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THE (NORMALIZED) LAPLACIAN EIGENVALUE OF SIGNED GRAPHS

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Abstract. A signed graph $\Gamma = (G, \sigma)$ consists of an unsigned graph $G = (V, E)$ and a mapping $\sigma : E \to \{+, -\}$. Let Γ be a connected signed graph and $L(\Gamma), \mathcal{L}(\Gamma)$ be its Laplacian matrix and normalized Laplacian matrix, respectively. Suppose $\mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n \geq 0$ and $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0$ are the Laplacian eigenvalues and the normalized Laplacian eigenvalues of Γ, respectively. In this paper, we give two new lower bounds on λ_1 which are both stronger than Li's bound [8] and obtain a new upper bound on λ_n which is also stronger than Li's bound [8]. In addtion, Hou [6] proposed a conjecture for a connected signed graph Γ: $\sum\limits_{i=1}^{k} \mu_i > \sum\limits_{i=1}^{k} d_i$ $(1 \leq k \leq n-1)$. We investigate $\sum\limits_{i=1}^{k} \mu_i$ $(1 \leq k \leq n-1)$ and partly solve the conjecture.

1. INTRODUCTION

A signed graph $\Gamma = (G, \sigma)$ consists of an unsigned graph $G = (V, E)$ and a mapping $\sigma : E \to \{+, -\}$. The graph G is called the underlying graph of Γ. Signed graphs were introduced by Harary [5] in connection with the study of social balance in social psychology. They have been extensively studied because they come up naturally in many areas such as topological graph theory, group theory and so on. More results on signed graphs can be founded in [1].

Let $\Gamma = (G, \sigma)$ be a signed graph with the vertex set $V = V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$. For $v_i \in V(\Gamma)$, the degree of the vertex v_i, denoted by $d(v_i)$ (or d_i), is the number of vertices adjacent to v_i. Without loss of generality, we may suppose $d_1 \geq d_2 \geq \cdots \geq d_n$ throughout the paper. Let $D = D(G) = \text{diag}\{d_1, \ldots, d_n\}$ be a diagonal matrix of G. We often use the notation $v_i \sim v_j$ (or $i \sim j$) to mean that v_i (or i) is adjacent to v_j (or j) in Γ.

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The Laplacian matrix $L(\Gamma)$ (or L) of a signed graph $\Gamma = (G, \sigma)$ is defined to have entries

$$L_{ij} = \begin{cases} d_i & \text{if } i = j; \\ -\sigma(ij) & \text{if } i \sim j; \\ 0 & \text{otherwise.} \end{cases}$$

The normalized Laplacian $\mathcal{L}(\Gamma)$ (or $\mathcal{L}$) of a signed graph $\Gamma = (G, \sigma)$ is defined by $\mathcal{L}(\Gamma) = D^{-\frac{1}{2}} L(\Gamma) D^{-\frac{1}{2}}$; that is, $\mathcal{L}$ has the entries

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -\sigma(ij)\dfrac{1}{\sqrt{d_i d_j}} & \text{if } i \sim j; \\ 0 & \text{otherwise.} \end{cases}$$

It is known that both of L and $\mathcal{L}$ are positive semidefinite matrices. Let $\mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n \geq 0$ and $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0$ be the Laplacian eigenvalues and the normalized Laplacian eigenvalues of Γ, respectively.

The adjacency matrix and the Laplacian matrix have been more widely investigated than the normalized Laplacian matrix. One reason for this is that the normalized Laplacian is a rather new tool which has been popularized by Chung [2]. In some situations, the normalized Laplacian matrix is a more natural tool that works better than the adjacency matrix or Laplacian matrix. We can obtain much information about the graph from the normalized Laplacian eigenvalues. Let $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$ be the normalized Laplacian eigenvalues of a graph G. In [2], Chung proved $\lambda_{n-1} \leq \dfrac{n}{n-1}$ $(n \geq 2)$ with equality holding if and only if G is a complete graph K_n and a graph which is not a complete graph, then $\lambda_{n-1} \leq 1$. In 2008 and 2011, Li etc. [7, 9] gave some results on λ_{n-1} about the effect by grafting edges. In 2003 and 2009, Hou etc. [6] and Li etc. [8] introduced the notion of the Laplacian and the normalized Laplacian of signed graphs, respectively. They extended some fundamental concepts of Laplacian and normalized Lapalcian from graphs to signed graphs, respectively. In this paper, we give two new lower bounds on λ_1 which are both stronger than Li's bound [8] and obtain a new upper bound on λ_n which is also stronger than Li's bound [8]. In addition, Hou [6] proposed a conjecture for a connected signed graph Γ: $\sum_{i=1}^{k} \mu_i > \sum_{i=1}^{k} d_i$ $(1 \leq k \leq n-1)$. We investigate $\sum_{i=1}^{k} \mu_i$ $(1 \leq k \leq n-1)$ and partly solve the conjecture.

