

ON THE COMPACTNESS OF COMMUTATORS FOR ROUGH MARCINKIEWICZ INTEGRAL OPERATORS

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Abstract. Let $b \in \text{BMO}(\mathbb{R}^n)$ and \mathcal{M}_Ω be the Marcinkiewicz integral operator with kernel $\frac{\Omega(x)}{|x|^{n-1}}$, where Ω is homogeneous of degree zero, integrable and has mean value zero on the unit sphere S^{n-1} . In this paper, by means of Fourier transform estimates and approximation to the operator \mathcal{M}_Ω with integral operators having smooth kernels we show that if $b \in \text{CMO}(\mathbb{R}^n)$ and Ω satisfies a certain weak size condition, then the commutator $\mathcal{M}_{\Omega,b} = [b, \mathcal{M}_\Omega]$ generated by b and \mathcal{M}_Ω is a compact operator on $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$.

1. INTRODUCTION

We aim to prove that the following (sub-linear) operator, which is called the commutator of the rough Marcinkiewicz integral operator,

$$(1.1) \quad \mathcal{M}_{\Omega,b}f(x) := \left(\int_0^\infty \left| \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x) - b(y)) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2}$$

is an $L^p(\mathbb{R}^n)$ -compact operator for certain $1 < p < \infty$, if $b \in \text{CMO}(\mathbb{R}^n)$, the $\text{BMO}(\mathbb{R}^n)$ -closure of $C_c^\infty(\mathbb{R}^n)$, and Ω is homogeneous of degree zero, integrable, satisfies a certain weak integrability condition proposed by Grafakos and Stefanov [15] (see (1.3) below), and has mean value zero on the unit sphere S^{n-1} . When $n = 1$, then Ω is a constant times the signature function, and this case is rather simple. Thus, we suppose, here and below, that $n \geq 2$.

Received December 13, 2014, accepted April 29, 2015.

Communicated by Duy-Minh Nhieu.

2010 *Mathematics Subject Classification*: Primary 42B20; Secondary 11L15, 11P05.

Key words and phrases: Commutators, Marcinkiewicz integral operators, Compactness, Approximation. Supported by the NNSF of China (Nos. 11371295, 11471041) and the NSF of Fujian Province of China (No. 2015J01025).

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The commutator $\mathcal{M}_{\Omega,b}$ is closely related to the Marcinkiewicz integral operator \mathcal{M}_{Ω} defined by

$$\mathcal{M}_{\Omega}f(x) := \left(\int_0^{\infty} |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$(1.2) \quad F_{\Omega,t}(x) := \int_{|y|\leq t} \frac{\Omega(y)}{|y|^{n-1}} f(x-y) dy \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n).$$

Stein initiated to the study of \mathcal{M}_{Ω} in [18], where Ω was assumed to belong to a certain Lipschitz class $\text{Lip}_{\alpha}(S^{n-1})$ with $0 < \alpha < 1$. Subsequently, Benedek, Calderon and Panzone [3] showed that \mathcal{M}_{Ω} is bounded on $L^p(\mathbb{R}^n)$ provided $1 < p < \infty$ and $\Omega \in C^1(S^{n-1})$. For more than six decades, many authors studied this operator under many kinds of weak conditions on Ω ; see [1, 5, 11, 12, 22, 23] for a sample of this work. In particular, from [22, 23], we have the following result, which will be used below.

Theorem A. [22, 23] *Suppose that real parameters p and α satisfy*

$$\alpha > \frac{1}{2} \quad \text{and} \quad 1 + \frac{1}{2\alpha} < p < 1 + 2\alpha.$$

Let Ω be homogeneous of degree zero, integrable on S^{n-1} and have mean value zero. Suppose that $\Omega \in G_{\alpha}(S^{n-1})$, that is,

$$(1.3) \quad \|\Omega\|_{G_{\alpha}(S^{n-1})} := \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')| \left(\log \frac{1}{|\langle y', \xi \rangle|} \right)^{\alpha} d\sigma(y') < \infty.$$

Then \mathcal{M}_{Ω} is bounded on $L^p(\mathbb{R}^n)$.

We remark that the condition (1.3) was originally introduced in Walsh's paper [21] and developed by Grafakos and Stefanov [15] in the study of L^p -boundedness of singular integrals with rough kernels. It follows from [15] that $\bigcup_{q>1} L^q(S^{n-1}) \subsetneq G_{\alpha}(S^{n-1})$ for any $\alpha > 0$, and

$$(1.4) \quad G_{\alpha_1}(S^{n-1}) \subsetneq G_{\alpha_2}(S^{n-1}) \quad \text{for } 0 < \alpha_2 < \alpha_1;$$

moreover,

$$(1.5) \quad \bigcap_{\alpha>1} G_{\alpha}(S^{n-1}) \not\subseteq H^1(S^{n-1}) \not\subseteq \bigcup_{\alpha>1} G_{\alpha}(S^{n-1}),$$

where $H^1(S^{n-1})$ denotes the Hardy space on the unit sphere S^{n-1} .

The definition of the operator $\mathcal{M}_{\Omega,b}$ dates back to the work of Torchinsky and Wang [19], who showed that $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and

$b \in \text{BMO}(\mathbb{R}^n)$, provided $\Omega \in \text{Lip}_\alpha(S^{n-1})$. Subsequently, many authors considered the boundedness properties of this operators; see [7, 13, 16, 17, 24], for examples. In particular, Hu [16] obtained the following result.

