

ON A NEW MULTIPLE CRITICAL POINT THEOREM AND SOME APPLICATIONS TO ANISOTROPIC PROBLEMS

Marek Galewski

Abstract. Using the Fenchel-Young duality and mountain pass geometry we derive a new multiple critical point theorem. In a finite dimensional setting it becomes three critical point theorem while in an infinite dimensional case we obtain the existence of at least two critical points. The applications to anisotropic problems show that one can obtain easily that all critical points are nontrivial.

1. INTRODUCTION

Let E be a real reflexive Banach space. Given two continuously Fréchet differentiable convex functionals $\Phi, H : E \rightarrow \mathbb{R}$ with derivatives $\varphi, h : E \rightarrow E^*$ respectively, we undertake the existence and multiplicity of solutions to

$$(1) \quad \varphi(u) = h(u), u \in E$$

under some geometric conditions related to the existence of a minimizer over a certain set, a mountain pass geometry and, in a finite dimensional context, a direct global maximization. We denote by $J : E \rightarrow \mathbb{R}$ the action functional connected with (1), i.e. $J(u) = \Phi(u) - H(u)$ and therefore solutions to (1) correspond in a 1 – 1 manner to critical points of J . Thus we will provide a type of a multiple critical point theorem with applications to anisotropic boundary value problems, i.e. containing the continuous and discrete variable exponent Laplacian.

The method of obtaining the existence of multiple critical points differs somehow from the scheme within which such results were considered by Ricceri, see [20], and his followers, since we do not employ a type of a min-max inequality. Instead the existence of a first critical point is sought on a set - which need not be open - with the aid of the Fenchel-Young transform. Such ideas originate from [18] but we give

Received September 17, 2014, accepted March 5, 2015.

Communicated by Eiji Yanagida.

2010 *Mathematics Subject Classification*: 49J27, 39A12, 39A10.

Key words and phrases: Critical point, Multiplicity, $p(x)$ -Laplacian, Discrete $p(k)$ -Laplacian, Fenchel-Young transform.

here much simpler approach which we put into the context of the existence of a critical point to some functional and not to the direct investigation of a boundary value problem. Later several authors worked on improving the methodology contained in [18], see for example [10, 11] and references therein. The result which we present in this work seems to provide a most applicable version of all those mentioned which is connected with the so called dual method.

The second critical point is obtained with the aid of a general type of a Mountain Pass Lemma. In a finite dimensional case, we obtain a third critical point through the direct variational method. Thus in a finite dimensional case we use somehow different methodology from the one applied typically in the context of a three critical point theorems and which is suggested by the following observation: get two local minima/extrema and obtain a third through a mountain pass. In this work apart from the mountain pass solution we have a global minimizer over some set and a global maximizer.

We started related investigations in [11] but in this work we put them into some different framework and we apply our results for more general boundary value problems since we illustrate our results by examining the solvability of a Dirichlet problem with discrete and continuous $p(x)$ -Laplacian. Anisotropic boundary value problems are known to be mathematical models of various phenomena arising in the study of elastic mechanics (see [25]), electrorheological fluids (see [21]) or image restoration (see [5]). Variational continuous anisotropic problems have been started by Fan and Zhang in [8] and later considered by many methods and authors- see [13] for an extensive survey of such boundary value problems. In the discrete setting see for example [14, 16, 22] for the most recent results. For a background on variational methods we refer to [15, 24] while for a background on difference equations to [1]. The ideas connected with three critical point theorems - different from those used in this work - are to be found for example in [2, 20]. Let us mention [19] for some recent results concerning a general type of critical point theorem and [17] for some recent related result.

2. A MULTIPLE CRITICAL POINT THEOREM

We start this section with necessary mathematical prerequisites which are needed for the proof of the main multiplicity result. The Fenchel-Young dual for a convex Fréchet differentiable function $H : E \rightarrow \mathbb{R}$, see [6], reads

$$H^*(v) = \sup_{u \in E} \{\langle u, v \rangle - H(u)\}, \quad H^* : E^* \rightarrow \mathbb{R},$$

$\langle u, v \rangle$ stands for the duality pairing. Note that H^* and H^{**} , where H^{**} is defined in an obvious manner, are always convex l.s.c. functionals. The derivative of H at u is the subdifferential of H at u in the sense of convex analysis. We have the following relations

$$H(u) + H^*(v) = \langle u, v \rangle \iff v = h(u),$$

where h stand for the Fréchet derivative, and Fenchel-Young inequality

$$\langle p, u \rangle \leq H(u) + H^*(p)$$

which is valid for any $p \in E^*, u \in E$.