2. Lower Bound on the Largest Normalized Laplacian Eigenvalue of Signed Graphs

Let M_1 and M_2 be two matrices of order n. We call two matrices M_1 and M_2 signature similar if there exists a signature diagonal matrix $S = \text{diag}\{s_1, \ldots, s_n\}$ with $s_i = \pm 1$ such that $M_2 = S M_1 S$.

Lemma 2.1. [6]. *Let* $\Gamma_1 = (G, \sigma_1)$ *and* $\Gamma_2 = (G, \sigma_2)$ *be signed graphs with the same underlying graph* G. *Then* $\Gamma_1 \sim \Gamma_2$ *if and only if* $\mathcal{L}(\Gamma_1)$ *and* $\mathcal{L}(\Gamma_2)$ *are signature similar.*

Let C be a cycle of a signed graph $\Gamma = (G, \sigma)$. The sign of C is denoted by $\text{sgn}(C) = \prod_{e \in C} \sigma(e)$. A cycle C is called positive (resp. negative) if $\text{sgn}(C) = +$ (resp. $\text{sgn}(C) = -$). A signed graph is balanced if all cycles are positive.

Lemma 2.2. [6]. *Let* $\Gamma = (G, \sigma)$ *be a connected signed graph. Then the following conditions are equivalent:*

(1) $\Gamma = (G, \sigma)$ *is balanced;*

(2) $(G, \sigma) \sim (G, +)$*;*

(3) *There exists a partition* $V(\Gamma) = V_1 \cup V_2$ *such that every edge between* V_1 *and* V_2 *is negative and every edge within* V_1 *or* V_2 *is positive.*

Lemma 2.3. [2, 8]. *Let* $\Gamma = (G, \sigma)$ *be a connected signed graph with* n *vertices.*

(1) $\lambda_1 > 1$ *and* $0 \le \lambda_n < 1$*;*

(2) *If* Γ *is balanced, then* $\lambda_1 \ge \frac{n}{n-1}$ *with equality holding if and only if* $\Gamma \sim (K_n, +)$.

We now introduce two new lower bounds on λ_1.

Theorem 2.1. *Let* $\Gamma = (G, \sigma)$ *be a connected signed graph with* n *vertices and the normalized Laplacian eigenvalues* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. *Then*

$$\lambda_1 \ge 1 + \frac{2}{n} \sum_{i \sim j} \frac{1}{d_i d_j}.$$

Moreover, equality holds if and only if $\Gamma = (G, \sigma) \sim (K_n, +)$.

Proof. We consider the trace of the matrix $(\mathcal{L} - xI)^2$ with $x = \frac{\lambda_1}{2}$,

$$\text{tr}(\mathcal{L} - xI)^2 = n\left(1 - \frac{\lambda_1}{2}\right)^2 + 2\sum_{i \sim j} \frac{1}{d_i d_j}. \tag{2.1}$$

On the other hand, since $(\mathcal{L} - xI)^2$ has eigenvalues $(\lambda_1 - x)^2, \ldots, (\lambda_n - x)^2$, we have $\text{tr}(\mathcal{L} - xI)^2 = \sum_{i=1}^{n} (\lambda_i - \frac{\lambda_1}{2})^2$.