Theorem B. [16] *Suppose that the real parameters p and α satisfy*

$$\alpha > \frac{3}{2} \quad \text{and} \quad \frac{4\alpha}{4\alpha - 3} < p < \frac{4\alpha}{3}.$$

Let Ω be homogeneous of degree zero, integrable on S^{n-1} and have mean value zero, and let $b \in \text{BMO}(\mathbb{R}^n)$. Then for any $\Omega \in G_\alpha(S^{n-1})$, $\mathcal{M}_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\text{BMO}(\mathbb{R}^n)}$.

In this paper, we will focus on the compactness of $\mathcal{M}_{\Omega,b}$. We first recall the compactness concept and some relevant results. We say that a mapping T from a Banach space X to a Banach space Y is compact if T is continuous and maps bounded subsets of X into strongly pre-compact subsets of Y (see [2]). Compactness of commutators dates back to Uchiyama's work [20]. Uchiyama considered the commutator $T_{\Omega,b}$ generated by b and T_Ω which is given by:

$$T_{\Omega,b}f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (b(x) - b(y)) f(y) dy,$$

where $b \in \text{BMO}$, and $\Omega \in L^1(S^{n-1})$ satisfies $\int_{S^{n-1}} \Omega(x) d\sigma(x) = 0$. In his remarkable work [20], Uchiyama proved that for $\Omega \in \text{Lip}_\alpha(S^{n-1})$ ($0 < \alpha < 1$), $1 < p < \infty$, $T_{\Omega,b}$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \text{CMO}(\mathbb{R}^n)$. It is known that $\text{CMO}(\mathbb{R}^n)$ coincides with the space of the functions with vanishing mean oscillation; see [4, 10]. Recently, Chen and Hu [6] established the following result.

Theorem C. [6] *Suppose that the parameters α and p satisfy*

$$\alpha > 2 \quad \text{and} \quad \frac{\alpha}{\alpha - 1} < p < \alpha.$$

If $b \in \text{CMO}(\mathbb{R}^n)$, Ω is homogeneous of degree zero and has mean value zero on S^{n-1} , and $\Omega \in G_\alpha(S^{n-1})$, then the operator $T_{\Omega,b}$ is compact on $L^p(\mathbb{R}^n)$.

An interesting and important fact is that many nonlinear operators arising in mathematical physics and differential geometry are compact. Although many authors studied compact linear operators, the literature is not so rich regarding the compactness of nonlinear compact operators, which contain $\mathcal{M}_{\Omega,b}$ as a typical example. Recently, several attentions have been paid to the investigation on the non-linear compact operators, see [7, 8, 9, 14] et al. and therein references. In particular, Chen and Ding [7] showed the compactness of $\mathcal{M}_{\Omega,b}$ on $L^p(\mathbb{R}^n)$, provided that Ω satisfies certain regularity conditions of L^q -Dini type. Inspired by Theorem B, it is natural to ask whether $\mathcal{M}_{\Omega,b}$ is compact on $L^p(\mathbb{R}^n)$ under the assumption of that $\Omega \in G_\alpha(S^{n-1})$ for certain $\alpha > 0$ and $1 < p < \infty$. This question will be addressed by our the following theorem.

Theorem 1.1. *Suppose that the parameters α and p satisfy*

$$\alpha > \frac{3}{2} \quad \text{and} \quad \frac{4\alpha}{4\alpha - 3} < p < \frac{4\alpha}{3}.$$

If $b \in \text{CMO}(\mathbb{R}^n)$, Ω is homogeneous of degree zero and has mean value zero on S^{n-1} , and $\Omega \in G_\alpha(S^{n-1})$, then the operator $\mathcal{M}_{\Omega,b}$, which is given by (1.1), is compact on $L^p(\mathbb{R}^n)$.

Remark 1.1. Clearly, our theorem shows that $\mathcal{M}_{\Omega,b}$ is compact under the same assumptions as in Theorem B, which is new and interesting. Comparing with Theorem C, we know that the compactness of $\mathcal{M}_{\Omega,b}$ is better than one of $T_{\Omega,b}$ since the range of α is extended from $(2, \infty)$ to $(3/2, \infty)$, which implies that the condition required by $\mathcal{M}_{\Omega,b}$ is weaker than one by $T_{\Omega,b}$ according to (1.4); moreover, the range of p in our theorem, $(4\alpha/(4\alpha - 3), 4\alpha/3)$, is larger than $(\alpha/(\alpha - 1), \alpha)$, the range of p in Theorem B, for the same value of α satisfying $\alpha > 2$.

We shall use the following conventions:

- C always denotes a positive constant that is independent of main parameters involved but whose value may differ from line to line.
- We use the symbol $A \lesssim B$ to indicate that there exists a positive constant C such that $A \leq CB$.
- For a set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function.
- For $p \in [1, \infty)$, we use p' to denote the dual exponent of p , namely, $p' = \frac{p}{p-1}$.
- For a suitable function f , \hat{f} denotes the Fourier transform of f given by:

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

- Finally, $\langle \cdot, \cdot \rangle$ stands for the standard inner product on \mathbb{R}^n .

The rest of this paper is organized as follows: In Section 2 we will establish some auxiliary lemmas. The main ingredient is to establish the approximation to the Marcinkiewicz integrals operator, which will play key roles in our later proofs. Finally, we will prove our main theorem in Section 3.

