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# A CLASS OF NEW BI-INVARIANT METRICS ON THE HAMILTONIAN DIFFEOMORPHISM GROUPS

Guangcun Lu* and Tie Sun

**Abstract.** In this paper, we construct infinitely many bi-invariant metrics on the Hamiltonian diffeomorphism group and study their basic properties and corresponding generalizations of the Hofer inequality and Sikorav one.

## 1. Introduction and Main Results

### 1.1. The Hofer metric

In 1989, H. Hofer [12] constructed a remarkable bi-invariant Finsler metric on the group of compactly supported Hamiltonian diffeomorphisms $\mathrm{Ham}(M, \omega)$ of a symplectic manifold $(M, \omega)$, nowadays known as Hofer metric. Since then the intrinsic geometry of it has been being a very active and fruitful research field in symplectic topology and Hamiltonian dynamics (see the books [14, 18, 27], and the surveys [10, 19, 24, 28] and references therein for current progress situation).

Especially, a recent celebrated result made by Buhovsky and Ostrover [5] is a positive answer to the uniqueness question of the Hofer metric raised by Eliashberg and Polterovich [9]. They showed that up to equivalence of metrics the Hofer metric is the only bi-invariant Finsler metric on the group of Hamiltonian diffeomorphisms of a closed symplectic manifold under a natural assumption. For studies of non-Finslerian bi-invariant metrics on $\mathrm{Ham}(M, \omega)$ the readers may refer to [22, 29, 31].

Let us briefly review the construction of the Hofer metric following the notations in [27] without special statements. The readers who are familiar with it may directly

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*Corresponding author.

read the next section. Let $(M^{2n}, \omega)$ be a connected symplectic manifold of dimension $2n$ without boundary. Denote by $\mathcal{A}(M)$ the space of all smooth functions on $M$ with compact support (resp. zero-mean with respect to the canonical volume form $\omega^n$) if $M$ is open (resp. closed). A (time-dependent) smooth Hamiltonian function $F$ on $M \times I$, where $I \subset \mathbb{R}$ is an interval, is called **normalized** if $F_t = F(\cdot, t)$ belongs to $\mathcal{A}(M)$ for all $t$, and $\cup_{t\in I}\mathrm{supp}(F_t)$ is contained in a compact subset of $M$ in the case when $M$ is open. Such a normalized $F$ determines a (time-dependent) Hamiltonian vector field $X_{F_t}$ on $M$ via $i_{X_{F_t}}\omega = -dF_t$, and when $I = [0, 1]$ the corresponding flow $\{f_t\}$ starting from the identity is called a **Hamiltonian isotopy** generated by $F$ and is also denoted by $\{\phi_F^t\}$ for convenience. A diffeomorphism of $M$ is said to be **Hamiltonian** if it can be represented as a time-one map of some Hamiltonian isotopy. Denote by $\mathrm{Ham}(M, \omega)$ the set of all Hamiltonian diffeomorphisms on $(M, \omega)$. It is a subgroup of $\mathrm{Symp}(M, \omega)$. The Lie algebra of $\mathrm{Ham}(M, \omega)$ can be naturally identified with $\mathcal{A}(M)$ because of the following important fact by Banyaga (cf. [27, Prop.1.4.B]).

**Lemma 1.1.** [1]. *For every smooth path* $\{f_t\}$ *in* $\mathrm{Ham}(M, \omega)$, $t \in [a, b]$, *there exists a unique (time-dependent) normalized Hamiltonian function* $F : M \times [a, b] \to \mathbb{R}$ *such that*

$$\frac{d}{dt} f_t(x) = X_{F_t}(f_t(x)) \quad \forall (x, t) \in M \times [a, b]. \tag{1.1}$$

Hence the tangent vector of the path $\{f_t\}$ at $t = s$ is the function $F_s$. The adjoint action of $\mathrm{Ham}(M, \omega)$ on $\mathcal{A}(M)$ is given by $\mathrm{Ad}_f G = G \circ f^{-1}$ for $f \in \mathrm{Ham}(M, \omega)$ and $G \in \mathcal{A}(M)$. Any adjoint invariant norm $\|\cdot\|$ on $\mathcal{A}(M)$, i.e., $\|\mathrm{Ad}_f G\| = \|G \circ f^{-1}\| = \|G\|$ for any $f \in \mathrm{Ham}(M, \omega)$ and $G \in \mathcal{A}(M)$, defines a Finsler structure on $\mathrm{Ham}(M, \omega)$, and thus the length of a Hamiltonian path $\{f_t\}$, $t \in [a, b]$ with (unique) normalized Hamiltonian $F$ by

$$\mathrm{Length}\{f_t\} = \int_a^b \|F_t\| dt, \tag{1.2}$$

which does not depend on the parametrization. Without loss of generality we could fix $a = 0, b = 1$ in the definition. For arbitrary $\phi, \varphi \in \mathrm{Ham}(M, \omega)$, their pseudo-distance is defined by

$$d(\phi, \varphi) = \inf\{\mathrm{Length}(\alpha)\} \tag{1.3}$$

where the infimum is taken over all smooth Hamiltonian path $\alpha : [a, b] \to \mathrm{Ham}(M, \omega)$ with $\alpha(a) = \phi$ and $\alpha(b) = \varphi$. It is a bi-invariant pseudo-metric. (If the norm $\|\cdot\|$ on

$\mathcal{A}(M)$ is not adjoint invariant, $d$ is only right-invariant.) The non-degeneracy axiom, $d(\phi,\varphi)>0$ for $\phi\neq\varphi$, is not satisfied in general.

When the norm $\|\cdot\|$ is chosen as the $L_\infty$-norm,

$$\|H\|_\infty := \max_x H - \min_x H, \quad \text{for} \quad H\in\mathcal{A}(M),$$

Hofer showed in [12] that the corresponding pseudo-metric $d_H$ is a genuine metric in the case $M=\mathbb{R}^{2n}$. Later, this result was generalized to some larger class of symplectic manifolds by Polterovich [26], and finally to general manifolds by Lalonde and McDuff [17]. Nowadays this bi-invariant Finsler metric on the group $\mathrm{Ham}(M,\omega)$ is called the **Hofer metric**, and the function $\|\cdot\|_H = d_H(\cdot,\mathrm{id}_M):\mathrm{Ham}(M,\omega)\to\mathbb{R}$ is called the **Hofer norm**. Let

$$\mathcal{H}=\mathcal{H}(M)=\{H\in C^\infty(M\times[0,1])\,|\,H_t=H(\cdot,t)\in\mathcal{A}(M)\ \forall t\in[0,1]\},$$

$$\mathcal{F}=\mathcal{F}(M)=\{H\in C^\infty(M\times\mathbb{R}/\mathbb{Z})\,|\,H_t=H(\cdot,t)\in\mathcal{A}(M)\ \forall t\in\mathbb{R}\}.$$

Every $\phi\in\mathrm{Ham}(M,\omega)$ can be written as $\phi_F^1$ with $F\in\mathcal{F}$. Moreover it holds that

$$d_H(\phi,\varphi)=\inf\Big\{\int_0^1\|H_t-K_t\|_\infty dt\,\Big|\,H\in\mathcal{H}\text{ generates }\varphi\ \&\ K\in\mathcal{H}\text{ generates }\phi\Big\},$$

$$d_H(\mathrm{id}_M,\varphi)=\inf\Big\{\max_t\|F_t\|_\infty\,\Big|\,F\in\mathcal{F}\text{ generates }\varphi\Big\}, \tag{1.4}$$

see [14, (5.9)] for the first one, and [27, Lemma 5.1.C] for the second.

### 1.2. New bi-invariant metrics

Our new bi-invariant metrics on $\mathrm{Ham}(M,\omega)$ will be constructed in a similar way to Hofer's. For a smooth path $f:[0,1]\to\mathrm{Ham}(M,\omega)$ generated by a (time-dependent) normalized Hamiltonian function $F$, and each integer $k=0,1,2,\ldots$, we define the **k-length** of $f$ by

$$\mathrm{Length}_k(f) := \sum_{i=0}^k\int_0^1\Big\|\frac{\partial^i F_t}{\partial t^i}\Big\|_\infty dt = \sum_{i=0}^k\int_0^1\Big(\max_x\frac{\partial^i F_t}{\partial t^i}-\min_x\frac{\partial^i F_t}{\partial t^i}\Big)dt.$$

Clearly, $\mathrm{Length}_0$ is the same as that of (1.2). However, unlike $\mathrm{Length}_0$ the $k$-length ($k\geq 1$) strongly depends on the choice of parametrization.

Call a continuous path $f:[0,1]\to\mathrm{Ham}(M,\omega)$ **piecewise smooth** if there exists a division $0=t_0<t_1<\cdots<t_n=1, n\in\mathbb{N}$, such that $f_i=f|_{[t_{i-1},t_i]}$ is smooth for $i=1,\ldots,n$. Let $F=\{F^l\}_{l=1}^n$ be the corresponding normalized Hamiltonian function, that is,

$$\frac{d}{dt}f_t(x)=X_{F_t^i}(f_t(x))\quad\forall(x,t)\in M\times[t_{i-1},t_i], i=1,\ldots,n.$$

Define its $k$-length by

$$\text{Length}_k(f) := \sum_{i=1}^{n} \text{Length}_k(f_i) = \sum_{l=1}^{n}\sum_{i=0}^{k}\int_{t_{l-1}}^{t_l} \left\| \frac{\partial^i F_t^l}{\partial t^i} \right\|_\infty dt. \tag{1.5}$$

For $\phi, \varphi \in \text{Ham}(M, \omega)$, let $\Omega(\phi, \varphi)$ denote the space of all continuous and piecewise smooth paths $f : [0, 1] \to \text{Ham}(M, \omega)$ from $f(0) = \phi$ to $f(1) = \varphi$. Then we define pseudo-distances between $\phi$ and $\varphi$ by

$$d_k(\phi, \varphi) = \inf \left\{ \text{Length}_k(f) \,\middle|\, f \in \Omega(\phi, \varphi) \right\}, \quad k = 0, 1, \ldots. \tag{1.6}$$

**Lemma 1.2.** $d_0 = d_H$ *on* $\text{Ham}(M, \omega)$.

Clearly, $d_0 \le d_H$ since (1.2) does not depend on the parametrization. The converse inequality easily follows from the definition of $d_0$ and the triangle inequality for $d_H$.

Let us make some comments on the definition of $d_k$.

**Remark 1.3.** (i) For a positive integer $k$, the function $\text{Length}_k(\cdot)$ depends on the choice of parametrization of the path, and the derivative terms will in fact vanish when we take the infimum with respect to the path space with variant parametrization intervals. In fact, suppose that a smooth Hamiltonian path $\alpha : [0, 1] \to \text{Ham}(M, \omega)$ is generated by a normalized Hamiltonian $F_t$. For any $b > 0$, the reparametrized path

$$\beta_b(t) := \alpha(t/b) : [0, b] \to \text{Ham}(M, \omega)$$

is generated by the Hamiltonian function $G(x, t) = \frac{1}{b} F(x, \frac{t}{b})$, and hence

$$\int_0^b \left\| \frac{\partial G_t}{\partial t} \right\|_\infty dt = \int_0^b \frac{1}{b^2} \left\| \frac{\partial F}{\partial t}(x, \frac{s}{b}) \right\|_\infty ds = \frac{1}{b} \int_0^1 \left\| \frac{\partial F}{\partial t}(x, t) \right\|_\infty dt \to 0$$

as $b \to +\infty$. This fact still holds for the higher order derivatives. It follows that

$$b \to +\infty \Longrightarrow \text{Length}_k(\beta_b) \to \text{Length}_0(\alpha).$$

Thus if we define $d_k$ as in (1.3) no new thing can be obtained. This is why we fix the parametrization interval of paths, $[a, b] = [0, 1]$.

(ii) Since the definition of $k$-length involves the derivatives of a Hamiltonian function until $k$ order the quasi-triangle inequality cannot be obtained if we restrict to smooth paths from $[0, 1]$ to $\text{Ham}(M, \omega)$ in the definition of $d_k$. It is this reason that we extend the space of all smooth Hamiltonian paths to include the piecewise smooth ones.

As expected $d_k$ has the following properties.

**Theorem 1.4.** *$d_H = d_0 \le d_1 \le d_2 \le \cdots$, and*

($i$) (symmetry) $d_k(\phi,\varphi) = d_k(\varphi,\phi)$,

($ii$) (quasi-triangle inequality) $d_k(\phi,\varphi) \le 2^k(d_k(\phi,\theta) + d_k(\theta,\varphi))$,

($iii$) (non-degeneracy) $d_k(\phi,\varphi) \ge 0$, *and* $d_k(\phi,\varphi) = 0 \Longleftrightarrow \phi = \varphi$,

($iv$) (bi-invariance) $d_k(\phi,\varphi) = d_k(\phi\theta,\varphi\theta) = d_k(\theta\phi,\theta\varphi)$,

*for any* $\phi,\varphi,\theta \in \mathrm{Ham}(M,\omega)$ *and* $k = 0,1,\ldots$.

This theorem shows that (1.6) gives a sequence of bi-invariant quasi-metrics $\{d_k\}_{k=0}^{\infty}$ on $\mathrm{Ham}(M,\omega)$. Recall that a **quasidistance** on a nonempty set $X$ is a function $\rho : X\times X \to [0,+\infty)$ such that (i) $\rho(x,y) = \rho(y,x)$ for all $x,y\in X$, (ii) $\rho(x,y) = 0$ if and only if $x = y$, (iii) and there exists a finite constant $c \ge 1$ (quasi-triangle constant) such that $\rho(x,y) \le c(\rho(x,z) + \rho(z,y))$ for every $x,y,z \in X$. Such a pair $(X,\rho)$ is called a **quasimetric** space. (See [20]). A **group norm** (resp. **quasinorm**) on a group $G$ is a symmetric, nondegenerate and nonnegative function $\psi$ which is subadditive (resp. $C$-subadditive for some finite constant $C \ge 1$, that is, $\psi(xy) \le C(\psi(x)+\psi(y))$ for all $x,y\in G$). (See [20, page 113]).