Since

$$-\frac{\lambda_1}{2} \le 0 - \frac{\lambda_1}{2} \le \lambda_i - \frac{\lambda_1}{2} \le \lambda_1 - \frac{\lambda_1}{2} = \frac{\lambda_1}{2},$$

we have

$$\text{tr}(\mathcal{L} - xI)^2 = \sum_{i=1}^{n} \left(\lambda_i - \frac{\lambda_1}{2}\right)^2 \leq n\left(\frac{\lambda_1}{2}\right)^2. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain $\lambda_1 \geq 1 + \frac{2}{n}\sum_{i\sim j}\frac{1}{d_i d_j}$, where equality holds if and only if $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \dfrac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{n} = 1$. It is a contradiction with $\lambda_n < 1$ from (1) of Lemma 2.3. If Γ is balanced, then $\lambda_n = 0$ (from Lemma 2.2) and equality holds if and only if $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = \dfrac{\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1}}{n} = \dfrac{n}{n-1}$. Then the second part of the theorem follows from (2) of Lemma 2.3. ■

We can view the eigenvectors g of $\mathcal{L}(\Gamma)$ as functions which assign to each vertex v_i of Γ a real value $g(i)$. In particular, if $V = \{v_1, v_2, \ldots, v_n\}$ and $g = (g(1), g(2), \ldots, g(n))^T$, then g can be viewed as a function which assigns to each vertex v_i the real value $g(i)$. By letting $g = D^{1/2}f$, we have

$$\frac{g^T\mathcal{L}g}{g^Tg} = \frac{f^TD^{1/2}\mathcal{L}D^{1/2}f}{(D^{1/2}f)^TD^{1/2}f} = \frac{f^TLf}{f^TDf} = \frac{\sum_{i\sim j}(f(i) - \sigma(ij)f(j))^2}{\sum_i f^2(i)d_i}.$$

Then

$$\lambda_1 = \sup_f \frac{\sum_{i\sim j}(f(i) - \sigma(ij)f(j))^2}{\sum_i f^2(i)d_i}, \quad \lambda_n = \inf_f \frac{\sum_{i\sim j}(f(i) - \sigma(ij)f(j))^2}{\sum_i f^2(i)d_i},$$

Recall that $d_1 \geq d_2 \geq \cdots \geq d_n$ denotes the degree sequence of G.

Theorem 2.2. *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. Then*

$$\lambda_1 \geq 1 + \frac{1}{d_1}.$$

Proof. Let v_i be the vertex with $d(v_i) = d_i$. We can define f as follows:

$$f(u) = \begin{cases} \sum_{j:j\sim i} d_j & \text{if } u = v_i; \\ -\sigma(v_iv_j)d_i & \text{if } u \sim v_i; \\ 0 & \text{otherwise.} \end{cases}$$

Note that with the choice of f, we have $(D\mathbf{1})^T f = 0$ ($\mathbf{1}$ denotes the constant vector with the value 1 on each vertex) and

$$\lambda_1 \geq \frac{\sum_{i\sim j}(f(i)-\sigma(ij)f(j))^2}{\sum_i f^2(i)d_i} \geq \frac{d_i\left(\sum_{j:j\sim i} d_j + d_i\right)^2}{d_i\left(\sum_{j:j\sim i} d_i\right)^2 + d_i^2\sum_{j:j\sim i} d_i} = \frac{\sum_{j:j\sim i} d_j + d_i}{\sum_{j:j\sim i} d_j} = 1 + \frac{d_i}{\sum_{j:j\sim i} d_j}.$$

Since v_i has the degree d_i, other vertices have degrees at most d_1. Thus $\sum_{j:j\sim i} d_j \leq \sum_{j:j\sim i} d_1 = d_1 d_i$, from which Theorem 2.2 follows. ∎

Remark 1. One can check that both of Theorems 2.1 and 2.2 are stronger than (1) of Lemma 2.3. Theorem 2.1 is in general not comparable with Theorem 2.2. When Γ has the underlying graph K_n, both bounds are equal to $\frac{n}{n-1}$. Table 1 below with the underlying graph G_1 (cycle C_{n-1} adding a pendent edge) showing Theorem 2.2 is the strongest, and the other underlying graph G_2 (cycle C_n adding an edge which does not exist in C_n) showing Theorem 2.2 is the weakest.

Table 1:

Graph	Theorem 2.1	Theorem 2.2
$\Gamma = (G_1, \sigma)$	$(9n-1)/6n$	$3/2$ (stronger)
$\Gamma = (G_2, \sigma)$	$(27n-8)/18n$	$4/3$ (weaker)

Combining $d_1 \leq n-1$ and Theorem 2.2, we have the following corollary.

Corollary 2.1. *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. Then $\lambda_1 \geq \frac{n}{n-1}$.*

3. Least Normalized Laplacian Eigenvalues of Signed Graphs

In this section, we discuss some properties on λ_n for signed graphs. First, we give a new upper bound on λ_n which is stronger than Li's bound [8].