We also recall the following version of a Mountain Pass Lemma from [9]. Functional $J : E \rightarrow \mathbb{R}$ satisfies the Palais-Smale condition (PS-condition for short) if every sequence (u_n) such that $\{J(u_n)\}$ is bounded and $J'(u_n) \rightarrow 0$, has a convergent subsequence.

Lemma 1. (Mountain Pass Lemma, MPL Lemma). [9] *Let E be a Banach space and assume that $J \in C^1(E, \mathbb{R})$ satisfies the PS-condition. Let S be a closed subset of E which disconnects E . Let x_0 and x_1 be points of E which are in distinct connected components of $E \setminus S$. Suppose that J is bounded below in S , and in fact the following condition is verified for some b*

$$(2) \quad \inf_{x \in S} J(x) \geq b \text{ and } \max\{J(x_0), J(x_1)\} < b.$$

If we denote by Γ the family of continuous paths $\gamma : [0, 1] \rightarrow E$ joining x_0 and x_1 , then

$$c := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} J(\gamma(s)) \geq \max\{J(x_0), J(x_1)\} > -\infty$$

is a critical value and J has a non-zero critical point x at level c .

Now we can state our main result.

Theorem 2. *Let E be a infinite dimensional reflexive Banach space.*

(i) *Let $X \subset E$ and let there exist $u, v \in X$ satisfying $\varphi(v) = h(u)$, and such that*

$$J(u) \leq \inf_{x \in X} J(x).$$

Then u is a critical point to J , and thus it solves (1).

(ii) *Let S be a set disconnecting E such that*

$$J(u) \leq \inf_{x \in X} J(x) < \inf_{x \in S} J(x).$$

Assume that there exists $w \in E$ with $\lim_{t \rightarrow \infty} J(tw) = -\infty$ and that J satisfies the PS-condition. If u belongs to a bounded component of $E \setminus S$, then there exists a non-zero critical point z different from u .

Proof. The proof that u is a critical point, i.e. the proof of part (i) follows by Theorem 1 from [11] but we provide it for readers convenience shortening and

simplifying some arguments used there. We put $p = \varphi(v) = h(u)$. Since $\frac{d}{du}\Phi = \varphi$, $\frac{d}{du}H = h$ we have by the definition of p

$$(3) \quad \Phi(v) = \langle v, p \rangle - \Phi^*(p) \text{ and } H(u) = \langle u, p \rangle - H^*(p).$$

By the Fenchel-Young inequality $-H(v) \leq H^*(p) - \langle p, v \rangle$ and by the first relation in (3) we have

$$\begin{aligned} \Phi(u) - H(u) &= J(u) \leq J(v) = \Phi(v) - H(v) \\ &= \langle v, p \rangle - \Phi^*(p) - H(v) \leq H^*(p) - \Phi^*(p). \end{aligned}$$

So by the Fenchel-Young inequality,

$$\langle u, p \rangle \leq \Phi(u) + \Phi^*(p) \leq H(u) + H^*(p) = \langle u, p \rangle.$$

Thus $\langle u, p \rangle = \Phi(u) + \Phi^*(p)$, and so, recalling the definition of p we see that $p = \frac{d}{du}\Phi(u) = \varphi(u) = h(u)$. This means that u is a critical point.

In order to prove part (ii), i.e. in order to get the second critical point, we will use Lemma 1. Since $\lim_{t \rightarrow \infty} J(tw) = -\infty$, so there exists some w_1 such that $J(w_1) \leq \inf_{x \in X} J(x) < \inf_{x \in S} J(x)$. Thus we have condition (2) satisfied taking $x_0 = u$ and $x_1 = w_1$. The existence of a second non-zero critical point readily follows. ■

In a finite dimensional context, we get easily the existence of a third critical point as follows:

Theorem 3. *Let E be a finite dimensional Banach. Let $X \subset E$ and let there exist $u, v \in X$ satisfying $\varphi(v) = h(u)$, and such that*

$$J(u) \leq \inf_{x \in X} J(x).$$

Then u is a critical point to J , and thus it solves (1). Let S be a set disconnecting E and assume that J is anti-coercive and that u belongs to a bounded component of $E \setminus S$. If moreover,

$$J(u) \leq \inf_{x \in X} J(x) < \inf_{x \in S} J(x)$$

then there exists additional two distinct critical points, both different from u , one of which is non-zero.