For every $d_k$, let us define a function

$$\|\cdot\|_k = d_k(\cdot,\mathrm{id}_M). \tag{1.7}$$

From Theorem 1.4 one easily derives:

**Theorem 1.5.** *$\|\cdot\|_0 = \|\cdot\|_H$, and for every $k\in\mathbb{N}$, $\|\cdot\|_k$ is a quasinorm, precisely speaking it satisfies:*

($i$) (Symmetry) $\|\phi^{-1}\|_k = \|\phi\|_k$,

($ii$) (The quasi-triangle inequality) $\|\phi\varphi\|_k \le 2^k(\|\phi\|_k + \|\varphi\|_k)$,

($iii$) (Non-degeneracy) $\|\phi\|_k \ge 0$, *and* $\|\phi\|_k = 0 \Longleftrightarrow \phi = \mathrm{id}_M$,

($iv$) (Conjugate invariance) $\|\theta\varphi\theta^{-1}\|_k = \|\varphi\|_k$,

*where* $\phi,\varphi,\theta \in \mathrm{Ham}(M,\omega)$ *are arbitrary.*

This shows that every $\|\cdot\|_k$ is a conjugate invariant quasinorm on the group $\mathrm{Ham}(M,\omega)$. Apply Theorem A.1 to $G = \mathrm{Ham}(M,\omega)$ and $\psi = \|\cdot\|_k$, $k\in\mathbb{N}$, we get

**Theorem 1.6.** *Define the function* $|||\cdot|||_k : \mathrm{Ham}(M,\omega) \to [0,\infty)$ *by*

$$\begin{aligned}&|||\phi|||_k\\ &:= \inf\left\{\left(\sum_{i=1}^{N}\|\phi_i\|_k^{\frac{1}{k+1}}\right)^{1+k}\;\middle|\; \begin{array}{l} N\in\mathbb{N},\ (\phi_1,\ldots,\phi_N)\in\mathrm{Ham}(M,\omega)^{(N)},\\ \phi=\phi_1\cdots\phi_N\end{array}\right\},\end{aligned} \tag{1.8}$$

*where* $\mathrm{Ham}(M,\omega)^{(N)} = \underbrace{\mathrm{Ham}(M,\omega)\times\cdots\times\mathrm{Ham}(M,\omega)}_{N}$, $N\in\mathbb{N}$. *Then*

$$\|\!|\phi|\!\|_k = \|\!|\phi^{-1}|\!\|_k, \quad \forall\phi\in\mathrm{Ham}(M,\omega), \tag{1.9}$$

$$\|\!|\theta\phi\theta^{-1}|\!\|_k = \|\!|\phi|\!\|_k, \quad \forall\theta,\phi\in\mathrm{Ham}(M,\omega), \tag{1.10}$$

$$4^{-(1+k)}\|\phi\|_k \le \|\!|\phi|\!\|_k \le \|\phi\|_k, \quad \forall\phi\in\mathrm{Ham}(M,\omega), \tag{1.11}$$

*and*

$$\|\!|\phi\varphi|\!\|_k^\beta \le \|\!|\phi|\!\|_k^\beta + \|\!|\varphi|\!\|_k^\beta, \quad \forall\phi,\varphi\in\mathrm{Ham}(M,\omega) \tag{1.12}$$

*for each* $\beta\in(0,\frac{1}{1+k}]$. *Also, for each* $N\in\mathbb{N}$, $\beta\in(0,\frac{1}{1+k}]$, *and* $\phi_i\in\mathrm{Ham}(M,\omega)$, $i=1,\dots,N$, *it holds that*

$$\|\phi_1\cdots\phi_N\|_k \le 4^{k+1}\left\{\sum_{i=1}^N \|\phi_i\|_k^\beta\right\}^{\frac{1}{\beta}}, \tag{1.13}$$

*and hence for each sequence* $(\phi_i)_{i\in\mathbb{N}}\subset\mathrm{Ham}(M,\omega)$,

$$\sup_{N\in\mathbb{N}}\|\phi_1\cdots\phi_N\|_k \le 4^{k+1}\left\{\sum_{i=1}^\infty \|\phi_i\|_k^\beta\right\}^{\frac{1}{\beta}}. \tag{1.14}$$

(1.9) can be derived from (1.8) and the symmetry of $\|\cdot\|_k$, and (1.10) can be obtained by Remark A.2 and the conjugate invariance of $\|\cdot\|_k$.

**Corollary 1.7.** *The function* $\|\!|\cdot|\!\|_k$ *defined in (1.8) is a conjugate invariant quasi-norm on* $\mathrm{Ham}(M,\omega)$ *which is equivalent to* $\|\cdot\|_k$*; and for every* $\beta\in(0,\frac{1}{1+k}]$, $\|\!|\cdot|\!\|_k^\beta$ *is a conjugate invariant norm on* $\mathrm{Ham}(M,\omega)$. *Thus*

$$\tilde{d}_k(\phi,\varphi) := \|\!|\phi\varphi^{-1}|\!\|_k \tag{1.15}$$

*is a bi-invariant quasimetric on* $\mathrm{Ham}(M,\omega)$*; and for each* $\beta\in(0,\frac{1}{1+k}]$,

$$\tilde{d}_k^\beta(\phi,\varphi) := (\tilde{d}_k(\phi,\varphi))^\beta \tag{1.16}$$

*is a bi-invariant metric on* $\mathrm{Ham}(M,\omega)$. *They all induce the same topology as* $d_k$.

Consider the commutator of two elements $\varphi$ and $\psi$ in $\mathrm{Ham}(M,\omega)$, $[\varphi,\psi] := \varphi\psi\varphi^{-1}\psi^{-1}$. It follows from Theorem 1.5 that

$$\|[\varphi,\psi]\|_k \le 2^{k+1}\min\{\|\varphi\|_k, \|\psi\|_k\}. \tag{1.17}$$

Similarly (1.9)-(1.12) lead to

$$|||[\varphi,\psi]|||_k^\beta \le 2\min\{|||\varphi|||_k^\beta, |||\psi|||_k^\beta\} \tag{1.18}$$

for each $\beta \in (0, \frac{1}{1+k}]$. For a non-empty subset $A \subset M$ let $e_k(A)$ (resp. $\tilde{e}_k(A)$) denote the **displacement energy** of it with respect to $\|\cdot\|_k$ (resp. $|||\cdot|||_k$), that is,

$$e_k(A) = \inf\{\|\vartheta\|_k \,|\, \vartheta \in \mathrm{Ham}(M,\omega) \ \&\ A\cap\vartheta(A) = \emptyset\}, \tag{1.19}$$

$$\tilde{e}_k(A) = \inf\{|||\vartheta|||_k \,|\, \vartheta \in \mathrm{Ham}(M,\omega) \ \&\ A\cap\vartheta(A) = \emptyset\}. \tag{1.20}$$

As in the proof of [9, Lemma 2.3.B] we may obtain:

**Theorem 1.8.** *Let $U \subset M$ be a non-empty open subset. Then for any $\varphi,\psi \in \mathrm{Ham}(M,\omega)$ with $\mathrm{supp}(\varphi) \subset U$ and $\mathrm{supp}(\psi) \subset U$ it holds that*

$$\|[\varphi,\psi]\|_k \le 4^{k+1}e_k(U) \quad \textit{and} \quad |||[\varphi,\psi]|||_k^\beta \le 4(\tilde{e}_k(U))^\beta \tag{1.21}$$

*for each $\beta \in (0, \frac{1}{1+k}]$.*

Motivated by the so-called "coarse" Hofer norm, for $f = \{f_t\} \in \Omega(\phi,\psi)$, if $F = \{F^l\}_{l=1}^n$ is the corresponding normalized Hamiltonian function, we use

$$\mathrm{Length}_k^*(f) := |||F|||_k := \sum_{i=0}^k \max_{1\le l\le n}\max_{t_{l-1}\le t\le t_l}\left\|\frac{\partial^i F^l}{\partial t^i}(x,t)\right\|_\infty \tag{1.22}$$

(which is independent of the choices of divisions) to replace (1.5), and obtain another sequence of bi-invariant quasimetrics

$$d_k^*(\phi,\varphi) = \inf\left\{\mathrm{Length}_k^*(f)\,\middle|\, f\in\Omega(\phi,\varphi)\right\}, \quad k = 0,1,\cdots \tag{1.23}$$

as in (1.6). Clearly, $d_k \le d_k^*$ for any $k \in \mathbb{N}\cup\{0\}$. (1.4) implies $d_0^* = d_0 = d_H$. Let $\|\cdot\|_k^* = d_k^*(\cdot,\mathrm{id}_M)$. Correspondingly, we have also $|||\cdot|||_k^*$ as in (1.8), and $\tilde{d}_k^*$, $e_k^*$ and $\tilde{e}_k^*$.

**Theorem 1.9.** *All conclusions from Theorem 1.4 to Theorem 1.8 still hold for $\|\cdot\|_k^*$, $d_k^*$, $|||\cdot|||_k^*$ and $\tilde{d}_k^*$, $e_k^*$ and $\tilde{e}_k^*$, but we need to add a factor 2 for the coefficients of inequalities in (ii) of Theorems 1.4, 1.5, (1.17) and the factor 4 in the first inequality of (1.21).*

The following result shows that the Hofer inequality in [13] also holds for each $d_k^*$.

**Theorem 1.10.** *For every* $\varphi, \psi \in \mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$,

$$d_k^*(\varphi,\psi) := \|\varphi\psi^{-1}\|_k^* \le C\,\mathrm{diameter}\big(\mathrm{supp}(\varphi\psi^{-1})\big)|\varphi-\psi|_{C^0},$$

*where* $C$ *is a constant and* $C \le 2^{3k+8}(k+1)^2\big(1+2^{k+1}+2^{2k+2}+2^{3k+3}\big)$. *(*Note*: if* $k=0$ *the constant* $C$ *can be chosen as* $128$ *as in the Hofer inequality.)*

Similarly, for any subset $S \subset \mathbb{R}^{2n}$, we define the **coarse proper displacement** $k$**-energy** $e_{p,k}^*(S)$ of it as

$$e_{p,k}^*(S) = \big\{a>0 \;\big|\; \text{for every bounded subset } A \subset S\ \exists\, \psi \in \mathrm{Ham}(\mathbb{R}^{2n},\omega_0) \text{ such that } \|\psi\|_k^* \le a \text{ and } A \text{ and } \psi(A) \text{ are properly separated}\big\},$$

and get the following generalization of the Sikorav inequality.

**Theorem 1.11.** *If* $H \in \mathcal{H}(\mathbb{R}^{2n})$ *satisfies* $\mathrm{supp}(H) \subset U \times [0,1]$, *then*

$$\|\varphi_H\|_k^* \le 2^{2k+4}(k+1)\big(1+2^{k+1}+2^{2k+2}+2^{3k+3}\big)e_{p,k}^*(U).$$

Finally, let us discuss the corresponding question investigated by Eliashberg and Polterovich [9]. When $0<p<\infty$ the function

$$\mathcal{A}(M) \ni H \to \|H\|_p = \left(\int_M |H|^p\omega^n\right)^{\frac{1}{p}}$$

is an adjoint invariant quasinorm because

$$\int_M |G(f^{-1}(x))|^p\omega^n = \int_M (f^{-1})^*(|G(x)|^p\omega^n) = \int_M |G(x)|^p\omega^n \tag{1.24}$$

for any $G \in \mathcal{A}(M)$ and $f \in \mathrm{Symp}(M,\omega)$, and

$$\|H+G\|_p \le K_p(\|H\|_p + \|G\|_p), \quad \forall H, G \in \mathcal{A}(M),$$

where $K_p$ is equal to $1$ for $p \ge 1$, and $2^{\frac{1-p}{p}}$ for $0<p<1$ (so $\|\cdot\|_p$ is only a quasinorm on $\mathcal{A}(M)$ in this case).

For a smooth path $f : [0,1] \to \mathrm{Ham}(M,\omega)$ generated by a (time-dependent) normalized Hamiltonian function $F_t$, and each integer $k = 0,1,2,\ldots$, we define the **(k,p)-length** of $f$ by

$$\mathrm{Length}_{(k,p)}(f) := \sum_{i=0}^k \int_0^1 \left\|\frac{\partial^i F_t}{\partial t^i}\right\|_p dt, \tag{1.25}$$

and the **(k,p)-length** of $f \in \Omega(\phi, \varphi)$ by the sum of $(k,p)$-lengths of all smooth pieces of it. By the same proofs as those of Theorem 1.4 it is readily verified that

$$d_{(k,p)}(\phi, \varphi) = \inf \left\{ \mathrm{Length}_{(k,p)}(f) \,\middle|\, f \in \Omega(\phi, \varphi) \right\} \tag{1.26}$$

defines a pseudo quasimetric $d_{(k,p)}$ on $\mathrm{Ham}(M, \omega)$ for any $p > 0, k = 0, 1, \ldots$. Note that

$$d_{(0,p)}(\phi, \varphi) = \inf \left\{ \mathrm{Length}_{(0,p)}(f) \,\middle|\, f \in \Omega(\phi, \varphi) \text{ is smooth} \right\}. \tag{1.27}$$

Eliashberg and Polterovich [9, 27] showed for each $p \in [1, \infty)$ that the pseudo-distance $d_{(0,p)}$ is degenerate, and vanishes if $M$ is closed. We have the following extension.