Theorem 3.1. *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. Then*

$$\lambda_n \leq 1 - \frac{2}{d_s + d_t},$$

where v_s is adjacent to v_t. Moreover, equality holds if and only if all cycles of length 3 containing v_s and v_t are negative.

Proof. Suppose $v_s, v_t \in V(\Gamma)$ and $v_s \sim v_t$. We can define f as follows:

$$f(u) = \begin{cases} 1 & \text{if } u = v_s; \\ \sigma(st) & \text{if } u = v_t; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\lambda_n \leq \frac{\sum_{i \sim j}(f(i) - \sigma(ij)f(j))^2}{\sum_i f^2(i)d_i} = \frac{d_s + d_t - 2}{d_s + d_t} = 1 - \frac{2}{d_s + d_t}. \tag{3.1}$$

From (3.1), we know equality holds if and only if $g = (\sqrt{d_s}, \sigma(st)\sqrt{d_t}, 0, \ldots, 0)^T$ is the eigenvector corresponding to $1 - \frac{2}{d_s + d_t}$. Suppose $v_i \in V \setminus \{v_s, v_t\}$ and v_i is adjacent to v_s and v_t, since $L(\Gamma)g = \frac{d_s + d_t - 2}{d_s + d_t} g$, then we have $\sigma(it) + \sigma(it)\sigma(st) = 0$, that is to say, all cycles of length 3 containing v_s and v_t are negative. ■

Combining (3.1) and $d_s, d_t \leq n - 1$, we have the following corollary.

Corollary 3.1. *Let $\Gamma = (G, \sigma)$ be a connected signed graph with n vertices. Then* $\lambda_n \leq \frac{n-2}{n-1}$.

Remark 2. In fact, Corollary 3.1 is sharp. Let $\Gamma_1 = (K_n, +)$ and $\Gamma_2 = (K_n, -)$. We know $\mathcal{L}(\Gamma_1) = \left\{ \left(\frac{n}{n-1} \right)^{(n-1)}, 0 \right\}$ (where exponents denote multiplicities), and $\mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2) = 2I_n$ (I_n is an identity diagonal matrix). So $\mathcal{L}(\Gamma_2) = \left\{ 2, \left(\frac{n-2}{n-1} \right)^{(n-1)} \right\}$.

Next, we characterize the upper bound of λ_n according to the number of positive (resp.negative) edges of Γ. we introduce some definitions in [2]. For a subset $S \subseteq V(\Gamma)$, we define vol$S$, the volume of S, to be the sum of the degree of the vertices in S:

$$\text{vol}S = \sum_{v_i \in S} d_i.$$

In particular, vol$G = \sum_{v_i \in V} d_i$. In addition, we define some parameters:

$$E(S) = \{e = uv : u, v \in S\}, \quad E^+(S) = \{e : e \in E(S) \text{ and } \sigma(e) = +\};$$
$$E(S, \overline{S}) = \{e = uv : u \in S \text{ and } v \in \overline{S}\}, \quad E(S, \overline{S}) = \{e : e \in E(S, \overline{S}) \text{ and } \sigma(e) = +\}.$$

Theorem 3.2. *Let* $\Gamma = (G, \sigma)$ *be a connected signed graph and* X, Y *be disjoint subsets of* $V(\Gamma)$. *Then*

$$\lambda_n \leq \frac{4\left|E^-(X)\right| + 4\left|E^-(Y)\right| + 4\left|E^+(X,Y)\right| + |E(X \cup Y, V \setminus (X \cup Y))|}{\mathit{vol}(X \cup Y)}.$$

Proof. Let $X, Y \subseteq V(\Gamma)$ be disjoint subsets and f be defined as follows:

$$f(u) = \begin{cases} 1 & \text{if } u \in X; \\ -1 & \text{if } u \in Y; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} & \frac{\sum\limits_{i\sim j}(f(i) - \sigma(ij)f(j))^2}{\sum\limits_{i} f^2(i)d_i} \\ = & \frac{4\left|E^-(X)\right| + 4\left|E^-(Y)\right| + 4\left|E^+(X,Y)\right| + |E(X, V \setminus (X \cup Y))| + |E(Y, V \setminus (X \cup Y))|}{\mathrm{vol}(X) + \mathrm{vol}(Y)} \\ = & \frac{4\left|E^-(X)\right| + 4\left|E^-(Y)\right| + 4\left|E^+(X,Y)\right| + |E(X \cup Y, V \setminus (X \cup Y))|}{\mathrm{vol}(X \cup Y)}, \end{aligned}$$

and the desired inequality follows. ■

Let $Y = \overline{X} = V \setminus X$ satisfy Theorem 3.2. We can obtain the following corollary.

