value can be rewritten as

$$\sum_{p+q=k}(p+1)(p+2)\binom{p}{j}\sum_{|\boldsymbol{\alpha}|=q+r+1}\zeta(\{1\}^{p+1},\alpha_0+1,\ldots,\alpha_q+1)\binom{\alpha_q-\delta_{0q}}{\ell+1}$$
$$+\sum_{p+q=k}(p+1)(q+1)\binom{p}{j}\sum_{|\boldsymbol{\alpha}|=q+r+2}\zeta(\{1\}^{p},\alpha_0+1,\ldots,\alpha_q,\alpha_{q+1}+1)\binom{\alpha_{q+1}}{\ell+1}.$$

To evaluate the integration over $D_2: 0<u<t<1$, we simply exchange the roles of j,ℓ with $k-j, r-\ell$ and obtain the values as

$$\sum_{p+q=k}(p+1)(p+2)\binom{p}{k-j}\sum_{|\boldsymbol{\alpha}|=q+r+1}\zeta(\{1\}^{p+1},\alpha_0+1,\ldots,\alpha_q+1)\binom{\alpha_q-\delta_{0q}}{r-\ell+1}$$
$$+\sum_{p+q=k}(p+1)(q+1)\binom{p}{k-j}\sum_{|\boldsymbol{\alpha}|=q+r+2}\zeta(\{1\}^{p},\alpha_0+1,\ldots,\alpha_q,\alpha_{q+1}+1)\binom{\alpha_{q+1}}{r-\ell+1}.$$

Therefore, we complete the proof. □

On the simplex $D_1: 0<t<u<1$, we make a change of variables:

$$t_1=1-u,\quad t_2=1-t.$$

Then the integration over D_1 is the transformed into

$$\frac{1}{j!\ell!(k-j)!(r-\ell)!}\iint_{0<t_1<t_2<1}\left(\log\frac{1}{1-t_1}\right)^{\ell}\left(\log\frac{1}{1-t_2}\right)^{r-\ell}$$
$$\times\left(\log\frac{1}{t_1}\right)^{k-j+1}\left(\log\frac{1}{t_2}\right)^{j+1}\frac{dt_1}{1-t_1}\frac{dt_2}{1-t_2}.$$

In a similar way as above, the value of the above is equal to

$$(j+1)(k-j+1)\sum_{g+h=r}\binom{g}{\ell}\sum_{|\boldsymbol{\beta}|=h+k+3}\zeta(\{1\}^g,\beta_0,\ldots,\beta_n,\beta_{n+1}+1)\binom{\beta_{n+1}}{j+1}.$$

So we obtain the following corollary.

Corollary 2.2. *For nonnegative integers k, j, r, ℓ such that*

$$0\le j\le k \quad \textit{and} \quad 0\le \ell\le r,$$

we have

$$\begin{aligned}&\sum_{p+q=k}(p+1)(p+2)\binom{p}{j}\sum_{|\boldsymbol{\alpha}|=q+r+1}\zeta(\{1\}^{p+1},\alpha_0+1,\ldots,\alpha_q+1)\binom{\alpha_q-\delta_{0q}}{\ell+1}\\ &+\sum_{p+q=k}(p+1)(q+1)\binom{p}{j}\sum_{|\boldsymbol{\alpha}|=q+r+2}\zeta(\{1\}^{p+1},\alpha_0+1,\ldots,\alpha_q,\alpha_{q+1}+1)\binom{\alpha_{q+1}}{\ell+1}\\ =&(j+1)(k-j+1)\sum_{g+h=r}\binom{g}{\ell}\sum_{|\boldsymbol{\beta}|=h+k+3}\zeta(\{1\}^g,\beta_0,\ldots,\beta_n,\beta_{n+1}+1)\binom{\beta_{n+1}}{j+1}.\end{aligned}$$