2. PRELIMINARY LEMMAS AND APPROXIMATION

2.1. Dyadic decomposition

For each $l \in \mathbb{Z}$, $t \in \mathbb{R}_+$, we define $\sigma_{l,t}$ by

$$(2.1) \quad \sigma_{l,t}(x) := \frac{\Omega(x)}{2^{lt}|x|^{n-1}} \chi_{[2^{lt}, 2^{l+1}t]}(x) \quad (x \in \mathbb{R}^n).$$

So, the Fourier transform is given by:

$$(2.2) \quad \widehat{\sigma}_{l,t}(\xi) = \frac{1}{2^l t} \int_{2^l t \leq |y| \leq 2^{l+1} t} \frac{\Omega(y)}{|y|^{n-1}} e^{-2\pi i \langle y, \xi \rangle} dy, \quad \xi \in \mathbb{R}^n.$$

Observe that

$$(2.3) \quad \sigma_{l,t} = \sigma_{0,2^l t}.$$

It follows from (2.1) that $\Omega(x)\chi_{\{|y|<t\}}(x) = |x|^{n-1} \sum_{l=-\infty}^{-1} 2^l t \sigma_{l,t}(x)$. Thus, we have

$$(2.4) \quad \begin{aligned} \frac{F_{\Omega,t}(x)}{t} &= \sum_{l=-\infty}^{-1} 2^l \sigma_{l,t} * f(x) \text{ and } \mathcal{M}_{\Omega} f(x) \\ &= \left(\int_0^{\infty} \left| \sum_{l=-\infty}^{-1} 2^l \sigma_{l,t} * f(x) \right|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Lemma 2.1. *Let $t > 0$, $l \in \mathbb{Z}$ and $\Omega \in G_{\alpha}(S^{n-1})$ for $\alpha > 1/2$. Then*

- (i) $|\widehat{\sigma}_{l,t}(\xi)| \lesssim \min(1, |2^l t \xi|)$;
- (ii) $|\widehat{\sigma}_{l,t}(\xi)| \lesssim \min((\log(3 + |2^l t \xi|))^{-\alpha}, |2^l t \xi|)$.

Proof. Write $\xi' = \xi/|\xi|$ and $y' = y'/|y|$. Then we have

$$\widehat{\sigma}_{l,t}(\xi) = \int_{S^{n-1}} \Omega(y') \left(\int_1^2 e^{-2\pi i r \cdot 2^l t |\xi| \langle y', \xi' \rangle} dr \right) d\sigma(y')$$

by a change of variables. By the $L^1(S^{n-1})$ -integrability and the vanishing moment of Ω , it is easy to verify that:

$$|\widehat{\sigma}_{l,t}(\xi)| \lesssim \min(1, |2^l t \xi|).$$

Let $|2^l t \xi| > e^{\alpha}$. Note that

$$\left| \int_1^2 e^{-2\pi i r \cdot 2^l t |\xi| \langle y', \xi' \rangle} dr \right| = \left| \left[\frac{e^{-2\pi i r \cdot 2^l t |\xi| \langle y', \xi' \rangle}}{2\pi r \cdot 2^l t |\xi| \langle y', \xi' \rangle} \right]_{r=1}^{r=2} \right| \leq \frac{1}{2^l \pi t |\xi| |\langle \xi', y' \rangle|},$$

and the following trivial estimate:

$$\left| \int_1^2 e^{-2\pi i r \cdot 2^l t |\xi| \langle y', \xi' \rangle} dr \right| \leq \int_1^2 |e^{-2\pi i r \cdot 2^l t |\xi| \langle y', \xi' \rangle}| dr = 1.$$

We get from the assumption $2^l t |\xi| > e^{\alpha}$ that

$$\left| \int_1^2 e^{-2\pi i r 2^l t |\xi| \langle y', \xi' \rangle} dr \right| \lesssim \min \left(1, \frac{e^\alpha |\langle y', \xi' \rangle|^{-1}}{2^l t |\xi|} \right),$$

since the identity:

$$\frac{d}{dt} \frac{t}{\log^a t} = \frac{1}{\log^a t} - \frac{a}{\log^{a+1} t} = \frac{\log t - a}{\log^{a+1} t}$$

shows, $t/\log^a t$ is increasing in (e^a, ∞) for any $a > 0$, or equivalently,

$$\frac{t}{s} \leq \frac{\log^a t}{\log^a s} \quad (s \geq t \geq e^a).$$

Thus, we can deduce that for $\alpha > 1/2$,

$$\left| \int_1^2 e^{-2\pi i r 2^l t |\xi| \langle y', \xi' \rangle} dr \right| \lesssim \frac{\log^\alpha(e^\alpha |\langle y', \xi' \rangle|^{-1})}{\log^\alpha(2^l t |\xi|)},$$

provided $|2^l t \xi| > e^\alpha$. Then, (ii) holds thanks to (1.3). This completes the proof of Lemma 2.1. ■

We define the modified Marcinkiewicz integral operator \mathcal{M}_Ω^K by:

$$\mathcal{M}_\Omega^K f(x) := \left(\int_0^\infty |F_{\Omega, 2^k t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $F_{\Omega, t}$ is given by (1.2).

Lemma 2.2. *Let $\alpha > 1/2$. If we set*

$$M_K = 2^K \sup_{\xi \in \mathbb{R}^n} \left(\int_0^\infty \sum_{l=-\infty}^K \min(|2^l t \xi|^2, (\log(3 + |2^l t \xi|))^{-2\alpha}) \frac{dt}{t} \right)^{1/2} < \infty,$$

then $M_K = 2^K M_0$ is finite and

$$\|\mathcal{M}_\Omega^K f\|_{L^2} \lesssim M_K \|f\|_{L^2}$$

for all $f \in L^2(\mathbb{R}^n)$.