**Theorem 1.12.** *For each $k \in \mathbb{N} \cup \{0\}$ and each $0 < p < 1/k$, the pseudo quasimetric $d_{(k,p)}$ is degenerate, and vanishes if $M$ is closed.*

Similarly, for a closed embedded Lagrangian submanifold $L$ of $(M,\omega)$ let $\mathcal{L}(M,\omega,L)$ denote the space of Lagrangian submanifolds of $(M, \omega)$ which is Hamiltonian isotopic to $L$. For each $k = 0, 1, 2, \ldots$, define $\delta_k : \mathcal{L}(M, \omega, L) \times \mathcal{L}(M, \omega, L) \to \mathbb{R} \cup \{+\infty\}$ by

$$\delta_k(L_1, L_2) = \inf\{\|\phi\|_k \,|\, \phi \in \mathrm{Ham}(M, \omega) \;\&\; \phi(L_1) = L_2\}.$$

Then $\delta_k(L_1, L_2) = \delta_k(L_2, L_1)$ and $\delta_k(L_1, L_2) \le 2^k\big(\delta_k(L_1, L_3) + \delta_k(L_2, L_3)\big)$ for any $L_i \in \mathcal{L}$, $i = 1, 2, 3$. If $(M, \omega)$ is a tame symplectic manifold, Chekanov showed in [8] that $\delta_0 = \delta_H$ is non-degenerate, and so each $\delta_k$ is a $\mathrm{Ham}(M, \omega)$-invariant quasimetric.

The paper is organized as follows. In Section 2 we give proofs of Theorem 1.4, 1.10, 1.11, 1.12. Extensions of our metrics onto the group of symplectic diffeomorphisms will be discussed in Section 3. Finally, Section 4 outlines our constructions on the group of strictly contact diffeomorphisms as a concluding remark.

## 2. Proofs

### 2.1. Proof of Theorem 1.4

In the following we always assume $k > 0$.

(i) For $\Omega(\phi, \varphi) \ni f : [0, 1] \to \mathrm{Ham}(M, \omega)$, let $\bar{f} \in \Omega(\varphi, \phi)$ be defined by $\bar{f}(t) = f(1 - t)$, $\forall t \in [0, 1]$. Then $\Omega(\phi, \varphi) \ni f \to \bar{f} \in \Omega(\varphi, \phi)$ is a bijection. By the assumption there exists a division $0 = t_0 < t_1 < \cdots < t_n = 1, n \in \mathbb{N}$, such that $f_i = f|_{[t_{i-1}, t_i]}$ is smooth for $i = 1, \ldots, n$. By Lemma 1.1, for each $j = 1, \ldots, n$ there exists a (time-dependent) normalized Hamiltonian function $F_j : M \times [t_{j-1}, t_j] \to \mathbb{R}$ such that

$$\frac{d}{dt} f_j(t)(x) = X_{F_j}\big(f_j(t)(x), t\big) \tag{2.1}$$

for all $x \in M$ and $t \in [t_{j-1}, t_j]$. Set $s_i = 1 - t_{n-i}$, $i = 0, \dots, n$. Then $0 = s_0 < s_1 < \cdots < s_n = 1$ is a division of $[0, 1]$, and for any $t \in [s_{j-1}, s_j]$ it holds that

$$\bar{f}_j(t) = \bar{f}|_{[s_{j-1}, s_j]}(t) = f(1-t) = f|_{[t_{n-j}, t_{n-j+1}]}(1-t) = f_{n-j+1}(1-t).$$

From this and (2.1) it follows that

$$\begin{aligned} \frac{d}{dt}\bar{f}_j(t)(x) &= -\frac{d}{ds} f_{n-j+1}(s)(x)|_{s=1-t} \\ &= -X_{F_{n-j+1}}\big(f_{n-j+1}(s)(x), s\big)|_{s=1-t} = X_{G_j}\big(\bar{f}_j(t)(x), t\big), \end{aligned}$$

where $G_j : M \times [s_{j-1}, s_j] \to \mathbb{R}$ is given by $G_j(x, t) = -F_{n-j+1}(x, 1-t)$. Hence

$$\frac{\partial^i (G_j(x,s))}{\partial s^i} = (-1)^{i+1} \times \, (F_{n-j+1})_2^{(i)}(x, 1-s) \quad \forall s \in [s_{j-1}, s_j]$$

for $i = 0, \dots, k$, where $(F_{n-j+1})_2^{(i)}$ stands for the $i$th partial derivative of $F_{n-j+1}$ with respect to the second variable. By definition we have

$$\begin{aligned} \mathrm{Length}_k(\bar{f}) &= \sum_{j=1}^{n} \mathrm{Length}_k(\bar{f}_j) \\ &= \sum_{j=1}^{n}\sum_{i=0}^{k}\int_{s_{j-1}}^{s_j} \left\| \frac{\partial^i (G_j(x,s))}{\partial s^i} \right\|_\infty ds \\ &= \sum_{j=1}^{n}\sum_{i=0}^{k}\int_{s_{j-1}}^{s_j} \left\| (-1)^{i+1} (F_{n-j+1})_2^{(i)}(x, 1-s) \right\|_\infty ds \\ &= \sum_{j=1}^{n}\sum_{i=0}^{k}\int_{s_{j-1}}^{s_j} \left\| (F_{n-j+1})_2^{(i)}(x, 1-s) \right\|_\infty ds \\ &= \sum_{j=1}^{n}\sum_{i=0}^{k}\int_{t_{n-j}}^{t_{n-j+1}} \left\| (F_{n-j+1})_2^{(i)}(x, t) \right\|_\infty dt \\ &= \sum_{j=1}^{n}\sum_{i=0}^{k}\int_{t_{j-1}}^{t_j} \left\| (F_j)_2^{(i)}(x, t) \right\|_\infty dt \\ &= \sum_{j=1}^{n} \mathrm{Length}_k(f_j) = \mathrm{Length}_k(f). \end{aligned}$$

Thus $d_k(\phi, \varphi) = d_k(\varphi, \phi)$.

(ii) Let $\Omega(\phi,\theta)\ni f:[0,1]\to \mathrm{Ham}(M,\omega)$, $\Omega(\theta,\varphi)\ni g:[0,1]\to \mathrm{Ham}(M,\omega)$. We define the product path $g\sharp f:[0,1]\to \mathrm{Ham}(M,\omega)$ of $f$ and $g$ by

$$g\sharp f(t)=\begin{cases} f(2t) & 0\le t\le 1/2\\ g(2t-1) & 1/2\le t\le 1,\end{cases}$$

then $g\sharp f$ is a piecewise smooth Hamiltonian path connecting $\phi$ and $\varphi$, i.e. $g\sharp f\in \Omega(\phi,\varphi)$.

By assumption there exist divisions

$$0<t_0<t_1<\cdots<t_n=1 \text{ and } 0<t'_0<t'_1<\cdots<t'_m=1,$$

such that for $i=1,\dots,n$, $f_i=f|_{[t_{i-1},t_i]}$ is smooth; for $j=1,\dots,m$, $g_j=g|_{[t'_{j-1},t'_j]}$ is also smooth. Denote the Hamiltonian functions generating $\{f_i\}_{i=1}^n$, $\{g_j\}_{j=1}^m$ by $\{F_i\}_{i=1}^n$ and $\{G_j\}_{j=1}^m$ respectively. For $0\le i\le n$, set $s_i=\frac{t_i}{2}$, for $n+1\le i\le n+m$, set $s_i=\frac{t'_{i-n}+1}{2}$, then

$$0=s_0<s_1<\cdots<s_n=\frac{1}{2}<s_{n+1}<\cdots<s_{n+m}=1$$

is a division of $[0,1]$, and at this time $(g\sharp f)_i=g\sharp f|_{[s_{i-1},s_i]}$ is smooth for $i=1,\dots,n+m$. Denote the Hamiltonian function generating $(g\sharp f)_i$ by $H_i:M\times[s_{i-1},s_i]\to\mathbb{R}$, then

$$H_i(x,s)=2F_i(x,2s),\quad 1\le i\le n,$$

$$H_i(x,s)=2G_{i-n}(x,2s-1),\quad n+1\le i\le n+m.$$

By definition we have

$$\begin{aligned}
d_k(\phi,\varphi)&\le \mathrm{Length}_k(g\sharp f)=\sum_{j=1}^{n+m}\mathrm{Length}_k\Big((g\sharp f)_j\Big)\\
&=\sum_{j=1}^{n+m}\sum_{i=0}^{k}\int_{s_{j-1}}^{s_j}\left\|\frac{\partial^i(H_j(x,s))}{\partial s^i}\right\|_\infty ds\\
&=\sum_{j=1}^{n}\sum_{i=0}^{k}\int_{s_{j-1}}^{s_j}\left\|\frac{\partial^i(2F_j(x,2s))}{\partial s^i}\right\|_\infty ds\\
&\quad+\sum_{j=n+1}^{n+m}\sum_{i=0}^{k}\int_{s_{j-1}}^{s_j}\left\|\frac{\partial^i(2G_{j-n}(x,2s-1))}{\partial s^i}\right\|_\infty ds\\
&=\sum_{j=1}^{n}\sum_{i=0}^{k}\int_{\frac{t_{j-1}}{2}}^{\frac{t_j}{2}}2^{i+1}\left\|(F_j)_2^{(i)}(x,2s)\right\|_\infty ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=n+1}^{n+m}\sum_{i=0}^{k}\int_{\frac{t'_{j-1-n}+1}{2}}^{\frac{t'_{j-n}+1}{2}} 2^{i+1}\left\|(G_{j-n})_2^{(i)}(x,2s-1)\right\|_\infty ds\\
&=\sum_{j=1}^{n}\sum_{i=0}^{k}\int_{t_{j-1}}^{t_j} 2^i\left\|(F_j)_2^{(i)}(x,s)\right\|_\infty ds+\sum_{j=1}^{m}\sum_{i=0}^{k}\int_{t'_{j-1}}^{t'_j} 2^i\left\|(G_j)_2^{(i)}(x,s)\right\|_\infty ds\\
&\le 2^k\left(\sum_{j=1}^{n}\sum_{i=0}^{k}\int_{t_{j-1}}^{t_j}\left\|\frac{\partial^i(F_j(x,s))}{\partial s^i}\right\|_\infty ds+\sum_{j=1}^{m}\sum_{i=0}^{k}\int_{t'_{j-1}}^{t'_j}\left\|\frac{\partial^i(G_j(x,s))}{\partial s^i}\right\|_\infty ds\right)\\
&=2^k\left(\sum_{j=1}^{n}\mathrm{Length}_k(f_j)+\sum_{j=1}^{m}\mathrm{Length}_k(g_j)\right)\\
&=2^k\left(\mathrm{Length}_k(f)+\mathrm{Length}_k(g)\right).
\end{aligned}
$$

Take the infimum for all $f\in\Omega(\phi,\theta)$ and $g\in\Omega(\theta,\varphi)$ at the right hand of the above equation respectively, we obtain the desired triangle inequality $d_k(\phi,\varphi)\le 2^k(d_k(\phi,\theta)+d_k(\theta,\varphi))$.
(iii) Because $d_k\ge d_0=d_{\mathrm{H}}$, the non-degeneracy of $d_k$ is obvious.
(iv) Firstly, we prove the right-invariance of $d_k$. Let $f\in\Omega(\phi,\varphi)$ be generated by $\{F_i, 1\le i\le n\}$ as above. Then $(f\circ\theta)(t):=f(t)\theta$ defines an element $f\circ\theta$ in $\Omega(\phi\theta,\varphi\theta)$, and $f\circ\theta|_{[t_{i-1},t_i]}$ also correspond to (time-dependent) Hamiltonian functions $F_i$, $i=1,\dots,n$. So $\mathrm{Length}_k(f)=\mathrm{Length}_k(f\circ\theta)$. Hence,

$$
\begin{aligned}
d_k(\phi,\varphi)&=\inf_f\{\mathrm{Length}_k(f)\}=\inf_f\{\mathrm{Length}_k(f\circ\theta)\}\\
&=\inf_{g\in\Omega(\phi\theta,\varphi\theta)}\{\mathrm{Length}_k(g)\}=d_k(\phi\theta,\varphi\theta).
\end{aligned}
$$

Next we prove the left-invariance of $d_k$. Let $f$ as above. Then $\theta\circ f(t):=\theta(f_t(x))$ defines an element $\theta\circ f$ in $\Omega(\theta\phi,\theta\varphi)$, and $\theta\circ f|_{[t_{i-1},t_i]}$ correspond to Hamiltonian functions $F_i(\theta^{-1}(\cdot),t)$, $i=1,\dots,n$. So $\mathrm{Length}_k(f)=\mathrm{Length}_k(\theta\circ f)$. By the same argument as above, we get $d_k(\phi,\varphi)=d_k(\theta\phi,\theta\varphi)$. ■

### 2.2. Proof of Theorems 1.10, 1.11

Following [30, 13] we first prove

**Lemma 2.1.** *Assume* $\psi_1,\psi_2,\dots,\psi_m\in\mathrm{Ham}(\mathbb{R}^{2n},\omega_0)$ *have properly separated supports. Then*

$$\|\psi_1\psi_2\cdots\psi_m\|_k^*\le 2(k+1)\max_j\|\psi_j\|_k^*.$$

*Proof.* Given $\varepsilon > 0$, by (1.23) we have $f_j \in \Omega(\mathrm{id}, \psi_j)$ such that $\mathrm{Length}_k^*(f_j) < \|\psi_j\|_k^* + \varepsilon$, $j = 1, \dots, m$. Let $H_j$ be the corresponding normalized Hamiltonian functions of $f_j$, $j = 1, \dots, m$. Then $|||H_j|||_k = \mathrm{Length}_k^*(f_j) \le \|\psi_j\|_k^* + \varepsilon$. Through a refinement, we could suppose that there exists a common division of time $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $H_j^i := H_j|_{[t_{i-1}, t_i]}$ is smooth for every $1 \le j \le m, 1 \le i \le n$.