3. The general case

To produce the shuffle product formula of

$$\binom{j+m}{m}\zeta(\{1\}^{j+m-1},r-\ell+2) \quad \text{and} \quad \binom{k-j+n}{n}\zeta(\{1\}^{k-j+n-1},\ell+2)$$

for positive integers m and n, we express their product by (2.1) as the following separable double integral

$$\begin{aligned}\frac{1}{j!(k-j)!m!n!\ell!(r-\ell)!}\int_0^1\int_0^1&\left(\log\frac{1}{1-t}\right)^{j+m}\left(\log\frac{1}{1-u}\right)^{k-j+n}\\ &\times\left(\log\frac{1}{t}\right)^{r-\ell}\left(\log\frac{1}{u}\right)^{\ell}\frac{dt}{t}\frac{du}{u}.\end{aligned}$$

As mentioned earlier, we decompose the region of integration $I^2=[0,1]\times[0,1]$ into two simplices of dimension two as

$$D_1: 0<t<u<1 \quad \text{and} \quad D_2: 0<u<t<1.$$

During the evaluation of the integration over $D_1: 0<t<u<1$, we substitute the factor

$$\left(\log\frac{1}{1-u}\right)^{k-j+n}$$

by the product of

$$\sum_{a+b=k-j} \frac{(k-j)!}{a!b!} \left(\log \frac{1}{1-t}\right)^a \left(\log \frac{1-t}{1-u}\right)^b$$

and

$$\sum_{g+h=n} \frac{n!}{g!h!} \left(\log \frac{1}{1-t}\right)^g \left(\log \frac{1-t}{1-u}\right)^h.$$

Also, we substitute the factor $\left(\log \frac{1}{t}\right)^{r-\ell}$ by its binomial expansion

$$\sum_{c+d=r-\ell} \frac{(r-\ell)!}{c!d!} \left(\log \frac{u}{t}\right)^c \left(\log \frac{1}{u}\right)^d,$$

so that by Proposition 1.2 the value of the integration over D_1 is

$$\sum_{a+b=k-j} \sum_{g+h=n} \binom{j+m+a+g}{j,m,a,g} \binom{b+h}{b} \sum_{c+d=r-\ell} \binom{d+\ell}{d} \\ \times \sum_{|\boldsymbol{\alpha}|=b+h+c+1} \zeta(\{1\}^{j+m+a+g-1}, \alpha_0+1, \ldots, \alpha_{b+h}+d+\ell+1).$$

Let $p = a+j$, $q = b$ be new dummy variables. Then the above value can be rewritten as

$$\sum_{p+q=k} \sum_{g+h=n} \binom{p}{j} \binom{m+p+g}{m,p,g} \sum_{c+d=r-\ell} \binom{d+\ell}{d} \\ \times \sum_{|\boldsymbol{\alpha}|=q+h+c+1} \zeta(\{1\}^{m+p+g-1}, \alpha_0+1, \ldots, \alpha_{q+h}+d+\ell+1),$$

or

$$\sum_{p+q=k} \sum_{g+h=n} \binom{p}{j} \binom{m+p+g}{m,p,g} \binom{q+h}{h} \\ \times \sum_{|\boldsymbol{\alpha}|=q+h+r+1} \zeta(\{1\}^{m+p+g-1}, \alpha_0+1, \ldots, \alpha_{q+h}+1) \binom{\alpha_{q+h}-\delta_{0,q+h}}{\ell+1}.$$

Again, exchange the roles of j, ℓ with $k-j$ and $r-\ell$, we obtained the value of the integration over D_2 in the following:

$$\sum_{p+q=k} \sum_{g+h=n} \binom{p}{k-j} \binom{n+p+g}{n,p,g} \binom{q+h}{h} \\ \times \sum_{|\boldsymbol{\alpha}|=q+h+r+1} \zeta(\{1\}^{n+p+g-1}, \alpha_0+1, \ldots, \alpha_{q+h}+1) \binom{\alpha_{q+h}-\delta_{0,q+h}}{r-\ell+1}.$$

Therefore, we have the following general theorem.