Proof. The fact that M_0 is finite follows from the change of variables:

$$M_0 = \left(\int_0^\infty \sum_{l=-\infty}^{-1} 4^l \min(t^2, (\log(3 + |t|))^{-2\alpha}) \frac{dt}{t} \right)^{1/2} < \infty. \quad \blacksquare$$

2.2. Approximation

Next, as we announced, we shall construct an approximation of Ω . Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative function having integral 1 and supported on a small ball $\{x : |x| \leq 1/4\}$. For $l \in \mathbb{Z}$, let $\phi_l(x) := 2^{-nl} \phi(2^{-l}x)$. We then have for $\xi \in \mathbb{R}^n$,

$$(2.5) \quad |\widehat{\phi}_l(\xi) - 1| = |\widehat{\phi}(2^l \xi) - 1| \lesssim \min(1, |2^l \xi|).$$

For a positive integer j , let

$$(2.6) \quad \sigma_{l,t}^j(x) := \sigma_{l,t} * \phi_{l-j}(x),$$

and

$$\Omega(x) = |x|^{n-1} \sum_{l=-\infty}^{-1} 2^l \sigma_{l,t}^j(x) = |x|^{n-1} \sum_{l=-\infty}^{-1} \int_{\mathbb{R}^n} \frac{\Omega(y) \chi_{[2^l t, 2^{l+1} t]}(y)}{|y|^{n-1}} \phi_{l-j}(x-y) dy.$$

Note that

$$(2.7) \quad \sigma_{l,t}^j = \sigma_{l,t} * \phi_{l-j} = \sigma_{0,2^l t} * \phi_{l-j} = \sigma_{0,2^l t}^{j-l}.$$

Motivated by (2.4), we define the approximation operator \mathcal{M}_Ω^j by:

$$(2.8) \quad \mathcal{M}_\Omega^j f(x) := \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \sigma_{l,t}^j * f(x) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

By the Minkowski inequality, we obtain the following pointwise estimate:

Lemma 2.3. *Keep to the same notations above. Then*

$$(2.9) \quad \mathcal{M}_\Omega f(x) \leq 2 \left(\int_1^2 \sum_{l \in \mathbb{Z}} |\sigma_{l,t} * f(x)|^2 \frac{dt}{t} \right)^{1/2};$$

$$(2.10) \quad \mathcal{M}_\Omega^j f(x) \leq 2 \left(\int_1^2 \sum_{l \in \mathbb{Z}} |\sigma_{l,t}^j * f(x)|^2 \frac{dt}{t} \right)^{1/2};$$

$$(2.11) \quad \begin{aligned} & |\mathcal{M}_\Omega f(x) - \mathcal{M}_\Omega^j f(x)| \\ & \leq \sum_{l=-\infty}^{-1} 2^l \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\sigma_{k,t} * f(x) - \sigma_{k,t}^{j-l+k} * f(x)|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Proof. The proofs of (2.9) and (2.10) are simpler than that of (2.11). So we concentrate on (2.11). Note that

$$\begin{aligned} & |\mathcal{M}_\Omega f(x) - \mathcal{M}_\Omega^j f(x)| \\ & = \left| \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \sigma_{l,t} * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} - \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \sigma_{l,t}^j * f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \right| \\ & \leq \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l (\sigma_{l,t} * f(x) - \sigma_{l,t}^j * f(x)) \right|^2 \frac{dt}{t} \right)^{1/2}, \end{aligned}$$

by (2.3), (2.7) and the change of variables, we have

$$\begin{aligned} & |\mathcal{M}_\Omega f(x) - \mathcal{M}_\Omega^j f(x)| \\ & \leq \sum_{l=-\infty}^{-1} 2^l \left(\int_0^\infty |\sigma_{l,t} * f(x) - \sigma_{l,t}^j * f(x)|^2 \frac{dt}{t} \right)^{1/2} \\ & = \sum_{l=-\infty}^{-1} 2^l \left(\int_0^\infty |\sigma_{0,t} * f(x) - \sigma_{0,t}^{j-l} * f(x)|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Applying dyadic decomposition of $(0, \infty)$, we get

$$\begin{aligned} & |\mathcal{M}_\Omega f(x) - \mathcal{M}_\Omega^j f(x)| \\ & \leq \sum_{l=-\infty}^{-1} 2^l \left(\sum_{k \in \mathbb{Z}} \int_{2^k}^{2^{k+1}} |\sigma_{0,t} * f(x) - \sigma_{0,t}^{j-l} * f(x)|^2 \frac{dt}{t} \right)^{1/2} \\ & = \sum_{l=-\infty}^{-1} 2^l \left(\sum_{k \in \mathbb{Z}} \int_1^2 |\sigma_{k,t} * f(x) - \sigma_{k,t}^{j-l+k} * f(x)|^2 \frac{dt}{t} \right)^{1/2}. \quad \blacksquare \end{aligned}$$

One of the important observations for the proof of Theorem 1.1 is the following lemma:

Lemma 2.4. *Suppose that α and p satisfy*

$$\alpha > \frac{1}{2} \quad \text{and} \quad 1 + \frac{1}{2\alpha} < p < 1 + 2\alpha.$$

Let Ω be homogeneous of degree zero, have mean value zero and $\Omega \in G_\alpha(S^{n-1})$. Then the operator \mathcal{M}_Ω^j defined as (2.8) is bounded on $L^p(\mathbb{R}^n)$ and the operator norm is bounded by a constant independent of j .