Proof. Note that in a finite dimensional setting an anti-coercive functional necessarily satisfies the PS-condition. Thus the existence of two distinct solutions, u and some $z \neq 0$, follows by Theorem 2. Since J is anti-coercive and continuous it has an argument of a maximum which we denote by w . Since J is differentiable it follows that w is a critical point. Since

$$\max\{J(z), J(u)\} \leq \inf_{x \in X} J(x) < \inf_{x \in S} J(x) \leq \sup_{x \in S} J(x) \leq \sup_{x \in E} J(x)$$

we see that w is a third critical point distinct from the previous ones. ■

Remark 4. Somehow related results are also contained in [4], where it is proved that if $J : E \rightarrow \mathbb{R}$ satisfies the PS-condition and if $X \subset E$ is an open set such that

$$\inf_{x \in \overline{X}} J(x) < \inf_{x \in \partial X} J(x)$$

then there is some u which is a critical point to J such that $\inf_{x \in \overline{X}} J(x) = J(u)$. The difference between this result and ours is that we do not assume the PS-condition to be satisfied and we do not use the Ekelend’s variational principle in the proof, but instead we impose convexity and use the Fenchel-Young transform. Moreover, we do not need to assume that X is open and u need not to be global minimizer.

The multiplicity result in [4] is also obtained with the aid of Mountain Pass Lemma in the following context. Let X be an open ball centered at 0 with radius r and let $J(0) = 0$, $J : E \rightarrow \mathbb{R}$ satisfies the PS-condition, there exists an element $e \in E \setminus X$ such that $J(e) \leq 0$. If additionally

$$-\infty < \inf_{x \in \overline{X}} J(x) < 0 < \inf_{x \in \partial X} J(x),$$

then J has two critical points. Compared with our results we do not need to take X as a ball and also we do need to know that $\inf_{x \in \partial X} J(x) > 0$.

By using methods mentioned in the above remark in [23] the Author obtains the existence of at least two non-zero solutions for some periodic and Neumann problems with the discrete $p(k)$ –Laplacian.

3. APPLICATION TO THE DISCRETE ANISOTROPIC EQUATIONS

For fixed a, b such that $a < b < \infty$, $a \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$ we denote $\mathbb{N}(a, b) = \{a, a + 1, \dots, b - 1, b\}$. Consider the following anisotropic discrete problem

$$(4) \quad \begin{aligned} -\Delta (\phi_{p(k-1)} (\Delta x(k - 1))) &= \lambda f(k, x(k)), \quad k \in \mathbb{N}(1, T), \\ x(0) &= x(T + 1) = 0 \end{aligned}$$

where $\lambda > 0$ is a numerical parameter, $\phi_{p(k)}(t) = |t|^{p(k)-2}t$, $p : \mathbb{N}(0, T) \rightarrow \mathbb{R}$, $p^+ = \max_{k \in \mathbb{N}(0, T)} p(k) \geq 2$; Δ is the forward difference operator defined by $\Delta x(k) = x(k + 1) - x(k)$. Let $F(t, \xi) = \int_0^\xi f(t, s) ds$ for $(t, \xi) \in (\mathbb{N}(1, T) \times \mathbb{R})$. We also denote $p^- = \min_{k \in \mathbb{N}(0, T)} p(k)$.

We will employ the following assumptions.

H0 $f \in C(\mathbb{N}(1, T) \times \mathbb{R}; \mathbb{R})$;

H1 function $x \rightarrow F(k, x)$ is convex on \mathbb{R} for all $k \in \mathbb{N}(1, T)$;

H2 there exist constants $\mu > p^+$, $c_1 > 0$, $c_2 \in \mathbb{R}$, $d > 0$ and $m > d$ such that

$$F(k, x) \geq c_1|x|^\mu + c_2$$

for all $k \in \mathbb{N}(1, T)$ and all $|x| \geq m$.

The assumptions employed here are not very restrictive. There are many functions satisfying both **H1** and **H2**. See for example $F(k, x) = c_1|x|^\mu + c_2$ with even and sufficiently large μ . By a solution x of (4) we mean such a function $x : \mathbb{N}(0, T + 1) \rightarrow \mathbb{R}$ which satisfies the given equation on $\mathbb{N}(1, T)$ and the given boundary conditions. Solutions to (4) will be investigated in a space E of functions $x : \mathbb{N}(0, T + 1) \rightarrow \mathbb{R}$ such that $x(0) = x(T + 1) = 0$; E is considered with the following equivalent norms

$$\|x\| = \left(\sum_{k=1}^{T+1} |\Delta x(k-1)|^2 \right)^{\frac{1}{2}} \text{ and } \|x\|_0 = \left(\sum_{k=1}^T |x(k)|^2 \right)^{\frac{1}{2}}.$$