Theorem 3.1. *For a pair of positive integers m, n and integers k, j, r, ℓ such that*

$$0 \le j \le k \quad and \quad 0 \le \ell \le r,$$

the shuffle product formula of multiple zeta values

$$\binom{j+m}{j}\zeta(\{1\}^{j+m-1}, r-\ell+2) \quad and \quad \binom{k-j+n}{n}\zeta(\{1\}^{k-j+n-1}, \ell+2)$$

is given by

$$\begin{aligned}
&\binom{j+m}{j}\zeta(\{1\}^{j+m-1}, r-\ell+2)\binom{k-j+n}{n}\zeta(\{1\}^{k-j+n-1}, \ell+2) \\
&= \sum_{p+q=k}\sum_{g+h=n}\binom{p}{j}\binom{m+p+g}{m,p,g}\binom{q+h}{h} \\
&\quad \times \sum_{|\boldsymbol{\alpha}|=q+h+r+1}\zeta(\{1\}^{m+p+g-1}, \alpha_0+1, \ldots, \alpha_{q+h}+1)\binom{\alpha_{q+h}-\delta_{0,q+h}}{\ell+1} \\
&\quad + \sum_{p+q=k}\sum_{g+h=n}\binom{p}{k-j}\binom{n+p+g}{n,p,g}\binom{q+h}{h} \\
&\quad \times \sum_{|\boldsymbol{\alpha}|=q+h+r+1}\zeta(\{1\}^{n+p+g-1}, \alpha_0+1, \ldots, \alpha_{q+h}+1)\binom{\alpha_{q+h}-\delta_{0,q+h}}{r-\ell+1}.
\end{aligned}$$

4. Applications

Multiply both sides of the shuffle relation in Theorem 2.1 by $(-1)^{j+\ell}$ and then sum over all j, ℓ with $0 \le j \le k$, $0 \le \ell \le r$, we obtain the following theorem.

Theorem 4.1. *For a pair of nonnegative integers k, r with $k+r$ even, then for $k \ge 1$*

$$\begin{aligned}
&(k+1)\zeta(k+r+4) - (k-1)\sum_{|\boldsymbol{\alpha}|=k+r+2}\zeta(1, \alpha_0, \ldots, \alpha_k+1) \\
&- 2\sum_{|\boldsymbol{\alpha}|=k+r+1}\zeta(1, 1, \alpha_0, \ldots, \alpha_{k-1}+1) \\
&= \frac{1}{2}\sum_{j=0}^{k}\sum_{\ell=0}^{r}(-1)^{j+\ell}(j+1)(k-j+1)\zeta(\{1\}^{j}, r-\ell+2)\zeta(\{1\}^{k-j}, \ell+2).
\end{aligned}$$

Remark 4.2. In light of the restriction sum formula [4–6, 9, 12, 13], we have

$$\sum_{|\boldsymbol{\alpha}|=k+r+2}\zeta(1, \alpha_0, \ldots, \alpha_k+1) = \sum_{|\boldsymbol{c}|=k+2}\zeta(c_1, c_2+r+2)$$

and

$$\sum_{|\boldsymbol{\alpha}|=k+r+1}\zeta(1, 1, \alpha_0, \ldots, \alpha_{k-1}+1) = \sum_{|\boldsymbol{d}|=k+2}\zeta(d_1, d_2, d_3+r+2).$$

Thus the assertion of the above theorem tells us that a sum of multiple zeta values of depth 3 and even weight can be written as a multiple of double Euler sums of even weight plus some single zeta values.

On the other hand, if we multiply both sides of the shuffle relation in Theorem 2.1 by $(-1)^\ell$ and then sum over all $0 \le j \le k$ and $0 \le \ell \le r$, we obtain the following theorem.