Proof. It suffices to prove that

$$\left\| \left(\int_1^2 \sum_{l \in \mathbb{Z}} |\sigma_{l,t}^j * f|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \leq C \|f\|_{L^p}$$

from Lemma 2.3.

Let $\psi \in C^\infty(\mathbb{R}^n)$ -function such that

$$\text{supp}(\psi) \subset B(4) \setminus B(1) \quad \text{and that} \quad \sum_{j=-\infty}^\infty \psi(2^{-j}\xi)^2 \equiv \chi_{\mathbb{R}^n \setminus \{0\}}(\xi).$$

Define $S_l f(x) := \mathcal{F}^{-1}[\psi(2^{-l}\cdot)\mathcal{F}f](x)$. Then we have

$$\sigma_{l,t}^j * f = \sum_{k=-\infty}^{\infty} S_{l-k}[\sigma_{l,t}^j * S_{l-k}f].$$

We recall the key observation by Wu [22, (3.6)]: For each (measurable) collection $\{g_{t;l,k}\}$ of functions

$$(2.12) \quad \left\| \left(\sum_{l=-\infty}^{\infty} \int_1^2 \left| \sum_{k=-\infty}^{\infty} S_{l-k} g_{t;l,k} \right|^2 dt \right)^{1/2} \right\|_{L^p}^q \lesssim \sum_{k=-\infty}^{\infty} \left\| \left(\sum_{l=-\infty}^{\infty} \int_1^2 |g_{t;l,k}|^2 dt \right)^{1/2} \right\|_{L^p}^q$$

for any q , where the implicit constant depends only on p and q . Thus, by letting $g_{t;j,k} = \sigma_{l,t}^j * S_{j-k}f$, we obtain

$$\left\| \left(\int_1^2 \sum_{l \in \mathbb{Z}} |\sigma_{l,t}^j * f|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p}^q \leq C \sum_{k=-\infty}^{\infty} \left\| \left(\sum_{l=-\infty}^{\infty} \int_1^2 |\sigma_{l,t}^j * S_{l-k}f|^2 dt \right)^{1/2} \right\|_{L^p}^q.$$

In [22, p. 294], it was proved that

$$\left\| \left(\sum_{l=-\infty}^{\infty} \int_1^2 |\sigma_{l,t}^j * S_{l-k}f|^2 dt \right)^{1/2} \right\|_{L^p} \leq C \left\| \left(\sum_{l=-\infty}^{\infty} |S_{l-k}f|^2 \right)^{1/2} \right\|_{L^p}.$$

Thus,

$$\left\| \left(\int_1^2 \sum_{l \in \mathbb{Z}} |\sigma_{l,t}^j * f|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p}^q \leq C \sum_{k=-\infty}^{\infty} \left\| \left(\sum_{l=-\infty}^{\infty} |S_{l-k}f|^2 \right)^{1/2} \right\|_{L^p}^q.$$

It remains to use the well-known Littlewood-Paley theory and the similar argument in [22, p. 294], we can gain that the operator \mathcal{M}_{Ω}^j is bounded on $L^p(\mathbb{R}^n)$ for $1 + 1/(2\alpha) < p < 1 + 2\alpha$. ■

Lemma 2.5. *Let Ω be homogeneous of degree zero, integrable and have mean value zero. Define \mathcal{M}_{Ω} , \mathcal{M}_{Ω}^j by (2.4) and (2.8) respectively. If $\Omega \in G_{\alpha}(S^{n-1})$ for some $\alpha \in (1/2, \infty)$. Then for any $p \in (1 + 1/(2\alpha), 1 + 2\alpha)$, there exists a constant $\delta = \delta_{\alpha,p} > 0$ such that*

$$(2.13) \quad \|\mathcal{M}_{\Omega}f - \mathcal{M}_{\Omega}^j f\|_{L^p(\mathbb{R}^n)} \leq j^{-\delta} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. For each $\xi \in \mathbb{R}^n \setminus \{0\}$ and positive integer j , let l_0 be the integer such that $2^{j/2-1} < |2^{l_0}\xi| < 2^{j/2}$, $t \in [1, 2]$. Then, by Plancherel’s theorem, we have

$$\begin{aligned} \|\mathcal{M}_\Omega f - \mathcal{M}_\Omega^j f\|_{L^2(\mathbb{R}^n)}^2 &\lesssim \left\| \left(\int_1^2 \sum_{l \in \mathbb{Z}} |\sigma_{l,t} * f - \sigma_{l,t}^j * f|^2 dt \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)}^2 \\ &= \int_1^2 \int_{\mathbb{R}^n} \sum_{l \in \mathbb{Z}} |\sigma_{l,t} * f(x) - \sigma_{l,t}^j * f(x)|^2 dx dt \\ &= \int_1^2 \int_{\mathbb{R}^n} \sum_{l \in \mathbb{Z}} |\widehat{\sigma_{l,t}^j}(\xi) - \widehat{\sigma_{l,t}}(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi dt. \end{aligned}$$