Note that for $c_b = \frac{1}{2}$ and $c_a = (T(T + 1))^{1/2}$ we have

$$(5) \quad c_b \|x\| \leq \|x\|_0 \leq c_a \|x\| \text{ for all } x \in E.$$

As in [3] we can use the Luxemburg norm

$$\|x\|_{p(\cdot)} = \inf \left\{ v > 0 : \sum_{k=1}^{T+1} \left| \frac{\Delta x(k-1)}{v} \right|^{p(k-1)} \leq 1 \right\}$$

such that there exist constants $L_1 > 0, L_2 > 1$

$$(6) \quad L_1 \|x\|_{p(\cdot)} \leq \|x\| \leq L_2 \|x\|_{p(\cdot)} \text{ for all } x \in E.$$

Now, if $\varphi : E \rightarrow \mathbb{R}$

$$\varphi(x) = \sum_{k=1}^{T+1} |\Delta x(k-1)|^{p(k-1)},$$

then we have the following inequalities

$$(7) \quad \|x\|_{p(\cdot)}^{p^-} \leq \varphi(x) \leq \|x\|_{p(\cdot)}^{p^+} \text{ for } \|x\|_{p(\cdot)} > 1.$$

Solutions to (4) correspond to the critical points to the following C^1 functional $\mathcal{I} : E \rightarrow \mathbb{R}$

$$\mathcal{I}(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^T F(k, x(k)).$$

Lemma 5. *Assume that **H0**, **H2** are satisfied. Then for any $\lambda > 0$ functional \mathcal{I} is anti-coercive, i.e. $\mathcal{I}(x) \rightarrow -\infty$ as $\|x\| \rightarrow +\infty$.*

Proof. From Hölder’s inequality, by **H2** and(5) we get for any $x \in E$

$$\sum_{k=1}^T F(k, x(k)) \geq c_1 T^{\frac{2-\mu}{2}} (c_b)^\mu \|x\|^\mu + c_2 T.$$

We see for $\|x\|_{p(\cdot)} > 1$ by relations (6), (7) that

$$\begin{aligned} \mathcal{I}(x) &\leq \frac{1}{p^-} \sum_{k=1}^{T+1} |\Delta x(k-1)|^{p(k-1)} - \lambda c_1 \sum_{k=1}^T |x(k)|^\mu - \lambda c_2 T \\ &\leq \frac{1}{p^-} \|x\|_{p(\cdot)}^{p^+} - \lambda c_1 T^{\frac{2-\mu}{2}} (L_1 c_b)^\mu \|x\|_{p(\cdot)}^\mu - \lambda c_2 T. \end{aligned}$$

Hence $\mathcal{I}(x) \rightarrow -\infty$ as $\|x\| \rightarrow +\infty$. ■

Theorem 6. *Assume that conditions **H0-H2** are satisfied. There exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$ problem (4) has at least three nontrivial solutions.*

Proof. By Lemma 5 functional \mathcal{I} is anticoercive. Therefore there is some element $z \in E$ such that $J(z) < 0$. Let us define a set $D \subset E$ as a ball with radius $r > 1$ with respect to the Luxemburg norm, i.e.

$$D = \left\{ x \in E : \|x\|_{p(\cdot)} \leq r \right\},$$

where $r > 1$ is chosen so that $z \in D$. Let $S = \partial D$. Denote by d the maximal value of functional $x \rightarrow \sqrt{\sum_{k=1}^T f^2(k, x(k))}$ over D . Put

$$\lambda^* = \frac{r^{p^- - 1}}{L_2 d c_a}$$

and fix $\lambda \in (0, \lambda^*]$. We shall apply Theorem 3. Define $\Phi, H : E \rightarrow \mathbb{R}$

$$\Phi(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)}, \quad H(x) = \lambda \sum_{k=1}^T F(k, x(k))$$

and note that these are convex C^1 functionals.