Using the same notations as those of [14], we set $S_j = \mathrm{supp}(\psi_j)$, choose $R > 0$ such that $\mathrm{supp}(H_j) \subset B_R(x_j^*) \times [0,1]$ with $x_j^* \in S_j$, and then vectors $v_j$ such that the sets $B_R(S_j + v_j)$ are disjoint. Let $\tau \in \mathrm{Ham}(M, \omega)$ be the map associated to the $S_j$ guaranteed by Lemma 8 on the page 175 of [14], and let $\hat{\psi}_j = \tau \psi_j \tau^{-1}$ and

$$\hat{f}_j(t) = \tau f_j(t) \tau^{-1},\ j = 1, \dots, m.$$

Then the corresponding normalized Hamiltonian function with $\hat{f}_j$ is given by

$$\hat{H}_j(x,t) = H_j(\tau^{-1}(x), t) = H_j(x - v_j, t),$$

and thus the corresponding normalized Hamiltonian function with

$$\hat{f}_1 \hat{f}_2 \cdots \hat{f}_m \in \Omega(\mathrm{id}, \hat{\psi}_1 \hat{\psi}_2 \cdots \hat{\psi}_m)$$

is $\hat{H} = \hat{H}_1 + \cdots + \hat{H}_m$. It follows that

$$\|\psi_1 \psi_2 \cdots \psi_m\|_k^* = \|\hat{\psi}_1 \hat{\psi}_2 \cdots \hat{\psi}_m\|_k^* \le |||\hat{H}|||_k.$$

By the definition in (1.22) we have $s_i \in [0,1]$, $i = 0, \dots, k$, such that

$$\begin{aligned}
|||\hat{H}|||_k &= \sum_{i=0}^{k} \max_{1 \le l \le n} \max_{t_{l-1} \le t \le t_l} \left( \sup_x \frac{\partial^i \hat{H}^l}{\partial t^i} - \inf_x \frac{\partial^i \hat{H}^l}{\partial t^i} \right) \\
&= \sum_{i=0}^{k} \left( \sup_x \frac{\partial^i \hat{H}}{\partial t^i}(x, s_i) - \inf_x \frac{\partial^i \hat{H}}{\partial t^i}(x, s_i) \right) \\
&= \sum_{i=0}^{k} \left[ \max_{1 \le j \le m} \left( \sup_x \frac{\partial^i \hat{H}_j}{\partial t^i}(x, s_i) \right) - \min_{1 \le j \le m} \left( \inf_x \frac{\partial^i \hat{H}_j}{\partial t^i}(x, s_i) \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{k} 2 \max_{1\le j\le m}\left(\left\|\frac{\partial^i \hat{H}_j}{\partial t^i}(\cdot, s_i)\right\|_\infty\right)\\
&\leq \sum_{i=0}^{k} 2 \max_{1\le j\le m}\left(\max_{t\in[0,1]}\left\|\frac{\partial^i \hat{H}_j}{\partial t^i}(\cdot, t)\right\|_\infty\right)\\
&\leq 2(k+1)\max_{1\le j\le m}\left(\sum_{i=0}^{k}\max_{t\in[0,1]}\left\|\frac{\partial^i \hat{H}_j}{\partial t^i}(\cdot, t)\right\|_\infty\right)\\
&= 2(k+1)\max_{1\le j\le m}|||\hat{H}_j|||_k\\
&\leq 2(k+1)\max_j\|\psi_j\|_k^* + 2(k+1)\varepsilon.
\end{aligned}
$$

This holds for every $\varepsilon > 0$ and the lemma is proved. ∎

*Proof of Theorem 1.11*. Given $\varepsilon > 0$, as in the proof of [30, 13] we can construct maps $\psi_j, 0 \le j \le N$ satisfying $d_k^*(\psi_j, \psi_{j+1}) < \varepsilon$, and maps $\varphi_j (0 \le j \le 2N)$, $\alpha_j (1 \le j \le N)$, $\beta_j (0 \le j \le N)$ with $\|\varphi_0\|_k^* \le e_{p,k}^*(U) + \varepsilon$. Then we have

$$
\begin{aligned}
\|\varphi_H\|_k^* &= \|\beta_N\|_k^* = \left\|\left(\prod_{j=1}^{N}\alpha_j\beta_j\right)\left(\prod_{j=0}^{N-1}\alpha_{j+1}\beta_j\right)^{-1}\right\|_k^*\\
&\leq 2^{k+2}(k+1)\left(\max_{1\le j\le N}\|\alpha_j\beta_j\|_k^* + \max_{0\le j\le N-1}\|\alpha_{j+1}\beta_j\|_k^*\right)
\end{aligned}
$$

(because $\|\phi\psi\|_k^* \le 2^{k+1}(\|\phi\|_k^* + \|\psi\|_k^*)$ by Theorem 1.9). We can estimate

$$
\begin{aligned}
\|\alpha_j\beta_j\|_k^* &\leq 2^{k+1}\left[\|\varphi_{2j-1}\|_k^* + 2^{k+1}(\|\varphi_{2j-1}^{-1}\varphi_{2j}\|_k^* + \|\varphi_{2j}\|_k^*)\right]\\
&\leq 2^{k+1}\|\varphi_0\|_k^* + 2^{3k+3}(\|\varphi_{2j-1}\|_k^* + \|\varphi_{2j}\|_k^*) + 2^{2k+2}\|\varphi_0\|_k^*\\
&= (2^{k+1} + 2^{2k+2} + 2^{3k+4})\|\varphi_0\|_k^*.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\|\alpha_{j+1}\beta_j\|_k^* &\leq 2^{k+1}\left[\|\varphi_{2j+1}\|_k^* + 2^{k+1}(\|\psi_{j+1}\psi_j^{-1}\psi_j\varphi_{2j+1}^{-1}\varphi_{2j}\psi_j^{-1}\|_k^* + \|\varphi_{2j}\|_k^*)\right]\\
&\leq 2^{k+1}\|\varphi_0\|_k^* + 2^{3k+3}(\|\psi_{j+1}\psi_j^{-1}\|_k^* + \|\varphi_{2j+1}^{-1}\varphi_{2j}\|_k^*) + 2^{2k+2}\|\varphi_0\|_k^*\\
&\leq (2^{k+1} + 2^{2k+2} + 2^{4k+5})\|\varphi_0\|_k^* + 2^{3k+3}d_k^*(\psi_{j+1}, \psi_j).
\end{aligned}
$$

Summing up, we obtain

$$
\begin{aligned}
\|\varphi_H\|_k^* &\leq 2^{k+2}(k+1)(2^{k+2} + 2^{2k+3} + 2^{3k+4} + 2^{4k+5})\|\varphi_0\|_k^* + 2^{4k+5}(k+1)\varepsilon\\
&\leq 2^{2k+4}(k+1)(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3})e_{p,k}^*(U) +\\
&\quad 2^{2k+4}(k+1)(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3})\varepsilon + 2^{4k+5}(k+1)\varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, the proof is finished. ■

Carefully checking the proof of Proposition 6 in [13] (or [14, Lemma 10]) and replacing $E$ and $e_p$ therein we still have for our $\|\cdot\|_k^*$ and $e_{p,k}^*$:

**Lemma 2.2.** *Let* $\psi \in \mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$ *with* $\psi \neq \mathrm{id}$ *and let* $\delta > |\psi - \mathrm{id}|_{C^0}$. *For every* $Q \subset \mathbb{R}^{2n}$ *open and satisfying* $Q \cap \mathrm{supp}(\psi) \neq \emptyset$, *there exists a* $\varphi \in \mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$ *satisfying*

(*i*) $\varphi|Q = \psi|Q$

(*ii*) $\mathrm{supp}(\psi) \subset U$

(*iii*) $\|\varphi\|_k^* \le 2^{2k+4}(k+1)(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3})e_{p,k}^*(U)$,

*where* $U$ *is the intersection of* $B_\delta(Q)$ *with the convex hull of* $\mathrm{supp}(\psi)$, *and* $B_\delta(Q) = \{x|\mathrm{dist}(x, Q) < \delta\}$.

Similarly, corresponding to [13, Corollary 7] or [14, Lemma 11] we have

**Lemma 2.3.** *Let* $U = (a_1, a_2) \times (b_1, b_2) \oplus \mathbb{R}^{2n-2}$. *Then for* $\psi \in \mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$ *with* $\mathrm{supp}(\psi) \subset U$ *it holds that*

$$\|\varphi\|_k^* \le 2^{2k+4}(k+1)(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3})(a_2 - a_1)(b_2 - b_1).$$

*Proof of Theorem 1.10.* Let $\{\varphi_j\}_{j\in I}$ be as in the proof of [14, Theorem 9]. Corresponding to the inequality (iii) on the page 180 of [14], we have

$$\|\varphi_j\|_k^* \le 2^{2k+4}(k+1)(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3})e_{p,k}^*(U_j).$$

And similar to [14, (5.32)] we have estimate

$$\begin{aligned}\Big\| \prod_{j\in I} \varphi_j \Big\|_k^* &\le 2(k+1)\max_{j\in I} \|\varphi_j\|_k^* \\ &\le 2(k+1)2^{2k+4}(k+1)(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3}) \cdot 2(\varepsilon + \delta_1)R \\ &\le 2^{2k+6}(k+1)^2(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3})R\delta.\end{aligned}$$

Thus

$$\begin{aligned}\|\psi\|_k^* &\le 2^{k+1}(\|\psi\theta\|_k^* + \|\theta\|_k^*) \\ &= 2^{k+1}\left(\Big\| \prod \hat{\varphi}_j \Big\|_k^* + \Big\| \prod \varphi_j \Big\|_k^*\right) \\ &\le 2^{3k+8}(k+1)^2(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3})R\delta.\end{aligned}$$

This holds true for every $\delta > |\psi - \mathrm{id}|_{C^0}$, and we conclude that

$$\|\psi\|_k^* \le 2^{3k+8}(k+1)^2(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3})R|\psi - \mathrm{id}|_{C^0},$$

where $R = \mathrm{diameter}\big(\mathrm{supp}(\psi)\big)$. Finally we have the inequality:

$$\begin{aligned} & d_k^*(\varphi, \psi) \\ \le\ & 2^{3k+8}(k+1)^2(1 + 2^{k+1} + 2^{2k+2} + 2^{3k+3})\,\mathrm{diameter}\big(\mathrm{supp}(\varphi\psi^{-1})\big)|\varphi - \psi|_{C^0}, \quad \blacksquare \end{aligned}$$

### 2.3. Proof of Theorem 1.12

When $k = 0$ and $p \in [1, \infty)$ this is the result in [9]. We shall assume that either $k = 0$ and $0 < p < 1$ or $k \in \mathbb{N}$ and $0 < p < 1/k$ below.

The proof ideas are same as those of [9]. Given a bi-invariant pseudo quasimetric $\rho$ on $\mathrm{Ham}(M, \omega)$, the ($\rho$-)**displacement energy** of a subset $A \subset M$ is defined by

$$e_\rho(A) = \inf\{\rho(\mathrm{id}, g) |\ g \in \mathrm{Ham}(M, \omega) \text{ such that } g(A) \cap A = \emptyset\}. \tag{2.2}$$

As in the proof of [9, Th.2.2.A] we may obtain:

**Claim 2.4.** Every nonempty open subset of $M$ has positive displacement energy with respect to a bi-invariant quasimetric on $\mathrm{Ham}(M, \omega)$.

So it suffices to prove that the displacement energy associated with $d_{(k,p)}$ ($0 < p < 1/k$) vanishes for some embedded open ball in $M$. In fact, using Darboux theorem, we can choose a chart $M \supset U \ni w \to (x_1(w), \cdots, x_n(w), y_1(w), \cdots, y_n(w)) \in \mathbb{R}^{2n}$ so that the symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on it. Replacing $\omega$ by $N\omega$ for some large $N > 0$ we may assume

$$U \supset B(0, 4) = \left\{(x, y) \in \mathbb{R}^{2n} \;\Big|\; \sum_{i=1}^n (x_i^2 + y_i^2) < 16\right\}, \quad A = B(0, 1).$$

Consider a Hamiltonian isotopy $\{h_t\}$, $t \in [0, 1]$ such that when restricted to $U$, $h_t$ is simply a shift by $2t$ along the $y_1$ coordinate. Assume that $\{h_t\}$ is generated by $H$, then $H(x, y, t) = 2x_1$ on $U$ ($H$ is not normalized). Clearly $h_1(A) \cap A = \emptyset$.

Fixed a smooth cut-off function $\delta : \mathbb{R} \to [0, 1]$, such that $\delta \equiv 1$ for $|t| \le 1/4$, $\delta \equiv 0$ for $|t| \ge 3/4$. Set $\delta_m(t) = \delta(mt)$. When $M$ is open, we can define a sequence of functions $\{G_m\}_{m \in \mathbb{N}}$ as follows:

$$G_m(x, y, t) = \begin{cases} 2x_1\delta_m(\sqrt{|x|^2 + |y - (2t, 0, \ldots, 0)|^2} - 1) & (x, y) \in U \\ 0 & (x, y) \notin U. \end{cases} \tag{2.3}$$

$G_m$ is smooth on $M$. The support of $G_m$ is contained in a neighborhood of $h_t(\partial A)$ for each $t \in [0,1]$ and tends to $h_t(\partial A)$ as $m \to \infty$. Since for every $t$ the function $G_m(\cdot, t)$ coincides with $H$ near $h_t(\partial A)$, we conclude that the Hamiltonian vector field of $G_m$ is equal to $X_H$ near $h_t(\partial A)$ for every $m$. Hence the Hamilton isotopy $\{\phi^t_{G_m}\}$ of $G_m$ satisfies $\phi^t_{G_m}(\partial A) = h_t(\partial A)$ and so $\phi_{G_m}(\partial A) \cap \partial A = \emptyset$. But this obviously implies that $\phi_{G_m}(A) \cap A = \emptyset$.