A trivial computation involving the Lemma 2.1 with $t \in [1, 2]$ leads to that

$$\begin{aligned} \sum_{l \in \mathbb{Z}} |\widehat{\sigma_{l,t}^j}(\xi) - \widehat{\sigma_{l,t}}(\xi)|^2 &= \sum_{l \in \mathbb{Z}} \left(|\widehat{\sigma_{l,t}}(\xi)| |\widehat{\phi_{l-j}}(\xi) - 1| \right)^2 \\ &\lesssim \sum_{l \in \mathbb{Z}, l \leq l_0} |2^{l-j}\xi|^2 + \sum_{l \in \mathbb{Z}, l > l_0} \log^{-2\alpha}(|2^l\xi|) \\ &\lesssim 2^{-j} + j^{-2\alpha+1} \\ &\lesssim j^{-2\alpha+1}. \end{aligned}$$

Consequently,

$$\|\mathcal{M}_\Omega f - \mathcal{M}_\Omega^j f\|_{L^2(\mathbb{R}^n)} \leq j^{-\alpha+1/2} \|f\|_{L^2(\mathbb{R}^n)},$$

and

$$(2.14) \quad \|\mathcal{M}_\Omega^j f - \mathcal{M}_\Omega^{j+1} f\|_{L^2(\mathbb{R}^n)} \leq j^{-\alpha+1/2} \|f\|_{L^2(\mathbb{R}^n)},$$

since $\alpha > 3/2$. This implies that

$$(2.15) \quad \mathcal{M}_\Omega f - \mathcal{M}_\Omega^j f = \sum_{m=j}^{\infty} (\mathcal{M}_\Omega^{m+1} f - \mathcal{M}_\Omega^m f)$$

converges in the $L^2(\mathbb{R}^n)$ operator norm.

On the other hand, Lemma 2.4 tells us that for any positive integer m and $q \in (1 + 1/(2\alpha), 1 + 2\alpha)$,

$$(2.16) \quad \|\mathcal{M}_\Omega^m f - \mathcal{M}_\Omega^{m+1} f\|_{L^q(\mathbb{R}^n)} \leq \|\mathcal{M}_\Omega^m f\|_{L^q(\mathbb{R}^n)} + \|\mathcal{M}_\Omega^{m+1} f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)}.$$

Interpolation between the inequalities (2.14) and (2.16) then shows that for any $0 < \eta < 1$

$$(2.17) \quad \|\mathcal{M}_\Omega^m f - \mathcal{M}_\Omega^{m+1} f\|_{L^p(\mathbb{R}^n)} \lesssim j^{-\delta_{\alpha,p}} \|f\|_{L^p(\mathbb{R}^n)}$$

with $\delta_{\alpha,p} = (\alpha - 1/2)\eta$ and $1/p = \eta/2 + (1 - \eta)/(1 + 2\alpha)$ if $p \in (2, 1 + 2\alpha)$, or $1/p = \eta/2 + 2\alpha(1 - \eta)/(2\alpha + 1)$ if $p \in (1 + 1/(2\alpha), 2)$. Along with (2.15), a straightforward computation shows that $\delta_{\alpha,p} > 0$ when $p \in (1 + 1/(2\alpha), 1 + 2\alpha)$. This yields (2.13) and completes the proof of Lemma 2.5. ■

3. PROOF OF THEOREM 1.1

3.1. Reduction to the case when $b \in C_c^\infty(\mathbb{R}^n)$

First, let us justify that we can assume $b \in C_c^\infty(\mathbb{R}^n)$. To this end, we suppose that $b \in \text{CMO}(\mathbb{R}^n)$. Since for any $\varepsilon > 0$ there exists $b^\varepsilon \in C_c^\infty$ such that $\|b - b^\varepsilon\|_{\text{BMO}(\mathbb{R}^n)} < \varepsilon$, and that

$$\begin{aligned} & |\mathcal{M}_{\Omega,b}f(x) - \mathcal{M}_{\Omega,b^\varepsilon}f(x)| \\ & \leq \left\{ \int_0^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [(b - b^\varepsilon)(x) - (b - b^\varepsilon)(y)]f(y)dy \right|^2 \frac{dt}{t^3} \right\}^{1/2}. \end{aligned}$$

Then by Theorem B,

$$\|\mathcal{M}_{\Omega,b}f - \mathcal{M}_{\Omega,b^\varepsilon}f\|_{L^p(\mathbb{R}^n)} \lesssim \|b - b^\varepsilon\|_{\text{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Thus, to prove Theorem 1.1, we can suppose that $b \in C_c^\infty(\mathbb{R}^n)$.

3.2. Reduction to the case of smooth kernel

Let us assume $b \in C_c^\infty(\mathbb{R}^n)$ and define

$$\mathcal{M}_{\Omega,b}^j f(x) := \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} \sigma_{l,t} * \phi_{l-j}(x-y)(b(x) - b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}.$$

For $b \in C_c^\infty(\mathbb{R}^n)$, it is easy to see that

$$\begin{aligned} & |\mathcal{M}_{\Omega,b}^j f(x) - \mathcal{M}_{\Omega,b}f(x)| \\ & \leq \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} (\sigma_{l,t} * \phi_{l-j}(x-y) - \sigma_{l,t}(x-y))(b(x) - b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2} \\ & \lesssim \|b\|_{L^\infty(\mathbb{R}^n)} \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} (\sigma_{l,t} * \phi_{l-j}(x-y) - \sigma_{l,t}(x-y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2} \\ & \quad + \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} (\sigma_{l,t} * \phi_{l-j}(x-y) - \sigma_{l,t}(x-y))b(y)f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Thus by Lemma 2.5,

$$\begin{aligned} \|\mathcal{M}_{\Omega,b}^j f - \mathcal{M}_{\Omega,b} f\|_{L^p(\mathbb{R}^n)} &\lesssim j^{-\delta} \|b\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} + j^{-\delta} \|fb\|_{L^p(\mathbb{R}^n)} \\ &\lesssim j^{-\delta} \|b\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Therefore, we have only to prove the compactness of $\mathcal{M}_{\Omega,b}^j$.