Since \mathcal{I} is continuous and D is closed and bounded, we see that there exists an argument of a minimum of \mathcal{I} over D , which we denote by u . Note that u belongs to a bounded component of $E \setminus S$ and that $J(u) \leq J(z) < 0$ so that $u \neq 0$ since $J(0) = 0$. Consider the auxiliary Dirichlet problem

$$(8) \quad \begin{aligned} -\Delta (\phi_{p(k-1)} (\Delta x(k-1))) &= \lambda f(k, u(k)), \quad k \in \mathbb{N}(1, T), \\ x(0) = x(T+1) &= 0. \end{aligned}$$

Note that problem (8) is uniquely solvable by some $v \in E$. This follows since the action functional corresponding to (8)

$$J_1(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^T f(k, u(k))x(k)$$

is coercive, C^1 and strictly convex.

We shall prove that $v \in D$. Suppose $\|v\|_{p(\cdot)} \geq 1$. Multiplying (8) with $x = v$ by v and summing from 1 to T we have

$$(9) \quad \sum_{k=1}^{T+1} |\Delta v(k-1)|^{p(k-1)} = \lambda \sum_{k=1}^T f(k, u(k))v(k).$$

By relation (7) we see that $\sum_{k=1}^{T+1} |\Delta v(k-1)|^{p(k-1)} \geq \|v\|_{p(\cdot)}^{p^-}$. By Schwartz inequality and the definition of d we obtain by (5), (6) that

$$\sum_{k=1}^T f(k, u(k))v(k) \leq \sqrt{\sum_{k=1}^T f^2(k, u(k))} \|v\|_0 \leq L_2 d c_a \|v\|_{p(\cdot)}.$$

Since $\lambda \leq \frac{r^{p^- - 1}}{L_2 d c_a}$ we see from (9) that $\|v\|_{p(\cdot)}^{p^- - 1} \leq \lambda L_2 d \leq r$. If $\|v\|_{p(\cdot)} \leq 1$ the conclusion is immediate. Thus $v \in D$ and so Theorem 3 applies. ■

When convexity is not assumed, it is easy to see that problem (4) has at least one solution for an $\lambda > 0$ since functional J is anti-coercive and continuous and the space E is finite dimensional. Thus we have the following

Theorem 7. *Assume that conditions **H0-H2** are satisfied. Then for any $\lambda > 0$ problem (4) has at least one solution.*

4. APPLICATIONS TO CONTINUOUS ANISOTROPIC PROBLEMS

In this section we mean for applications to continuous problems. Let $p, q \in C([0, \pi], \mathbb{R}^+)$, $1/p(t) + 1/q(t) = 1$ for $t \in [0, \pi]$. We assume

$$p^- = \inf_{t \in [0, \pi]} p(t) > 1$$

and we let $p^+ = \sup_{t \in [0, \pi]} p(t)$. In the space $W_0^{1,p(t)}(0, \pi)$ (see [7, 8]) we consider the following norm

$$\|x\|_{W_0^{1,p(t)}} = \left\| \frac{d}{dt} x \right\|_{L^{p(t)}} = \inf \left\{ \lambda > 0 \mid \int_0^\pi \left| \frac{\frac{d}{dt} x(t)}{\lambda} \right|^{p(t)} dt \leq 1 \right\},$$

where $\frac{d}{dt}$ stands for an a.e. derivative. From [7] we see that there exists a constant $C_1 > 0$ such that (Poincaré inequality)

$$(10) \quad \|x\|_{L^{p(t)}} \leq C_1 \left\| \frac{d}{dt} x \right\|_{L^{p(t)}} \text{ for all } x \in W_0^{1,p(t)}(0, \pi).$$

The Hölder’s type inequality reads for $x \in L^{p(t)}(0, \pi)$ and $y \in L^{q(t)}(0, \pi)$

$$(11) \quad \int_0^\pi x(t) y(t) dt \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|x\|_{L^{p(t)}} \|y\|_{L^{q(t)}} \leq 2 \|x\|_{L^{p(t)}} \|y\|_{L^{q(t)}}.$$

The functional $x \rightarrow \int_0^\pi \left| \frac{d}{dt} x(t) \right|^{p(t)} dt$ is called a modular for $W_0^{1,p(t)}(0, \pi)$. We have the following relation between a modular and a norm

$$\int_0^\pi \left| \frac{d}{dt} x(t) \right|^{p(t)} dt \geq \min \left\{ \left\| \frac{d}{dt} x \right\|_{L^{p(t)}}^{p^-}, \left\| \frac{d}{dt} x \right\|_{L^{p(t)}}^{p^+} \right\}$$

and

$$\int_0^\pi \left| \frac{d}{dt} x(t) \right|^{p(t)} dt \leq \max \left\{ \left\| \frac{d}{dt} x \right\|_{L^{p(t)}}^{p^-}, \left\| \frac{d}{dt} x \right\|_{L^{p(t)}}^{p^+} \right\}.$$