Clearly, for $0 < p < \infty$ we have $\int_0^1 \|G_m(\cdot, t)\|_p dt \to 0$ as $m \to \infty$.

Next we prove that for each $1 \le i \le k$,

$$\left\| \frac{\partial^i G_m}{\partial t^i}(\cdot, t) \right\|_p^p \to 0 \quad \text{uniformly in } t \in [0,1] \text{ as } m \to \infty$$

For simplicity we write $r = \sqrt{(|x|^2 + |y - (2t, 0, \ldots, 0)|^2)}$. In $U$ we have

$$\frac{\partial^i G_m}{\partial t^i}(x,y,t) = m^i x_1 \delta_m^{(i)}(r-1) \cdot 2^{i+1} \big(\frac{t - y_1}{r}\big)^i + m^{i-1} L(x,y,t).$$

Here $L$ is a combination of $\delta_m^{(j)}, 1 \le j \le i-1$. Its coefficients are rational functions of $x, y, t$ which are bounded in $C^0$-norm near $h_t(\partial A)$. Hence,

$$\begin{aligned} \left\| \frac{\partial^i G_m}{\partial t^i}(\cdot, t) \right\|_p^p &= \int_U m^{ip} \left| x_1 \delta_m^{(i)}(r-1) \cdot 2^{i+1} \left( \frac{t - y_1}{r} \right)^i + \frac{1}{m} L(x,y,t) \right|^p \omega^n \\ &\le C m^{ip} \int_{\Sigma_1 \cup \Sigma_2} \omega^n = C m^{ip} \big( \mathrm{Vol}(\Sigma_1) + \mathrm{Vol}(\Sigma_2) \big), \end{aligned}$$

where $C > 0$ is a constant depending on $\delta$, $\Sigma_1$ is a region bounded by two spheres whose radiuses are $1 + \frac{1}{4m}, 1 + \frac{3}{4m}$ respectively, and $\Sigma_2$ is a similar region with radiuses $1 - \frac{1}{4m}, 1 - \frac{3}{4m}$ respectively. So we have

$$\begin{aligned} &\left\| \frac{\partial^i G_m}{\partial t^i}(\cdot, t) \right\|_p^p \\ &\le \frac{\pi^n}{n!} C m^{ip} \left[ \left(1 + \frac{3}{4m}\right)^{2n} - \left(1 + \frac{1}{4m}\right)^{2n} + \left(1 - \frac{1}{4m}\right)^{2n} - \left(1 - \frac{3}{4m}\right)^{2n} \right] \\ &= \frac{\pi^n}{n!} C m^{ip} \left[ \frac{2}{m} + O\left(\frac{1}{m^2}\right) \right]. \end{aligned}$$

Since $ip \le kp < 1$, so the above expression tends to zero when $m \to \infty$.

When $M$ is closed, we could find an open set $V$ disjoint with the above $U$, and shrinking $U$ properly we can also assume that $V$ is symplectomorphic to $U$. Then we can define a function $\widetilde{G_m}$ on $V$ which is the same form as $G_m$ in (2.3) but with a minus in front additionally. Define $K_m$ to be the sum of $G_m$ and $\widetilde{G_m}$. Then $K_m$ is a normalized Hamiltonian function such that $\phi_{K_m}(A) \cap A = \emptyset$. Using the same estimates as above for $G_m$ and $\widetilde{G_m}$ respectively, we derive

$$\left\| \frac{\partial^i K_m}{\partial t^i}(\cdot, t) \right\|_p \to 0 \quad \text{uniformly in } t \in [0,1] \text{ as } \quad m \to \infty.$$

All these lead to the first conclusion.

Note that $\{\phi \in \mathrm{Ham}(M,\omega) \mid \rho(\mathrm{id}_M, \phi) = 0\}$ is also a normal subgroup of $\mathrm{Ham}(M,\omega)$ for any bi-invariant pseudo quasimetric $\rho$ on $\mathrm{Ham}(M,\omega)$. The second claim follows from Banyaga's theorem as in [27]. ■

## 3. Extensions

It is a natural question to extend bi-invariant pseudo-metric on $\mathrm{Ham}(M,\omega)$ to the group of symplectic diffeomorphisms. There exist different ways realizing this. We only use the method by Lalonde and Polterovich [16], and one by Banyaga [2].

### 3.1. The method by Lalonde and Polterovich

Each $\phi \in \mathrm{Symp}(M,\omega)$ induces an isometry with respect to the Hofer metric,

$$C_\phi : \mathrm{Ham}(M,\omega) \to \mathrm{Ham}(M,\omega),\ f \mapsto C_\phi f := \phi f \phi^{-1}.$$

For $\alpha \in (0, \infty]$ it was shown in [16, Proposition 1.2.A] that

$$r_\alpha(\phi) := \sup\{d_H(f, C_\phi f) \mid f \in \mathrm{Ham}(M,\omega) \ \&\ \|f\|_H \le \alpha\} \tag{3.1}$$

defines a bi-invariant function $r_\alpha$ and $r_\alpha(\phi) \le 2\alpha\ \forall \phi \in \mathrm{Symp}(M,\omega)$. In particular, $r_\alpha$ is an bi-invariant norm on $\mathrm{Symp}(M,\omega)$ if $\alpha \in (0,\infty)$. We say $\phi \in \mathrm{Symp}(M,\omega)$ to be **bounded** if the function $(0,\infty) \ni \alpha \mapsto r_\alpha(\phi)$ is bounded, or equivalientl $C_\phi$ is $C^0$-bounded, i.e.,

$$r_\infty(\phi) = \sup\{d_H(f, C_\phi f) \mid f \in \mathrm{Ham}(M,\omega)\} < \infty. \tag{3.2}$$

By the bi-invariance of the Hofer metric $d_H$ the set of all bounded symplectomorphism in $\mathrm{Symp}(M,\omega)$ form a normal subgroup of $\mathrm{Symp}(M,\omega)$, denoted by $\mathrm{BI}(M)$. Set $\mathrm{BI}_0(M) = \mathrm{BI}(M) \cap \mathrm{Symp}_0(\mathrm{M})$, where $\mathrm{Symp}_0(\mathrm{M})$ is the connected component of $\mathrm{Symp}(M,\omega)$ containing the identity map. Every $\phi \in \mathrm{Ham}(M,\omega)$ is bounded and $r_\infty(\phi) \le 2\|\phi\|_H$ because

$$d_H(f, C_\phi f) = d_H(\mathrm{id}, [\phi, f]) \le 2\|\phi\|_H, \quad \forall f \in \mathrm{Ham}(M,\omega). \tag{3.3}$$

Hence $\mathrm{Ham}(M,\omega)$ is a subgroup of $\mathrm{BI}_0(M)$. These motivated Lalonde and Polterovich to propose the so-called bounded isometry conjecture.

**Conjecture 1.** [16, Conjecture 1.3.A]. $\mathrm{BI}_0(M) = \mathrm{Ham}(M, \omega)$.

They proved it in [16] for some cases, for example, $M$ is any closed surface with area form or $M$ is a product of closed surfaces of genus greater than $0$ with product symplectic form. For recent progresses the reader may refer to [15, 11, 7, 25].

Now let us consider corresponding questions with metric $d_k$. For each $\phi \in \mathrm{Symp}(M, \omega)$ it is easy to check that $C_\phi$ is still an isometry of $\mathrm{Ham}(M, \omega)$ with respect to the quasi-norm $\|\cdot\|_k = d_k(\mathrm{id}, \cdot)$. Corresponding to (3.1) we define

$$r_{\alpha,k}(\phi) := \sup\{d_k(f, C_\phi f) \mid f \in \mathrm{Ham}(M, \omega) \;\&\; \|f\|_k \le \alpha\} \tag{3.4}$$

for each $\alpha \in (0, \infty]$. It is also bi-invariant and satisfies

$$r_{\alpha,k}(\phi) \le 2^{k+1}\alpha, \ \forall \phi \in \mathrm{Symp}(M, \omega). \tag{3.5}$$

Actually, the function $r_{\alpha,k}$ is a quasi-norm on $\mathrm{Symp}(M, \omega)$ by the following:

**Lemma 3.1.** *For every $k \ge 0$, $\alpha \in (0, \infty]$, the function $r_{\alpha,k}$ on $\mathrm{Symp}(M, \omega)$ defined above is conjugate invariant, assumes the value $0$ only at the identity, and satisfies the quasi-triangle inequality*

$$r_{\alpha,k}(\phi\psi) \le 2^k(r_{\alpha,k}(\phi) + r_{\alpha,k}(\psi)).$$

*Proof.* (i) The conjugate invariance of $r_{\alpha,k}$. For $\phi, \varphi \in \mathrm{Symp}(M, \omega)$ we have

$$\begin{aligned} r_{\alpha,k}(\varphi\phi\varphi^{-1}) &= \sup_{\{f \mid \|f\|_k \le \alpha\}} d_k(f, \varphi\phi\varphi^{-1} f \varphi\phi^{-1}\varphi^{-1}) \\ &= \sup_{\{f \mid \|f\|_k \le \alpha\}} d_k(\varphi^{-1} f \varphi, \phi\varphi^{-1} f \varphi\phi^{-1}) \\ &= \sup_{\{g \mid \|g\|_k \le \alpha\}} d_k(g, \phi g \phi^{-1}) \qquad \text{(setting } g = \varphi^{-1} f \varphi) \\ &= r_{\alpha,k}(\phi). \end{aligned}$$

(ii) The non-degeneracy of $r_{\alpha,k}$. If $r_{\alpha,k}(\phi) = 0$, then $d_k(f, \phi f \phi^{-1}) = 0$ for each $f \in \mathrm{Ham}(M, \omega)$, $\|f\|_k \le \alpha$. By the non-degeneracy of $d_k$, we have $f = \phi f \phi^{-1}$. Suppose that $f$ is generated by Hamiltonian function $F$. Then $\phi f \phi^{-1}$ is generated by $F \circ \phi^{-1}$. Hence $F = F \circ \phi^{-1}$. By the arbitrariness of $F$, we have $\phi = \mathrm{id}$.
(iii) The quasi-triangle inequality holds for $r_{\alpha,k}$.

$$\begin{aligned} r_{\alpha,k}(\phi\psi) &= \sup_{\{f \mid \|f\|_k \le \alpha\}} d_k(f, \phi\psi f \psi^{-1}\phi^{-1}) \\ &\le 2^k \left( \sup_{\{f \mid \|f\|_k \le \alpha\}} d_k(f, \phi f \phi^{-1}) + \sup_{\{f \mid \|f\|_k \le \alpha\}} d_k(\phi f \phi^{-1}, \phi\psi f \psi^{-1}\phi^{-1}) \right) \\ &= 2^k \left( r_{\alpha,k}(\phi) + r_{\alpha,k}(\psi) \right). \end{aligned}$$

■

If $r_{\infty,k}(\phi) < \infty$ we call $\phi$ a **k-bounded isometry**. Clearly, every $k$-bounded symplectomorphism is bounded. As in (3.3) for $\phi \in \mathrm{Ham}(M,\omega)$ we have

$$d_k(f, C_\phi f) = d_k(\mathrm{id}, [\phi, f]) \le 2^{k+1}\|\phi\|_k, \quad \forall f \in \mathrm{Ham}(M,\omega) \tag{3.6}$$

and thus $r_{\infty,k}(\phi) \le 2^{k+1}\|\phi\|_k$.

Let

$$\mathrm{BI}_k(M) = \{\phi \in \mathrm{Symp}(M,\omega) \mid r_{\infty,k}(\phi) < \infty\}$$

and

$$\mathrm{BI}_{k,0}(M) = \mathrm{BI}_k(M) \cap \mathrm{Symp}_0(\mathrm{M}),$$

which correspond to the sets $\mathrm{BI}(M)$ and $\mathrm{BI}_0(M)$ respectively. It is obvious that

$$\mathrm{BI}(M) \supseteq \mathrm{BI}_1(M) \supseteq \mathrm{BI}_2(M) \supseteq \cdots \supseteq \mathrm{BI}_k(M) \supseteq \cdots \supseteq \mathrm{Ham}(M,\omega)$$

because (3.6) and $\|\cdot\|_H = \|\cdot\|_0 \le \|\cdot\|_1 \le \cdots \le \|\cdot\|_k \le \cdots$. It is probably that $\mathrm{BI}_k(M) \ne \mathrm{BI}(M)$ for some $k \in \mathbb{N}$ and $M$. Corresponding to Conjecture 1 we have

**Conjecture 2.** For every symplectic manifold $M$ and every integer $m \in \mathbb{N} \cup \{0\}$,

$$\bigcap_{k=m}^{\infty} \mathrm{BI}_{k,0}(M) = \mathrm{BI}_{m-1,0}(M). \tag{BIC$_m$}$$

In particular, for every symplectic manifold $M$,

$$\bigcap_{k=0}^{\infty} \mathrm{BI}_{k,0}(M) = \mathrm{Ham}(M,\omega). \tag{WBIC}$$

Here (WBIC) means weak bounded isometry conjecture. Clearly, the proof of (WBIC) is more hopeful than one of Conjecture 1. In particular, all manifolds mentioned above satisfy $(\mathrm{BIC})_m$ and $(\mathrm{WBIC})$.

In the following we shall point out that many results in [16] can be generalized to the case of our quasi-metrics. Carefully checking the proof of Theorem 1.3.F in [16] we immediately obtain the following generalization of it.