3.3. Reduction by the use of the Fréchet-Kolmogorov theorem

So far, we reduced the matters to proving that $G \equiv \{\mathcal{M}_{\Omega,b} f : f \in F\}$ is strongly pre-compact in $L^p(\mathbb{R}^n)$, where F is a bounded set in $L^p(\mathbb{R}^n)$ and $b \in C_c^\infty(\mathbb{R}^n)$. We invoke the following criteria for compactness:

Theorem 3.1. (Fréchet-Kolmogorov, [25]). *A subset G of $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is strongly pre-compact if and only if it satisfies three conditions below:*

- (i) $\sup_{f \in G} \|f\|_{L^p(\mathbb{R}^n)} < \infty$;
- (ii) $\lim_{y \rightarrow 0} (\sup_{f \in G} \|f(\cdot + y) - f\|_{L^p(\mathbb{R}^n)}) = 0$;
- (iii) $\lim_{\beta \rightarrow \infty} (\sup_{f \in G} \|f \chi_{B(o,\beta)^c}\|_{L^p(\mathbb{R}^n)}) = 0$.

Since $\mathcal{M}_{\Omega,b}^j$ is bounded on $L^p(\mathbb{R}^n)$, (i) follows easily:

$$(3.1) \quad \sup_{f \in F} \|\mathcal{M}_{\Omega,b}^j f\|_{L^p(\mathbb{R}^n)} \lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} < \infty.$$

Therefore, the proof of Theorem 1.1 is completed once we prove the following lemma.

Lemma 3.2. *Suppose the parameters p and α satisfy the same condition as Theorem 1.1. Let $R > 0$ and F be a bounded subset of $L^p(\mathbb{R}^n)$. Assume that $b \in C_c^\infty(B(o, R))$ satisfies $\|b\|_{L^\infty(\mathbb{R}^n)} + \|\nabla b\|_{L^\infty(\mathbb{R}^n)} = 1$. Then*

$$(3.2) \quad \left(\sup_{f \in F} \|\chi_{B(o,A)^c} \mathcal{M}_{\Omega,b}^j f\|_{L^p(\mathbb{R}^n)} \right) \lesssim \left(\frac{R}{A} \right)^{p/p'} \|f\|_{L^p(\mathbb{R}^n)}$$

for all $A > 4R$ and

$$(3.3) \quad \lim_{z \rightarrow 0} \left(\sup_{f \in F} \|\mathcal{M}_{\Omega,b}^j f(\cdot + z) - \mathcal{M}_{\Omega,b} f\|_{L^p(\mathbb{R}^n)} \right) = 0.$$

The remaining part of this paper is the proof of this lemma.

3.4. Proof of (3.2)

It follows from Hölder’s inequality that for $x \in \mathbb{R}^n$ with $|x| > 4R$,

$$\begin{aligned} |\mathcal{M}_{\Omega,b}f(x)|^p &= \left\{ \int_0^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} [b(x) - b(y)] f(y) dy \right| \frac{dt}{t^3} \right\}^{p/2} \\ &= \left\{ \int_0^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} b(y) f(y) dy \right| \frac{dt}{t^3} \right\}^{p/2} \\ &\lesssim \left[\int_{\text{supp } b} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f(y)| \left(\int_{|x-y|\leq t} \frac{dt}{t^3} \right)^{1/2} dy \right]^p \\ &\lesssim \left(\int_{|y|\leq R} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy \right)^p \\ &\lesssim \left(\int_{|y|\leq R} |\Omega(x-y)| dy \right)^{p/p'} \frac{1}{|x|^{np}} \int_{|y|\leq R} |\Omega(x-y)| |f(y)|^p dy. \end{aligned}$$

Meanwhile, we have

$$\int_{|y|\leq R} |\Omega(x-y)| dy = \int_{B(x,R)} |\Omega(y)| dy \leq \int_{|x|-R < |y| < |x|+R} |\Omega(y)| dy \lesssim R|x|^{n-1}.$$

Inserting this estimate, we obtain

$$\begin{aligned} \int_{|x|>A} |\mathcal{M}_{\Omega,b}f(x)|^p dx &\lesssim R^{p/p'} \int_{|x|>A} \left(\int_{|y|<R} |f(y)|^p |\Omega(x-y)| dy \right) \frac{dx}{|x|^{n+p/p'}} \\ &\lesssim R^{p/p'} \|f\|_{L^p(\mathbb{R}^n)}^p \int_{|x|>(3A)/4} |\Omega(x)| \frac{dx}{|x|^{n+p/p'}} \\ &\lesssim \left(\frac{R}{A} \right)^{p/p'} \|f\|_{L^p(\mathbb{R}^n)}^p, \end{aligned}$$

which implies that (3.2) holds.