Let us consider an operator $L : W_0^{1,p(t)}(0, \pi) \rightarrow (W_0^{1,p(t)}(0, \pi))^*$ given by

$$(12) \quad \langle L(x), v \rangle = \int_0^\pi \left| \frac{d}{dt} x(t) \right|^{p(t)-2} \frac{d}{dt} x(t) \frac{d}{dt} v(t) dt$$

for $x, v \in W_0^{1,p(t)}(0, \pi)$. Then L is a homeomorphism [8, Theorem 3.1] and the Gâteaux derivative of $x \rightarrow \int_0^\pi \left| \frac{d}{dt} x(t) \right|^{p(t)} dt$ is given by (12).

For regularity of solutions to the problem under consideration we shall need a Fundamental Lemma of the Calculus of Variation in the form given by [26].

Lemma 8. [26] *If $g, h \in L^1(0, \pi)$ and*

$$\int_0^\pi \left(g(t) y(t) + h(t) \frac{d}{dt} y(t) \right) dt = 0$$

for all $y \in C_0^\infty(0, \pi)$, then $\frac{d}{dt} h = g$ a.e. on $[0, \pi]$ and $\frac{d}{dt} h \in L^1(0, \pi)$.

Now we may state the problem under consideration. We consider the problem of solving in $W_0^{1,p(t)}(0, \pi)$

$$(13) \quad \begin{aligned} -\frac{d}{dt} \left(\left| \frac{d}{dt} x(t) \right|^{p(t)-2} \frac{d}{dt} x(t) \right) &= \lambda f(t, x(t)) \\ x(0) &= x(\pi) = 0 \end{aligned}$$

with a numerical parameter $\lambda > 0$. Let $F(t, v) = \int_0^v f(t, \tau) d\tau$. We assume that

H3 $f : [0, \pi] \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function;

H4 there exists a constant $\theta > p^+$ such that for $v \in \mathbb{R}, v \neq 0$ and a.e. $t \in [0, \pi]$

$$0 < \theta F(t, v) \leq v f(t, v);$$

H5 there exist constants $\beta_1, \alpha > 0, \beta_2 \geq 0$ with $\alpha > p^+$ and such that for all $v \in \mathbb{R}$ and a.e. $t \in [0, \pi]$

$$|f(t, v)| \leq \beta_1 |v|^{\alpha-1} + \beta_2;$$

H6 $\lim_{v \rightarrow 0} \frac{|f(t, v)|}{|v|^{p^+-1}} = 0$ uniformly for a.e. $t \in [0, \pi]$;

H7 function $x \rightarrow F(t, x)$ is convex on \mathbb{R} for a.e. $t \in [0, \pi]$.

By convexity we get only that $F(t, v) \leq v f(t, v)$ for $v \in \mathbb{R}$ and a.e. $t \in [0, \pi]$ thus we need to assume the A-R condition. Relaxed version of the A-R conditions could also be assumed.

Following remarks about the modular and using the growth conditions we see that the action functional $J : W_0^{1,p(t)}(0, \pi) \rightarrow \mathbb{R}$ given by

$$(14) \quad J(x) = \int_0^\pi \frac{1}{p(t)} \left| \frac{d}{dt} x(t) \right|^{p(t)} - \lambda \int_0^\pi F(t, x(t)) dt$$

is continuously Gâteaux differentiable. Thus it is a C^1 functional. Weak solutions to (13) are critical points to J . By Lemma 8 it follows that any weak solution, i.e. a function x satisfying

$$\int_0^\pi \left| \frac{d}{dt} x(t) \right|^{p(t)-2} v(t) dt = \lambda \int_0^\pi f(t, x(t)) v(t) dt \text{ for all } v \in W_0^{1,q(t)}(0, \pi)$$

is a classical one, i.e. it is an absolutely continuous function such that $\frac{d^2}{dt^2} x$ exists for a.e. $t \in [0, \pi]$ and $\frac{d^2}{dt^2} x \in L^1(0, \pi)$. From [12] we get the two lemmas concerning the mountain geometry for (13). While in [12] these are given for the impulsive problem, it is to derive their counterparts for the non-impulsive one.