**Proposition 3.2.** *Suppose that a symplectic manifold* $(M,\omega)$ *satisfies* $\mathrm{BI}_{k,0}(M) = \mathrm{Ham}(M,\omega)$ *for some* $k \in \mathbb{N} \cup \{0\}$*. Then every symplectomorphism* $\phi \in \mathrm{Symp}(M,\omega)$ *which acts nontrivially on* $H^1_c(M,\mathbb{R})$ *is* $k$*-unbounded.*

Since a $k$-bounded symplectomorphism is also bounded, the following two propositions are, respectively, direct consequences of Theorems 1.4.A, 1.3.C in [16].

**Proposition 3.3.** *If a closed Lagrangian submanifold $L \subset (M, \omega)$ admits a Riemannian metric with non-positive sectional curvature, and the inclusion $L \hookrightarrow M$ induces an injection on fundamental groups, then $\phi(L) \cap L \neq \emptyset$ for every $k$-bounded $\phi \in \mathrm{Symp}(M, \omega)$.*

**Proposition 3.4.** *Let $S$ be a closed surface of genus greater than $0$ and let $(W, \omega_W)$ be closed and weakly exact (i.e., $\omega_W|_{\pi_2(W)} = 0$). Suppose that $\phi \times \psi$ be a $k$-bounded symplectomorphism of $(S \times W, \omega_S \oplus \omega_W)$. Then the symplectomorphism $\phi$ is Hamiltonian.*

Finally, we give the corresponding result of Theorem 5.1.A in [16].

**Theorem 3.5.** *For the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$, any compactly supported symplectic diffeomorphism $\phi$ of $\mathbb{R}^{2n}$ is $k$-bounded. Precisely, we have*

$$r_{\infty,k}(\phi) = \sup\{d_k(f, C_\phi f) \mid f \in \mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)\} \leq 2^{3k+2} E(\mathrm{supp}(\phi)),$$

*where $E(\mathrm{supp}(\phi))$ is the cylindrical capacity of $\mathrm{supp}(\phi)$. Recall that cylindrical capacity $E(X)$ of a bounded subset $X \subset \mathbb{R}^{2n}$ is definded by*

$$\inf\left\{c > 0 \mid \exists\, \phi \in \mathrm{Ham}(\mathbb{R}^{2n}, \omega_0) \text{ such that } \phi(X) \subset B^2(c) \times \mathbb{R}^{2n-2}\right\},$$

*where $B^2(c)$ stands for a disc of area $c$.*

This result can be proved along the proof lines of [16, Theorem 5.1.A]. For the sake of completeness we give the proof of it. Firstly, we give a corresponding lemma with Lemma 5.1.B in [16].

**Lemma 3.6.** *For all maps $\phi \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ and $f, g \in \mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$,*

$$d_k(f, C_\phi f) \leq 2^k d_k(f, C_{g\phi g^{-1}} f) + 2^{3k+2} \|g\|_k.$$

*Proof.* The quasi-triangle inequality implies that

$$d_k(f, C_\phi f) \leq 2^k (d_k(f, C_{g\phi g^{-1}} f) + d_k(C_\phi f, C_{g\phi g^{-1}} f)).$$

Write $I = d_k(C_\phi f, C_{g\phi g^{-1}} f)$. Then

$$\begin{aligned} I &= d_k(\phi f \phi^{-1}, g\phi g^{-1} f g \phi^{-1} g^{-1}) = d_k(f, \phi^{-1} g \phi g^{-1} f g \phi^{-1} g^{-1} \phi) \\ &= d_k(C_{[g,\phi^{-1}]} f, f), \end{aligned}$$

By Lemma 3.1 we have

$$\begin{aligned} I &\le r_{\infty,k}([g,\phi^{-1}]) = r_{\infty,k}(g\phi^{-1}g^{-1}\phi) \\ &\le 2^k(r_{\infty,k}(g) + r_{\infty,k}(\phi^{-1}g^{-1}\phi)) \\ &= 2^{k+1}r_{\infty,k}(g) \le 4^{k+1}\|g\|_k \end{aligned}$$

because of (3.6). ■

**Lemma 3.7.** *Suppose that $K$ is a compact subset in $\mathbb{R}^{2n}$, and that $L \subset \mathbb{R}^{2n}$ is a hyperplane so that $K$ lies on the left of $L$. Let $g$ be a compactly supported Hamiltonian diffeomorphism such that $g(K)$ sits in the right side of $L$, and let $L'$ be an arbitrary hyperplane parallel to $L$ such that $g(K)$ lies between $L$ and $L'$. Then there exists another compactly supported Hamiltonian diffeomorphism $g'$ such that $\|g'\|_k = \|g\|_k$ and $g'(K)$ lies on the right of $L'$.*

*Proof.* Let $(x_1, y_1, \dots, x_n, y_n)$ denote the coordinates in $\mathbb{R}^{2n}$. Without loss of generality we may assume that for some $v > 0$ and $\varepsilon > 0$,

$$L = \{x_1 = 0\} \quad \text{and} \quad L' = \{x_1 = v\},$$

$$x_1 < -\varepsilon \quad \forall (x_1, y_1, \dots, x_n, y_n) \in K.$$

Choose a cut off function $\eta(t) : \mathbb{R} \to [0, 1]$ satisfying

$$\eta|_{(-\infty,-\varepsilon]} = 0 \quad \text{and} \quad \eta|_{[0,+\infty)} = 1.$$

Let $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the Hamiltonian diffeomorphism generated by the function

$$\mathbb{R}^{2n} \ni (x_1, y_1, \dots, x_n, y_n) \mapsto H(x_1, y_1, \dots, x_n, y_n) = \eta(x_1) \cdot v y_1 \in \mathbb{R}.$$

(It is not compactly supported!) It is easily checked that

$$S(x_1, y_1, \dots, x_n, y_n) = (x_1, y_1, \dots, x_n, y_n) \quad \text{if } x_1 < -\varepsilon,$$

$$S(x_1, y_1, \dots, x_n, y_n) = (x_1 + v, y_1, \dots, x_n, y_n) \quad \text{if } x_1 > 0.$$

Set $g' := SgS^{-1}$. It is also a compactly supported Hamiltonian diffeomorphism and $\|g'\|_k = \|g\|_k$ by the bi-invariance of the quasi-metric $d_k$. Clearly

$$g'(K) = Sg(K) = g(K) + (v, 0, \dots, 0)$$

lies on the right of $L'$. ■

*Proof of Theorem 3.5.* By the definition of the cylindrical capacity, for a sufficiently small $\delta > 0$, there exists a Hamiltonian diffeomorphism $\psi$ such that

$$\psi(\mathrm{supp}(\phi)) \subset B^2(c) \times \mathbb{R}^{2n-2}, \quad \text{where} \quad c < E(\mathrm{supp}(\phi)) + \delta.$$

By composing with a suitable Hamiltonian diffeomorphism we may assume that $\psi(\mathrm{supp}(\phi))$ sits in $Q \times \mathbb{R}^{2n-2}$, where $Q$ is an open square in the $(x_1, y_1)$-plane with area $c$. Note that the displacement energy of $Q$ is just $c$. By the example shown in [27, p.17], there exists a Hamiltonian isotopy $\{g_t\} \subset \mathrm{Ham}(\mathbb{R}^2, \omega_0)$ with the time-1 map $g$ satisfying $\mathrm{Length}(g_t) = c$ and $Q \cap g(Q) = \emptyset$. Since the Hamiltonian of the flow $\{g_t\}$ may be chosen to be autonomous, we have $\|g\|_k \le \mathrm{Length}_k(g_t) = \mathrm{Length}(g_t) = c$. Then for $\widetilde{g} := g \times id : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, we have

$$\|\widetilde{g}\|_k \le \|g\|_k \le c < E(\mathrm{supp}(\phi)) + \delta.$$

Obverse that $\psi(\mathrm{supp}(\phi))$ and $\widetilde{g}\psi(\mathrm{supp}(\phi))$ have a positive distance. We can construct a hyperplane $L$ lying between $\psi(\mathrm{supp}(\phi))$ and $\widetilde{g}\psi(\mathrm{supp}(\phi))$ such that $\psi(\mathrm{supp}(\phi))$ strictly sits in the left side of $L$. For an arbitrary fixed $f \in \mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$, since $\mathrm{supp}(f)$ is a compact set of $\mathbb{R}^{2n}$ by the assumption we may choose a hyperplane $L'$ parallel to $L$ such that $\widetilde{g}\psi(\mathrm{supp}(\phi))$ strictly lies between $L$ and $L'$ and that $\psi(\mathrm{supp}(f))$ strictly sits in the left side of $L'$. Applying Lemma 3.7 to $K = \psi(\mathrm{supp}(\phi))$ we get a Hamiltonian diffeomorphism $\widetilde{g}'$ such that $\|\widetilde{g}'\|_k = \|\widetilde{g}\|_k < E(\mathrm{supp}(\phi)) + \delta$ and that $\widetilde{g}'(K)$ lies in the right side of $L'$ and hence

$$\widetilde{g}'(\psi(\mathrm{supp}(\phi))) \cap \psi(\mathrm{supp}(f)) = \emptyset.$$

It follows that $C_{(\psi^{-1}\widetilde{g}'\psi)\phi(\psi^{-1}\widetilde{g}'\psi)^{-1}} f = f$ since $f(\mathrm{supp}(f)) = \mathrm{supp}(f)$. So Lemma 3.6 leads to

$$d_k(f, C_\phi f) \le 2^{3k+2}\|\psi^{-1}\widetilde{g}'\psi\|_k = 2^{3k+2}\|\widetilde{g}'\|_k < 2^{3k+2}E(\mathrm{supp}(\phi)) + 2^{3k+2}\delta.$$

Since $\delta$ can be chosen arbitrarily small, the desired estimate is obtained. ■

Finally, as in [16, 5.2] using Lemma 3.1 and Proposition 3.5 we deduce that the functions

$$D_k(f, g) = r_{\infty,k}(g^{-1}f), \quad k = 0, 1, 2, \dots \tag{3.7}$$

give a sequence of non-degenerate bi-invariant quasi-metrics on the group $\mathrm{Symp}_c(\mathbb{R}^{2n}, \omega_0)$ of all compactly supported symplectomorphisms of $(\mathbb{R}^{2n}, \omega_0)$.

### 3.2. Banyaga's method

Let $(M, \omega)$ be a closed (i.e., compact and without boundary) symplectic manifold. For a smooth path $[a, b] \ni t \mapsto \phi_t \in \mathrm{Symp}(M, \omega)$, (which means the mapping $(x, t) \mapsto \phi_t(x)$ to be smooth), it determines a unique smooth family of symplectic vector fields,

$\dot{\phi}_t(x) = \frac{d\phi_t}{dt} \circ \phi_t^{-1}(x)$, whose dual 1-form $i_{\dot{\phi}_t}\omega$ is closed. When $[a,b] = [0,1]$ and $\phi_0 = \mathrm{id}$, $\Phi := \{\phi_t\}$ is called a **symplectic isotopy** on $(M,\omega)$.

Fix a Riemannian metric $g$ on $M$. A symplectic vector field $X$ on $M$ is said to be a **harmonic vector field** if $i_X\omega$ is a harmonic form; and a smooth path $[a,b] \ni t \mapsto \phi_t \in \mathrm{Symp}(M,\omega)$ is called **harmonic** if each form $i_{\dot{\phi}_t}\omega$ is harmonic. In particular, a symplectic isotopy $\Phi = \{\phi_t\}$ on $(M,\omega)$ is called a **harmonic isotopy** if it is a harmonic path in $\mathrm{Symp}(M,\omega)$.

**Lemma 3.8.** [2, Lemma 1]. *Any smooth path $[a,b] \ni t \mapsto \phi_t \in \mathrm{Symp}(M,\omega)$ can be decomposed in a unique way as $\phi_t = \rho_t\psi_t$, where $[a,b] \ni t \mapsto \rho_t \in \mathrm{Symp}(M,\omega)$ is a harmonic path and $[a,b] \ni t \mapsto \psi_t \in \mathrm{Ham}(M,\omega)$ is a (smooth) Hamiltonian path. In particular, if $\phi_t$ is a Hamiltonian path, then $\phi_t = \psi_t$ and $\rho_t = \mathrm{id}$.*

Fix a basis $\{h_1,\dots,h_r\}$ of harmonic 1-forms, where $r = \dim H^1(M,\mathbb{R})$. The space of harmonic 1-forms on $M$ is equipped with the following Euclidean metric:

$$|h| = \sum_i |\lambda_i| \quad \text{if } h = \sum_i \lambda_i h_i. \tag{3.8}$$

Banyaga defined in [2] the length of a symplectic isotopy $\Phi = \{\phi_t\}$ by

$$l_{HL}(\Phi) = \int_0^1 \big(|\mathcal{H}_t| + (\max_x U_t - \min_x U_t)\big)dt, \tag{3.9}$$

where $\mathcal{H}_t$ and $U_t$ are smooth families of harmonic 1-forms and functions respectively and satisfy the Hodge decomposition

$$i_{\dot{\phi}_t}\omega = \mathcal{H}_t + dU_t. \tag{3.10}$$

Clearly, for a Hamiltonian isotopy $\Phi$ the formula (3.9) reduces to (1.2). As for (1.2) we can prove that $l_{HL}(\Phi)$ is independent of the choice of parametrization of the path $\Phi$. However we don't have $l_{HL}(\Phi) = l_{HL}(\Phi^{-1})$ in general, where $\Phi^{-1} = \{\phi_t^{-1}\}$.