Theorem 4.3. *For a pair of nonnegative integers k, r with r even, then we have*

$$\begin{aligned}
&\sum_{p+q=k} 2^p(p+1)(p+2) \sum_{|\boldsymbol{\alpha}|=q+r+1} \zeta(\{1\}^{p+1}, \alpha_0+1, \dots, \alpha_q+1) \\
&\quad + \sum_{p+q=k} 2^p(p+1)(q+1) \sum_{|\boldsymbol{\alpha}|=q+r+2} \zeta(\{1\}^{p}, \alpha_0+1, \dots, \alpha_{q+1}+1) \\
&= \frac{1}{2}\sum_{j=0}^{k}\sum_{\ell=0}^{r}(-1)^\ell (j+1)(k-j+1)\zeta(\{1\}^j, r-\ell+2)\zeta(\{1\}^{k-j}, \ell+2).
\end{aligned}$$

Note that the shuffle product of two multiple zeta values of weight m and n will produce $\binom{m+n}{m}$ multiple zeta values of weight $m+n$. Therefore the shuffle product of

$$(j+1)\zeta(\{1\}^j, r-\ell+2) \quad \text{and} \quad (k-j+1)\zeta(\{1\}^{k-j}, \ell+2)$$

will produce

$$(j+1)(k-j+1)\binom{k+r+4}{j+r-\ell+2}$$

multiple zeta value of weight $k+r+4$. After a similar procedure as before, we obtain the following consequences from the previous two theorems.

Corollary 4.4. *For a pair of nonnegative integers k, r with $k+r$ even, we have*

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{k}\sum_{\ell=0}^{r}(-1)^{j+\ell}(j+1)(k-j+1)\binom{k+r+4}{j+r-\ell+2} \\
&= (k+r+3)\binom{k+r}{k}.
\end{aligned}$$

Corollary 4.5. *For a pair of nonnegative integers k, r with r even, we have*

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{k}\sum_{\ell=0}^{r}(-1)^{\ell}(j+1)(k-j+1)\binom{k+r+4}{j+r-\ell+2} \\
&= \sum_{p+q=k}\left\{2^p(p+1)(p+2)\binom{q+r}{q} + 2^p(p+1)(q+1)\binom{q+r+1}{q+1}\right\}.
\end{aligned}$$

At last, we multiply both sides of the shuffle relation by $(-1)^j$ and then sum over $0 \le j \le k$ and $0 \le \ell \le r$, we obtain the following theorem.

Theorem 4.6. *For a pair of nonnegative integers k, r with k even, then*

$$\begin{aligned}&(k+1)\sum_{|\boldsymbol{\alpha}|=k+r+2}\zeta(\alpha_0+1,\ldots,\alpha_{k+1}+1)(2^{\alpha_{k+1}}-1)\\&+2\sum_{|\boldsymbol{\alpha}|=k+r+1}\zeta(1,\alpha_0+1,\ldots,\alpha_k+1)(2^{\alpha_k-\delta_{0k}}-1)\\=&\frac{1}{2}\sum_{j=0}^{k}\sum_{\ell=0}^{r}(-1)^j(j+1)(k-j+1)\zeta(\{1\}^j,r-\ell+2)\zeta(\{1\}^{k-j},\ell+2).\end{aligned}$$

Also when both k and r are even, we have

$$\begin{aligned}&(k+1)\sum_{|\boldsymbol{\alpha}|=k+r+2}\zeta(\alpha_0+1,\ldots,\alpha_{k+1}+1)2^{\alpha_{k+1}}\\&+2\sum_{|\boldsymbol{\alpha}|=k+r+1}\zeta(1,\alpha_0+1,\ldots,\alpha_k+1)2^{\alpha_k-\delta_{0k}}\\=&\frac{1}{2}\sum_{j=0}^{k}\sum_{\substack{\ell=0\\ \ell:\text{even}}}^{r}(-1)^j(j+1)(k-j+1)\zeta(\{1\}^j,r-\ell+2)\zeta(\{1\}^{k-j},\ell+2).\end{aligned}$$

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Chung-Yie Chang
Department of Applied Mathematics, Tatung University, Taiwan
E-mail address: cychang@ttu.edu.tw