Lemma 9. *Suppose that **H3, H4, H5** hold. Then for any $\lambda > 0$ the functional J given by (14) satisfies the PS-condition.*

Lemma 10. *Suppose that **H3-H6** hold. Then for any $\lambda > 0$ there exist numbers $\eta, \xi > 0$ such that $J(x) \geq \xi$ for all $x \in W_0^{1,p(t)}(0, \pi)$ with $\|x\|_{W_0^{1,p(t)}} = \eta$. Moreover, there exists an element $z \in W_0^{1,p(t)}(0, \pi)$ with $\|z\|_{W_0^{1,p(t)}} > \eta$ and such that $J(z) < 0$.*

Using Mountain Pass Lemma 1 we get the following

Proposition 11. *Suppose that **H3-H6** hold. Then for any $\lambda > 0$ problem (13) has at least one nontrivial solution.*

Concerning the multiple solutions we have the main result of this section

Theorem 12. *Assume that conditions **H3-H7** are satisfied. Then there exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$ problem (13) has at least two nontrivial solutions.*

Proof. From Lemma 10 it follows that there exists an element $z \in W_0^{1,p(t)}(0, \pi)$ with $\|z\|_{W_0^{1,p(t)}} > \eta$ and such that $J(z) < 0$. Let us define a set $D \subset E$ as a closed ball with radius $r > 1$ centred at 0 and containing z . Let $S = \partial D$. Consider a functional $J_1 : W_0^{1,p(t)}(0, \pi) \rightarrow \mathbb{R}$

$$J_1(x) = \|f(\cdot, x(\cdot))\|_{L^q(t)}$$

over D . Since J_1 is considered on $W_0^{1,p(t)}(0, \pi)$, it is immediate that any function from D is continuous, thus we see by **H5** and classical Weierstrass Theorem that J_1 is bounded from the above on D . Let $\{x_k\}_{k=1}^\infty \subset D$ be a maximizing sequence, which, since D is closed, bounded and convex, has a weakly convergent subsequence in $W_0^{1,p(t)}(0, \pi)$ and this subsequence can be chosen so that it converges strongly in $C[0, \pi]$. Thus standard arguments show that J_1 has an argument of maximum u_1 over D and let us denote by d the corresponding maximal value.

Note that J is weakly l.s.c. and D is weakly compact, so J has an argument of a minimum over D which we denote by u . Note that $J(u) < 0$ since $z \in D$ such that $J(z) < 0$. Thus $u \neq 0$. Note that u belongs to a bounded component of $E \setminus S$. Put

$$\lambda^* = \frac{r^{p^+-1}}{C_1 \left(\frac{1}{p^-} + \frac{1}{q^-} \right) d}$$

and fix $\lambda \in (0, \lambda^*]$.

We shall apply Theorem 2. By Lemma 9 functional J satisfies the PS-condition. Put $\Phi, H : W_0^{1,p(t)}(0, \pi) \rightarrow \mathbb{R}$ by formulas

$$\Phi(x) = \int_0^\pi \frac{1}{p(t)} \left| \frac{d}{dt} x(t) \right|^{p(t)}, \quad H(x) = \lambda \int_0^\pi F(t, x(t)) dt$$

and note that these are convex C^1 functionals.

Functional Φ is strictly convex and coercive and so the direct method of the calculus of variations provides that the following auxiliary Dirichlet problem

$$(15) \quad \begin{aligned} -\frac{d}{dt} \left(\left| \frac{d}{dt} x(t) \right|^{p(t)-2} \frac{d}{dt} x(t) \right) &= \lambda f(t, u(t)) \\ x(0) = x(\pi) &= 0 \end{aligned}$$

is uniquely solvable by some $x \in W_0^{1,p(t)}(0, \pi)$. Multiplying the equation in (15) by a test function x and integrating by parts we get

$$\int_0^\pi \left| \frac{d}{dt} x(t) \right|^{p(t)} dt = \lambda \int_0^\pi f(t, u(t)) x(t) dt.$$

Suppose that $\|x\| \geq 1$ since otherwise we see that $x \in D$. By relation between a modular and a norm we see that

$$\int_0^\pi \left| \frac{d}{dt} x(t) \right|^{p(t)} dt \geq \left\| \frac{d}{dt} x \right\|_{L^{p(t)}}^{p^+}.$$

By Schwartz inequality, definition of d and (11), (10) we get

$$\begin{aligned} \int_0^\pi f(t, u(t)) x(t) dt &\leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|f(\cdot, u(\cdot))\|_{L^{q(t)}} \|x\|_{L^{p(t)}} \\ &\leq C_1 \left(\frac{1}{p^-} + \frac{1}{q^-} \right) d \left\| \frac{d}{dt} x \right\|_{L^{p(t)}}. \end{aligned}$$