In [2] the energy $e(\phi)$ of any $\phi \in \mathrm{Symp}_0(M,\omega)$ is defined by

$$e(\phi) = \inf_{\Phi}(l_{HL}(\Phi)),$$

where $\Phi$ runs over all symplectic isotopies connecting the identity and $\phi$. The so called **Hofer-like metric** is the map

$$\|\cdot\|_{HL} : \mathrm{Symp}_0(M,\omega) \to \mathbb{R}\cup\{\infty\},\ \phi \mapsto \frac{1}{2}\big(e(\phi) + e(\phi^{-1})\big),$$

which is actually a norm on $\mathrm{Symp}_0(M,\omega)$. Notice that the norm $\|\cdot\|_{HL}$ depends on the choice of the Riemannian metric $g$ on $M$ and the choice of the Euclidean norm

$|\cdot|$ on the space of harmonic 1-forms. However, different choices for $g$ and $|\cdot|$ yield equivalent metrics. The **Hofer-like distance** $d_{HL}$ on each connected component of $\mathrm{Symp}(M,\omega)$ is defined by $d_{HL}(\phi,\psi) := \|\phi\psi^{-1}\|_{HL}$. It is right invariant, but not left invariant. When $\Phi$ is a Hamiltonian isotopy, (3.9) reduces to (1.2), thus $\|\phi\|_{HL} \leq \|\phi\|_H \ \forall \phi \in \mathrm{Ham}(M,\omega)$. Moreover the subgroup $\mathrm{Ham}(M,\omega)$ is closed in $\mathrm{Symp}(M,\omega)$ endowed with the metric topology defined by $\|\cdot\|_{HL}$. As expected by Banyaga, Buss and Leclercq [6] showed that the restriction of the Hofer-like metric to $\mathrm{Ham}(M,\omega)$ is equivalent to the Hofer metric.

Given any smooth symplectic path $\alpha : [a,b] \to \mathrm{Symp}(M,\omega)$, we have a decomposition as (3.10), $i_{\dot\alpha(t)}\omega = \mathcal{H}_t + dU_t$. For every integer $k = 0,1,2,\ldots$, we define the $k$-**length** of $\alpha$ as

$$l_{HL,k}(\alpha) := \sum_{i=0}^{k} \int_a^b \left[ \left| \frac{\partial^i \mathcal{H}_t}{\partial t^i} \right| + \left( \max_x \frac{\partial^i U_t}{\partial t^i} - \min_x \frac{\partial^i U_t}{\partial t^i} \right) \right] dt. \tag{3.11}$$

where $|\cdot|$ is as in (3.8). Obviously, $l_{HL,0}(\alpha) = l_{HL}(\alpha)$. But when $k \geq 1$, $l_{HL,k}$ depends on the choice of parametrization of the path which is different from $l_{HL}$.

A continuous path $\Phi : [0,1] \to \mathrm{Symp}_0(M,\omega)$ is called a **piecewise smooth symplectic isotopy** if there exists a division $0 = t_0 < t_1 < \cdots < t_n = 1, n \in \mathbb{N}$, such that for each $i = 1,\ldots,n$, $\Phi_i = \Phi|_{[t_{i-1},t_i]}$ is smooth, and $\Phi(0) = \mathrm{id}$. We define the $k$-**length** of $\Phi$ as

$$l_{HL,k}(\Phi) := \sum_{i=1}^{n} l_{HL,k}(\Phi_i).$$

For any $\phi \in \mathrm{Symp}_0(M,\omega)$, let $\Omega(\phi)$ be the set of all piecewise smooth isotopies $\Phi : [0,1] \to \mathrm{Symp}_0(M,\omega)$ with $\Phi(1) = \phi$. Define the energy of $\phi$ by

$$e_k(\phi) := \inf \left\{ l_{HL,k}(\Phi) \,\middle|\, \Phi \in \Omega(\phi) \right\},$$

and the corresponding **Hofer-like** $k$-**metric** by

$$\|\phi\|_{HL,k} = \frac{1}{2}\big(e_k(\phi) + e_k(\phi^{-1})\big), \quad k = 0,1,\ldots.$$

We define the **Hofer-like** $k$-**distance** by

$$d_{HL,k}(\phi,\psi) := \|\phi\psi^{-1}\|_{HL,k}$$

on every connected component of $\mathrm{Symp}(M,\omega)$.

**Proposition 3.9.** *On a closed symplectic manifold $(M,\omega)$ with a fixed Riemannian metric $g$, we have*

$(i)$ $\|\cdot\|_{HL} = \|\cdot\|_{HL,0} \le \|\cdot\|_{HL,1} \le \|\cdot\|_{HL,2} \le \cdots$,

$(ii)$ *for each* $k$*,* $\|\cdot\|_{HL,k}$ *is a quasi-norm on* $\mathrm{Symp}_0(M,\omega)$.

*Proof.* (i) Notice that the corresponding Lemma 1.2 still hold in the current context. Hence for $\phi \in \mathrm{Symp}_0(M,\omega)$, we have $e_0(\phi) = e(\phi)$, so $\|\cdot\|_{HL,0} = \|\cdot\|_{HL}$.
(ii) Symmetry: By definition, if $\phi, \psi \in \mathrm{Symp}(M,\omega)$ is in the same connect component, then $d_{HL,k}(\phi,\psi) = d_{HL,k}(\psi,\phi)$.

Non-degeneracy: By (i) and the non-degeneracy of $\|\cdot\|_{HL}$, this is obvious.

The quasi-triangle inequality: Since the proof is similar to the one of property (ii) in Theorem 1.4, we skip some details and only outline the ideas of the proof. For any two symplectomorphisms $\phi, \psi \in \mathrm{Symp}_0(M,\omega)$, we choose

$$\Omega(\phi) \ni \Phi : [0,1] \to \mathrm{Symp}_0(M,\omega)$$

and

$$\Omega(\psi) \ni \Psi : [0,1] \to \mathrm{Symp}_0(M,\omega).$$

Define the concatenation product $\Phi * \Psi$ of $\Phi$ and $\Psi$ by

$$\Phi * \Psi(t) = \begin{cases} \Psi(2t) & 0 \le t \le 1/2 \\ \Phi(2t-1)\psi & 1/2 \le t \le 1. \end{cases}$$

then $\Phi * \Psi \in \Omega(\phi\psi)$. By definition we have

$$\begin{aligned} e_k(\phi\psi) &\le l_{HL,k}(\Phi * \Psi) \le 2^k(l_{HL,k}(\Psi) + l_{HL,k}(\Phi(t)\psi)) \\ &= 2^k(l_{HL,k}(\Phi) + l_{HL,k}(\Psi)) \end{aligned}$$

for all $\Phi \in \Omega(\phi)$, $\Psi \in \Omega(\psi)$. Taking the infimum respectively we get

$$\begin{aligned} e_k(\phi\psi) &\le 2^k(\inf\{l_{HL,k}(\Phi) \mid \Phi \in \Omega(\phi)\} + \inf\{l_{HL,k}(\Psi) \mid \Psi \in \Omega(\psi)\}) \\ &= 2^k(e_k(\phi) + e_k(\psi)). \end{aligned}$$

That is, $e_k$ (and so $\|\cdot\|_{HL,k}$) satisfies the quasi-triangle inequality. ■

**Proposition 3.10.** *Let* $(M,\omega)$ *be a closed symplectic manifold. Then for each* $k$ *the subgroup* $\mathrm{Ham}(M,\omega)$ *is closed in* $\mathrm{Symp}(M,\omega)$ *with respect to the metric topology defined by* $\|\cdot\|_{HL,k}$*.*

*Proof.* The ideas are similar to those of Theorem 14.2.A in [27]. Suppose there exists a sequence $\{f_n\} \subset \mathrm{Ham}(M,\omega)$ and $\phi \in \mathrm{Symp}(M,\omega)$, satisfying $d_{HL,k}(f_n,\phi) \to 0$ when $n \to \infty$. We intend to prove $\phi \in \mathrm{Ham}(M,\omega)$.

Since $\lim_{n\to\infty} \|f_n\phi^{-1}\|_{HL,k} = 0$, by the definition of $\|\cdot\|_{HL,k}$, for $\forall\varepsilon > 0$, $\exists N_0 > 0$, such that for each $N \geq N_0$ we have

$$\inf\{l_{HL,k}(\Phi)|\Phi \in \Omega(f_N\phi^{-1})\} < \varepsilon,$$

and so a $\Phi^N \in \Omega(f_N\phi^{-1})$ such that $l_{HL}(\Phi^N) \leq l_{HL,k}(\Phi^N) < \varepsilon$.

Assume the division of $\Phi^N$ is given by $0 = t_0 < t_1 < \cdots < t_n = 1$. For each $i = 1, \ldots, n$, $\Phi_i^N = \Phi^N|_{[t_{i-1},t_i]}$ is a smooth symplectic path. Next we translate $\Phi^N$ into a smooth symplectic isotopy through the precedure of reparametrization. We choose a increasing, surjective smooth function $s_i : [t_{i-1}, t_i] \to [t_{i-1}, t_i]$ for each $i$, and require $s_i$ is constant near the ends of its interval of definition. Define $\widetilde{\Phi}^N : [0,1] \to \mathrm{Symp}_0(M,\omega)$ by

$$\widetilde{\Phi}^N(t) = \Phi_i^N(s_i(t)), \quad \forall t \in [t_{i-1}, t_i],\ i = 1, \ldots, n,$$

then $\widetilde{\Phi}^N$ is a smooth symplectic isotopy. In fact if the harmonic 1-forms and Hamiltonian functions generated by $\Phi^N$ are $\{{}^i\mathcal{H}_t^N\}_{i=1}^n$ and $\{{}^iU_t^N\}_{i=1}^n$ respectively, and the harmonic 1-forms and Hamiltonian functions generated by $\widetilde{\Phi}^N$ are $\widetilde{\mathcal{H}}_t^N$ and $\widetilde{U}_t^N$ respectively, then when $t \in [t_{i-1}, t_i]$, we have

$$\widetilde{\mathcal{H}}_t^N = s_i'(t) \cdot {}^i\mathcal{H}_{s_i(t)}^N \ , \qquad \widetilde{U}_t^N = s_i'(t) \cdot {}^iU_{s_i(t)}^N. \tag{3.12}$$

By the change of variable formula, we get $l_{HL}(\widetilde{\Phi}^N) = l_{HL}(\Phi^N)$, in particular,

$$\int_0^1 \left|\widetilde{\mathcal{H}}_t^N\right| dt = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \left|{}^i\mathcal{H}_t^N\right| dt.$$

Since $l_{HL}(\Phi^N) < \varepsilon$, we get $\int_0^1 |\widetilde{\mathcal{H}}_t^N| dt < \varepsilon$.

Recall that the flux is a surjective homomorphism from the universal covering space $\widetilde{\mathrm{Symp}}_0(M,\omega)$ of $\mathrm{Symp}_0(M,\omega)$ to $H^1(M,\mathbb{R})$ given by

$$\mathrm{Flux}(\{\phi_t\}) = \left[\int_0^1 (i_{\dot{\phi}_t}\omega)dt\right] \in H^1(M,\mathbb{R}). \tag{3.13}$$

$\Gamma_\omega := \mathrm{Flux}\big(\pi_1(\mathrm{Symp}_0(M,\omega))\big) \subset H^1(M,\mathbb{R})$ is called the **flux group** of $(M,\omega)$, and is discrete as proved by Ono in [23]. Flux descends to a surjective homomorphism

$$\mathrm{flux} : \mathrm{Symp}_0(M,\omega) \to H^1(M,\mathbb{R})/\Gamma_\omega$$

with kernel $\mathrm{Ham}(M,\omega)$ (cf. [18]).

For any symplectic isotopy $\Phi$, let $\mathcal{H}(\Phi)$ denote the harmonic representation of the cohomology class $\mathrm{Flux}(\Phi)$. The decomposition (3.10) implies that

$$\mathcal{H}(\Phi) = \int_0^1 \mathcal{H}_t dt.$$

It follows from this that

$$\begin{aligned} \left|\mathcal{H}(\widetilde{\Phi}^N)\right| &= \left|\int_0^1 \widetilde{\mathcal{H}}_t^N dt\right| = \left|\int_0^1 \sum_{i=1}^r \widetilde{\lambda}_i^N(t) h_i dt\right| \\ &= \sum_{i=1}^r \left|\int_0^1 \widetilde{\lambda}_i^N(t) dt\right| \\ &\le \int_0^1 \sum_{i=1}^r |\widetilde{\lambda}_i^N(t)| dt = \int_0^1 |\widetilde{\mathcal{H}}_t^N| dt < \varepsilon, \end{aligned}$$

where $\widetilde{\mathcal{H}}_t^N$ is decomposed as $\sum_{i=1}^r \widetilde{\lambda}_i^N(t) h_i$.

Starting from $\widetilde{\Phi}^N$, we could construct a smooth symplectic path $\widetilde{\Phi}^N \circ \phi$ connecting $\phi$ and $f_N$. Obviously we have $\mathcal{H}(\widetilde{\Phi}^N) = \mathcal{H}(\widetilde{\Phi}^N \circ \phi)$. Choose any symplectic isotopy $\Psi$ from $\mathrm{id}$ to $\phi$, and any Hamiltonian isotopy $\alpha^N$ from $\mathrm{id}$ and $f_N$, we get a loop $(-\alpha^N)\sharp(\widetilde{\Phi}^N \circ \phi)\sharp\Psi$, whose flux has the harmonic representation

$$\mathcal{H}(\Psi) + \mathcal{H}(\widetilde{\Phi}^N \circ \phi) \in \Gamma_\omega$$

because a Hamiltonian path has zero flux. Note that $|\mathcal{H}(\widetilde{\Phi}^N)| < \varepsilon$, and that $\varepsilon$ is arbitrary small. We deduce that $\mathcal{H}(\Psi) \in \Gamma_\omega$ since $\Gamma_\omega$ is discrete. Hence $\phi \in \ker(\mathrm{flux}) = \mathrm{Ham}(M, \omega)$. ∎

**Remark 3.11.** By the method in [2], we can't obtain that $f_N^{-1}\phi$ is a Hamiltonian diffeomorphism for every $N$ large enough, but could only get the distance from $\mathrm{flux}(f_N^{-1}\phi)$ to $\Gamma_\omega$ trends to zero as $N \to \infty$.