3.5. Proof of (3.3)

We begin with the following decomposition of $\mathcal{M}_{\Omega,b}^j f(x) - \mathcal{M}_{\Omega,b}^j f(x+z)$:

$$\begin{aligned} &\mathcal{M}_{\Omega,b}^j f(x) - \mathcal{M}_{\Omega,b}^j f(x+z) \\ &= \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} \sigma_{l,t} * \phi_{l-j}(x-y) (b(x) - b(y)) f(y) dy \right|^2 \frac{dt}{t} \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
 & - \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} \sigma_{l,t} * \phi_{l-j}(x+z-y)(b(x)-b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
 & + \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} \sigma_{l,t} * \phi_{l-j}(x+z-y)(b(x)-b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
 & - \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} \sigma_{l,t} * \phi_{l-j}(x+z-y)(b(x+z)-b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
 & \leq \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} (\sigma_{l,t} * \phi_{l-j}(x+z-y) - \sigma_{l,t} * \phi_{l-j}(x-y))(b(x)-b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
 & \quad + |b(x) - b(x+z)| \mathcal{M}_\Omega^j f(x+z).
 \end{aligned}$$

Likewise,

$$\begin{aligned}
 & \mathcal{M}_{\Omega,b}^j f(x+z) - \mathcal{M}_{\Omega,b}^j f(x) \\
 & \leq \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} (\sigma_{l,t} * \phi_{l-j}(x+z-y) - \sigma_{l,t} * \phi_{l-j}(x-y))(b(x)-b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
 & \quad + |b(x) - b(x+z)| \mathcal{M}_\Omega^j f(x+z).
 \end{aligned}$$

As a result, we obtain

$$\begin{aligned}
 & |\mathcal{M}_{\Omega,b}^j f(x+z) - \mathcal{M}_{\Omega,b}^j f(x)| \\
 & \leq \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} (\sigma_{l,t} * \phi_{l-j}(x+z-y) - \sigma_{l,t} * \phi_{l-j}(x-y))(b(x)-b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2} \\
 & \quad + |b(x) - b(x+z)| \mathcal{M}_\Omega^j f(x+z).
 \end{aligned}$$

Since $\|\nabla b\|_{L^\infty(\mathbb{R}^n)} \leq 1$,

$$\begin{aligned}
 \|(b - b(\cdot + z))\mathcal{M}_\Omega^j f(\cdot + z)\|_{L^p(\mathbb{R}^n)} & \leq |z| \cdot \|\mathcal{M}_\Omega^j f(\cdot + z)\|_{L^p(\mathbb{R}^n)} \\
 & = |z| \cdot \|\mathcal{M}_\Omega^j f\|_{L^p(\mathbb{R}^n)} \\
 & \leq C|z| \cdot \|f\|_{L^p(\mathbb{R}^n)}.
 \end{aligned}$$

Then, the proof of Theorem 1.1 will be completed once we prove the following estimate:

Lemma 3.3. *Let us define*

$$\mathcal{M}_{\Omega,b}^{z,-\infty,j} f(x) := \left(\int_0^\infty \left| \sum_{l=-\infty}^{-1} 2^l \int_{\mathbb{R}^n} (\sigma_{l,t}^j(x+z-y) - \sigma_{l,t}^j(x-y))(b(x)-b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}.$$

Then we have

$$\|\mathcal{M}_{\Omega,b}^{z,-\infty,j}\|_{L^p \rightarrow L^p} = o(1) \quad (z \rightarrow 0)$$

for any p in Theorem 1.1.

Proof. We may suppose $p = 2$ by interpolation and Theorem B. Let us consider

$$\begin{aligned} & \mathcal{M}_{\Omega,b}^{z,K,j} f(x) \\ & := \left(\int_0^\infty \left| \sum_{l=K}^{-1} 2^l \int_{\mathbb{R}^n} (\sigma_{l,t}^j(x+z-y) - \sigma_{l,t}^j(x-y))(b(x) - b(y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}. \end{aligned}$$

Then we have

$$\|\mathcal{M}_{\Omega,b}^{z,-\infty,j}\|_{L^p \rightarrow L^p} \leq 2^K M_0 + \|\mathcal{M}_{\Omega,b}^{z,K,j}\|_{L^p \rightarrow L^p} \leq 2^K M_0 + 2\|b\|_{L^\infty} \|\mathcal{M}_{\Omega}^{z,K,j}\|_{L^p \rightarrow L^p}$$

from Lemma 2.2, where

$$\mathcal{M}_{\Omega}^{z,K,j} f(x) := \left(\int_0^\infty \left| \sum_{l=K}^{-1} 2^l \int_{\mathbb{R}^n} (\sigma_{l,t}^j(x+z-y) - \sigma_{l,t}^j(x-y))f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}.$$

As we did in Lemma 2.2, we have

$$\begin{aligned} & \|\mathcal{M}_{\Omega}^{z,K,j}\|_{L^p \rightarrow L^p} \\ & \lesssim \sup_{\xi \in \mathbb{R}^n} |1 - e^{i\xi z}| \left(\int_0^\infty \sum_{l=K}^{-1} |\mathcal{F}\phi(2^{-j+l}\xi)| \min(|2^l t \xi|^2, (\log(3 + |2^l t \xi|))^{-2\alpha}) \frac{dt}{t} \right)^{1/2} \\ & = o(1) \quad (z \rightarrow 0), \end{aligned}$$

where the implicit constant depends on K and j . Combining these estimates, we obtain the desired result and complete the proof of Theorem 1.1. ■

ACKNOWLEDGMENT

The authors would like to express their deep gratitude to the referee for his/her invaluable comments and suggestions.

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