Thus

$$\left\| \frac{d}{dt} x \right\|_{L^{p(t)}}^{p^+-1} \leq \lambda C_1 \left(\frac{1}{p^-} + \frac{1}{q^-} \right) d$$

and so $\left\| \frac{d}{dt} x \right\|_{L^{p(t)}} \leq \sqrt[p^+-1]{\lambda C_1 \left(\frac{1}{p^-} + \frac{1}{q^-} \right) d} \leq r$ by definition of λ^* . Thus $v \in D$ and therefore Theorem 2 applies. ■

REFERENCES

1. R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.
2. G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **75** (2012), 2992-3007.
3. C. Bereanu, P. Jebelean and C. Serban, Periodic and Neumann problems for discrete $p(\cdot)$ -Laplacian, *J. Math. Anal. Appl.*, **399** (2013), 75-87.
4. C. Bereanu, P. Jebelean and J. Mawhin, Multiple solutions for Neumann and periodic problems with singular ϕ -Laplacian, *J. Funct. Anal.*, **261** (2011), 3226-3246.
5. Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. Appl. Math.*, **66** (2006), 1383-1406.
6. I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
7. X. L. Fan and D. Zhao, On the Spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, *J. Math. Analysis Appl.*, **263** (2001), 424-446.
8. X. L. Fan and D. Zhang, Existence of solutions for $p(x)$ -Laplacian Dirichlet problem, *Nonl. Analysis*, **52** (2003), 1843-1852.
9. D. G. Figueredo, *Lectures on the Ekeland Variational Principle with Applications and Detours*, Preliminary Lecture Notes, SISSA, 1988.

10. M. Galewski, A. Nowakowski, D. O'Regan and A. Orpel, The dual variational method for n -th order ODEs with multipoint boundary conditions, *Appl. Anal.*, (2014), 957-971.
11. M. Galewski and E. Galewska, On a new critical point theorem and some applications to discrete equations, *Opusc. Math.*, **34(4)** (2014), 725-732.
12. M. Galewski and D. O'Regan, Impulsive boundary value problems for $p(t)$ -Laplacian's via critical point theory, *Czechoslovak Math. J.*, **62** (2012), 951-967
13. P. Harjulehto, P. Hästö, U. V. Le and M. Nuortio, Overview of differential equations with non-standard growth, *Nonlinear Anal.*, **72** (2010), 4551-4574.
14. A. Iannizzotto and V. Radulescu, Positive homoclinic solutions for the discrete p -Laplacian with a coercive potential, *Differential and Integral Equations*, **27** (2014), 35-44.
15. A. Kristály, V. Radulescu and C. Varga, *Variational Principles in Mathematical Physics, Geometry and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics, No. 136, Cambridge University Press, Cambridge, 2010.
16. N. Marcu and G. Molica Bisci, *Existence and multiplicity of solutions for nonlinear discrete inclusions*, Electron. J. Differential Equations 2012, No. 192, 13 pp.
17. Q. Meng, *Three periodic solutions for a class of ordinary p -Hamiltonian systems*, Bound. Value Probl. 2014, Article ID 150, electronic only (2014).
18. A. Nowakowski, A new variational principle and duality for periodic solutions of Hamilton's Equations, *J. Differential Equations*, **97** (1992), 174-188.
19. R. Precup, On a bounded critical point theorem of Schechter, *Stud. Univ. Babeş-Bolyai Math.*, **58** (2013), 87-95.
20. B. Ricceri, A three critical points theorem revisited, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **70** (2009), 3084-3089
21. M. Růžička, Electrorheological fluids: Modelling and mathematical theory, in: *Lecture Notes in Mathematics*, Vol. 1748, Springer-Verlag, Berlin, 2000.
22. C. Şerban, Existence of solutions for discrete p -Laplacian with potential boundary conditions, *J. Difference Equ. Appl.*, **19** (2013), 527-537.
23. C. Şerban, Multiplicity of solutions for periodic and Neumann problems involving the discrete (p)-Laplacian, *Taiwanese J. Math.*, **17(4)** (2013), 1425-1439
24. M. Struwe, *Variational Methods*, Springer, Berlin, 1996.
25. V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izv.*, **29** (1987), 33-66.
26. L. C. Young, *Lectures on the calculus of variations and optimal control theory*, W. B. Saunders Company, Philadelphia, 1969.

Marek Galewski
Institute of Mathematics
Technical University of Lodz
Wolczanska 215, 90-924 Lodz
Poland
E-mail: marek.galewski@p.lodz.pl