Han [11] also introduced a method constructing bi-invariant (quasi) metrics on $\mathrm{Symp}(M, \omega)$ from the Hofer metric. For a fixed positive number $K$, he defined $\|\phi\|_K = \min(\|\phi\|_H, K)$ if $\phi \in \mathrm{Ham}(M, \omega)$, and $K$ otherwise. However, when the above defined quasi-metrics (or metrics) $r_{\alpha,k}$, $\|\cdot\|_K$ are restricted back to $\mathrm{Ham}(M, \omega)$, the induced topologies are in general different from that of the Hofer metric.

## 4. Concluding Remarks

Extensions of the Hofer metric to contact geometry were also studied, see Banyaga and Donato [3], Banyaga and Spaeth [4] and Müller and Spaeth [21]. Our proceeding constructions can be completed in contact manifolds. Let $(N, \alpha)$ be a compact contact manifold of dimension $2n+1$. There exists a one-to-one correspondence between contact isotopies on $(N, \alpha)$ and elements of the space $C^\infty(N \times [0,1])$, $\{f_t\} \leftrightarrow H$, where $i_{X_t}\alpha = H_t$ with $H_t = H(\cdot, t)$ and $X_t = (\frac{d}{dt} f_t) \circ f_t^{-1}$; $H$ is called the contact Hamiltonian function of $\{f_t\}$. Call $\phi \in \mathrm{Diff}(N)$ a strictly contact diffeomorphism if $\phi^*\alpha = \alpha$. A contact isotopy is said to be strictly if each contact diffeomorphism in the isotopy is strictly. Denote by $G_\alpha(N)$ the group of strictly contact diffeomorphisms which are strictly contact isotopic to the identity.

Consider a surjective homomorphism from the universal cover $\widetilde{G_\alpha(N)}$ of $G_\alpha(N)$ to $\mathbb{R}$ given by

$$\{\phi_t\} \mapsto c(\{\phi_t\}) = \frac{1}{\mathrm{Vol}(N)} \int_0^1 \Big( \int_N H_t(x)\nu_\alpha \Big) dt,$$

where $H_t$ is the contact Hamiltonian of $\{\phi_t\}$ and the canonical volume form $\nu_\alpha := \alpha \wedge (d\alpha)^n$. For each $k = 0, 1, 2, \ldots$, we define $k$**-contact length** of $\phi_t$ by

$$\mathrm{Length}_{c,k}(\{\phi_t\}) := |c(\phi_t)| + \sum_{i=0}^k \int_0^1 \left( \max_{x\in N} \frac{\partial^i H}{\partial t^i}(x,t) - \min_{x \in N} \frac{\partial^i H}{\partial t^i}(x,t) \right) dt,$$

and $k$**-contact energy** of $\phi \in G_\alpha(N)$ by

$$E_{c,k}(\phi) = \inf_{\phi_t} \big( \mathrm{Length}_{c,k}(\{\phi_t\}) \big),$$

where $\{\phi_t\}$ takes over all piecewise smooth strictly contact isotopy from $\mathrm{id}$ to $\phi$. If $k=0$ it becomes the contact length and contact energy in [3, (17) and (20)]. Using the results in [3, 21] we may directly prove

**Theorem 4.1.** *For each $k = 0, 1, \ldots$, the mapping*

$$d_{c,k} : G_\alpha(N) \times G_\alpha(N) \to [0, \infty), \quad (\phi, \psi) \mapsto E_{c,k}(\phi\psi^{-1})$$

*is a bi-invariant quasimetric on $G_\alpha(N)$.*

When the contact manifold $(N, \alpha)$ is regular, that is, the Reeb field $R_\alpha$ of $\alpha$ generates a free $S^1$-action on $N$, the quotient manifold $B = N/S^1$ is a base of a principal $S^1$- bundle $\pi : N \to B$ and $B$ has a canonical symplectic form $\omega$ satisfying $\pi^*\omega = d\alpha$. In this case there exists an exact sequence

$$\{1\} \to S^1 \to G_\alpha(M) \xrightarrow{p} \mathrm{Ham}(B, \omega) \to \{1\}.$$

As in the proof of [4, Lemma 4.2] it is not hard to prove that $E_{c,k}(\phi) \geq \|p(\phi)\|_k$ for any $\phi \in G_\alpha(N)$.

As in Hofer geometry it is an important topic to study geodesics of our metrics.

## A. Appendix: Semigroupoid Metrization Theorem

Given a semigroupoid $(G,*)$, let $G^{(1)}=G$, $G^{(2)}=\{(a,b)\in G\times G: a*b$ is well-defined$\}$ and for each $N\in\mathbb{N}$, $N\ge 2$ let

$$G^{(N)}:=\big\{(a_1,\dots,a_N)\in G\times\cdots\times G\ |\ (a_j,a_{j+1})\in G^{(2)}\ \forall j\in\{1,\cdots,N-1\}\big\}.$$

In particular, if $(G,*)$ is a semigroup, $G^{(N)}$ is just the Cartesian product of $N$ copies of $G$.

**Theorem A.1.** ([20, Cor.3.33]). *Let $(G,*)$ be a semigroupoid, and assume that $\psi:G\to[0,\infty]$ is a function with the property that there exists a finite constant $C\ge 1$ such that*

$$\psi(a*b)\le C(\psi(a)+\psi(b)),\quad \text{for all } (a,b)\in G^{(2)}. \tag{A.1}$$

*Introduce*

$$\alpha:=\frac{1}{1+\log_2 C}\in(0,1] \tag{A.2}$$

*and define the function $\psi_\sharp:G\to[0,\infty]$ by*

$$\psi_\sharp(a):=\inf\left\{\left(\sum_{i=1}^N\psi(a_i)^\alpha\right)^{\frac{1}{\alpha}}:\ N\in\mathbb{N},\ (a_1,\dots,a_N)\in G^{(N)},\ a=a_1*\cdots*a_N\right\}. \tag{A.3}$$

*Then $\psi\approx\psi_\sharp$. More specifically, with $C$ the same constant as in (A.1), one has*

$$(2C)^{-2}\psi\le\psi_\sharp\le\psi\quad\text{on } G. \tag{A.4}$$

*In particular, $\psi^{-1}(\{0\})=\psi_\sharp^{-1}(\{0\})$. Furthermore, for every $\beta\in(0,\alpha]$ one has*

$$\psi_\sharp(a*b)^\beta\le\psi_\sharp(a)^\beta+\psi_\sharp(b)^\beta,\quad\forall(a,b)\in G^{(2)}, \tag{A.5}$$

*and $\psi_\sharp=\psi$ on $G$ if and only if $\psi_\sharp(a*b)^\alpha\le\psi_\sharp(a)^\alpha+\psi_\sharp(b)^\alpha$ for all $(a,b)\in G^{(2)}$. Finally, for each $N\in\mathbb{N}$ the original function $\psi$ satisfies*

$$\psi(a_1*\cdots*a_N)\le 4C^2\left\{\sum_{i=1}^N\psi(a_i)^\beta\right\}^{\frac{1}{\beta}} \tag{A.6}$$

*whenever $a_1,\dots,a_N\in G$ are such that*

$$(a_i, a_{i+1}) \in G^{(2)} \quad \text{for every } i \in \{1, \dots, N-1\}. \tag{A.7}$$

*In particular, if* $(a_i)_{i\in\mathbb{N}} \subset G$ *is a sequence with the property that (A.7) holds for every number* $N \in \mathbb{N}$ *with* $N \geq 2$*, then for each finite number* $\beta \in (0, \alpha]$ *one has*

$$\sup_{N\in\mathbb{N}} \psi(a_1 * \cdots * a_N) \leq 4C^2 \left\{ \sum_{i=1}^{\infty} \psi(a_i)^{\beta} \right\}^{\frac{1}{\beta}}. \tag{A.8}$$

**Remark A.2.** When $(G, *)$ is a group, a function $\psi : G \to [0, \infty]$ is said to be **conjugate invariant** provided $\psi(b * a * b^{-1}) = \psi(a)\ \forall a, b \in G$. In this case $\psi_\sharp$ is also conjugate invariant. In fact, for any $b \in G$, $(a_1, \dots, a_N) \in G^{(N)},\ a = a_1 * \cdots * a_N$, since $b * a * b^{-1} = (b * a_1 * b^{-1}) \cdots * (b * a_N * b^{-1})$ and

$$\psi(b * a_i * b^{-1}) = \psi(a_i) \quad \forall i = 1, \dots, N,$$

it follows from the definition of $\psi_\sharp(a)$ in $(A.3)$ that $\psi_\sharp(b * a * b^{-1}) = \psi_\sharp(a)$.

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## References

1. A. Banyaga, Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, *Comment. Math. Helv.* (*French*), **53(2)** (1978), 174-227.
2. A. Banyaga, *A Hofer-like metric on the group of symplectic diffeomorphisms*, Proceedings of the 2007 AMS-SIAM Summer Research Conference “Symplectic Topology and Measure presrving Dynamical Systems”, Snowbird, UT, Contemporary Math. 2010, pp. 1-23.
3. A. Banyaga and P. Donato, Lengths of contact isotopies and extensions of the Hofer metric, *Annals of Global Analysis and Geometry*, **30** (2006), 299-312.
4. A. Banyaga and P. Spaeth, *The group of strictly contact homeomorphisms*, arXiv:0812. 2461, 2008.
5. L. Buhovsky and Y. Ostrover, On the uniqueness of Hofer’s geometry, *Geom. Funct. Anal.*, **21** (2011), 1296-1330.
6. G. Buss and R. Leclercq, Pseudo-distances on symplectomorphism groups and applications to flux theory, *Math. Z.*, **272** (2012), 1001-1022.

7. C. Campos-Apanco and A. Pedroza, Bounded symplectic diffeomorphisms and split flux groups, *Proc. of Amer. Math. Soc.*, **140** (2012), 2883-2892.

8. Yu. V. Chekanov, Invariant Finsler metrics on the space of Lagrangian embeddings, *Math. Z.*, **234(3)** (2000), 605-619.

9. Y. Eliashberg and L. Polterovich, Bi-invariant metrics on the group of Hamiltonian diffeomorphisms, *Internat. J. Math.*, **4(5)** (1993), 727-738.

10. M. Entov, *Quasi-morphisms and quasi-states in symplectic topology*, preprint, arXiv: 1404.6408, 2014.

11. Z. Han, The bounded isometry conjecture for the Kodaira-Thurston manifold and 4-torus, *Israel J. Math.*, **176** (2010), 285-306.

12. H. Hofer, On the topological properties of symplectic maps, *Proc. Roy. Soc. Edinburgh Sect. A*, **115(1-2)** (1990), 25-38.

13. H. Hofer, Estimates for the energy of a symplectic map, *Comm. Math. Helv.*, **68** (1993), 48-72.

14. H. Hofer and E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Verlag, Basel, 1994.

15. F. Lalonde and C. Pestieau, Stabilisation of symplectic inequalities and applications, *American Mathematical Society Translations (2)*, **196** (1999), 63-72.

16. F. Lalonde and L. Polterovich, Symplectic diffeomorphisms as isometries of Hofer's norm, *Topology*, **36(3)** (1997), 711-727.

17. F. Lalonde and D. McDuff, The geometry of symplectic energy, *Ann. of Math. (2)*, **141(2)** (1995), 349-371.

18. D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Clarendon Press Oxford, 1995.

19. D. McDuff, *Loops in the Hamiltonian group: a survey*, Symplectic topology and measure preserving dynamical systems, 127-148, Contemp. Math., 512, Amer. Math. Soc., Providence, RI, 2010.

20. D. Mitrea, I. Mitrea, M. Mitrea and S. Monniaux, *Groupoid Metrization Theory*, Springer Science Business Media, New York, 2013.

21. S. Müller and P. Spaeth, *On topological contact dynamics I: symplectization and applications of the energy-capacity inequality*, arXiv:1110.6705, 2011.

22. Y.-G. Oh, Spectral invariants, analysis of the Floer moduli space, and geometry of the Hamiltonian diffeomorphism group, *Duke Math. J.*, **130** (2005), 199-295.

23. K. Ono, Floer-Novikov cohomology and the flux conjecture, *Geom. Funct. Anal.*, **16** (2006), 981-1020.

24. Y. Ostrover, *When symplectic topology meets Banach space geometry*, preprint, arXiv: 1404.6954,2014.

25. A. Pedroza, On the bounded isometry conjecture, *C. R. Acad. Sci. Paris, Ser. I*, **349** (2011), 1097-1100.

26. L. Polterovich, Symplectic displacement energy for Lagrangian submanifolds, *Ergodic Theory Dynam. Systems*, **13(2)** (1993), 357-367.

27. L. Polterovich, The geometry of the group of symplectic diffeomorphisms, *Lectures in Mathematics ETH Zürich*, Birkhüser Verlag, Basel, 2001.

28. L. Polterovich, *Geometry on the group of Hamiltonian diffeomorphisms*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 401-410 (electronic).

29. M. Schwarz, On the action spectrum for closed symplectically aspherical manifolds, *Pacific J. Math.*, **193** (2000), 1046-1095.

30. J. C. Sikorav, *Systémes hamiltoniens et topologie symplectique*, prepnnt, Dipartimento di Matematica dell' Universitá di Pisa, 1990.

31. C. Viterbo, Symplectic topology as the geometry of generating functions, *Math. Ann.*, **292** (1992), 685-710.

Guangcun Lu and Tie Sun
School of Mathematical Sciences
Beijing Normal University
Laboratory of Mathematics and Complex Systems
Ministry of Education
Beijing 100875
People's Republic of China
E-mail: gclu@bnu.edu.cn
suntie@mail.bnu.edu.cn