Corollary 3.2. *Let* $\Gamma = (G, \sigma)$ *be a connected signed graph and* $X \subseteq V(\Gamma)$. *Then*

$$\lambda_n \leq \frac{4\left|E^-(X)\right| + 4\left|E^-(\overline{X})\right| + 4\left|E^+(X,\overline{X})\right|}{\mathit{vol}(\Gamma)}.$$

If Γ is balanced, then we know $|E^-(X)| = \left|E^-(\overline{X})\right| = \left|E^+(X,\overline{X})\right| = 0$ from Lemma 2.2. So we have the following result.

Corollary 3.3. *Let* $\Gamma = (G, \sigma)$ *be a connected balanced signed graph. Then* $\lambda_n = 0$.

4. The Lower Bound of the Laplacian Eigenvalues Sum on Signed Graphs

For a connected unsigned graph G, let the degree sequence of G be $d_1 \geq d_2 \geq \cdots \geq d_n$ and $\nu_1 \geq \cdots \geq \nu_{n-1} \geq \nu_n = 0$ be the Laplacian eigenvalues of G, respectively. Grone [3] proved the following inequalities

$$\sum_{i=1}^{k} \nu_i \geq 1 + \sum_{i=1}^{k} d_i, \text{for all } k = 1, 2, \ldots, n-1. \tag{4.1}$$

Generally speaking, for connected signed graphs, the above inequalities do not hold for all $1 \le k \le n-1$. Hou [6] gave the following conjecture:

Conjecture 1. Let $\Gamma = (G, \sigma)$ be a connected signed graph, the degree sequence of Γ be $d_1 \ge d_2 \ge \cdots \ge d_n$ and $\mu_1 \ge \cdots \ge \mu_{n-1} \ge \mu_n$ be the Laplacian eigenvalues of Γ, respectively. Then

$$\sum_{i=1}^{k} \mu_i > \sum_{i=1}^{k} d_i, \text{for all } k = 1, 2, \ldots, n-1.$$

For $k = 1$, Hou [6] proved $\mu_1 \ge 1 + d_1$. In the following, we prove the conjecture 1 is also true for $k = 2, n-1$.

Lemma 4.1. *Let* $M = \begin{pmatrix} B & C \\ C^T & E \end{pmatrix}$ *be an* $n \times n$ *positive definite (resp. semidefinite) symmetric matrix where* B *is a* $k \times k$ *principal submatrix and* $C \neq 0$. *If* B *is a positive definite matrix, then* $E - C^T B^{-1} C$ *is also a positive definite (resp. semidefinite) matrix.*

Proof. Let

$$P = \begin{pmatrix} I_{k\times k} & -B^{-1}C \\ 0 & I_{(n-k)\times(n-k)} \end{pmatrix}.$$

Then

$$P^T \begin{pmatrix} B & C \\ C^T & E \end{pmatrix} P = \begin{pmatrix} B & 0 \\ 0 & E - C^T B^{-1} C \end{pmatrix}.$$

M is positive definite (resp. semidefinite), so that implies that $E - C^T B^{-1} C$ is a positive definite(resp. semidefinite) matrix. ∎

Theorem 4.1. *Let* $\Gamma = (G, \sigma)$ *be a connected signed graph,* $d_1 \ge d_2 \ge \cdots \ge d_n$ *and* $\mu_1 \ge \cdots \ge \mu_{n-1} \ge \mu_n$ *be the degree sequence and the Laplacian eigenvalues of* Γ, *respectively. Then*

$$\mu_1 + \mu_2 > d_1 + d_2.$$

Proof. Let $X = \{v_1, v_2\} \subseteq V(\Gamma)$. Partition $L(\Gamma)$ according to the vertex partition $(X, \overline{X})$ as follows:

$$L(\Gamma) = \begin{pmatrix} B & C \\ C^T & E \end{pmatrix}$$

where B is 2×2 principal submatrix of $L(\Gamma)$. Since Γ is connected, we have $C \neq 0$ and $d_1 d_2 \ge 2$. Since $B = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \begin{pmatrix} d_1 & -1 \\ -1 & d_2 \end{pmatrix}$ or $\begin{pmatrix} d_1 & 1 \\ 1 & d_2 \end{pmatrix}$, we can check B is positive definite. Then $E - C^T B^{-1} C$ is also positive semidefinite from Lemma 4.1. Let

$$F = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ C^T B^{-\frac{1}{2}} & G \end{pmatrix},$$

where $G = \left(E - C^T B^{-1} C\right)^{\frac{1}{2}}$. One can check $L(\Gamma) = FF^T$. Then $L(\Gamma)$ has the same eigenvalues as

$$K = F^T F = \begin{pmatrix} B + B^{-\frac{1}{2}} C C^T B^{-\frac{1}{2}} & * \\ * & * \end{pmatrix}$$

where

$$CC^T = \begin{pmatrix} \sum\limits_{\substack{v_i \sim v_1 \\ v_i \neq v_2}} 1 & 2s_1 - s \\ 2s_1 - s & \sum\limits_{\substack{v_i \sim v_2 \\ v_i \neq v_1}} 1 \end{pmatrix},$$

$s = |\{v : v \sim v_1 \text{ and } v \sim v_2\}|$ and $s_1 = |\{v : v \sim v_1, v \sim v_2 \text{ and } \sigma(vv_1) = \sigma(vv_2)\}|$. So

$$\begin{aligned} \mu_1 + \mu_2 \geq\ & \text{trace}(B + B^{-\frac{1}{2}} C C^T B^{-\frac{1}{2}}) \\ =\ & \text{trace}(B) + \text{trace}(B^{-\frac{1}{2}} C C^T B^{-\frac{1}{2}}) \\ =\ & \text{trace}(B) + \text{trace}(B^{-1} C C^T) \\ =\ & d_1 + d_2 + \text{trace}(B^{-1} C C^T) \end{aligned}$$

We only need to discuss trace$(B^{-1}CC^T) > 0$ according to the following two cases.

(1) v_1, v_2 are not adjacent. Then

$$B = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{pmatrix}.$$

We have

$$\text{trace}(B^{-1} C C^T) = \frac{1}{d_2} \sum_{\substack{v_i \sim v_1 \\ v_i \neq v_2}} 1 + \frac{1}{d_1} \sum_{\substack{v_i \sim v_2 \\ v_i \neq v_1}} 1 = \frac{d_1}{d_2} + \frac{d_2}{d_1} \geq 2.$$

(2) v_1, v_2 are adjacent.

Then

$$B = \begin{pmatrix} d_1 & -1 \\ -1 & d_2 \end{pmatrix} \text{ or } B = \begin{pmatrix} d_1 & 1 \\ 1 & d_2 \end{pmatrix}.$$

In fact, if $B = \begin{pmatrix} d_1 & 1 \\ 1 & d_2 \end{pmatrix}$, we let $\Gamma_1 = (G, \sigma_1)$ such that $\sigma_1(v_iv_j) = -\sigma(v_iv_j)$ for $v_iv_j \in E(\Gamma)$, that is to say, there exists a signature matrix $S = \text{diag}\{1, -1, 1, \ldots, 1\}$ such that $\Gamma = S\Gamma_1 S$. From Lemma 2.1, we know $L(\Gamma)$ and $L(\Gamma_1)$ have the same Laplacian eigenvalues. So we only discuss the case $B = \begin{pmatrix} d_1 & -1 \\ -1 & d_2 \end{pmatrix}$. We can obtain $B^{-1} = \begin{pmatrix} \frac{d_2}{d_1d_2-1} & \frac{1}{d_1d_2-1} \\ \frac{1}{d_1d_2-1} & \frac{d_1}{d_1d_2-1} \end{pmatrix}$. So

$$\begin{aligned}
\text{trace}(B^{-1}CC^T) &= \frac{d_2}{d_1d_2-1} \sum_{\substack{v_i \sim v_1 \\ v_i \neq v_2}} 1 + \frac{d_1}{d_1d_2-1} \sum_{\substack{v_i \sim v_2 \\ v_i \neq v_1}} 1 + 2\frac{2s_1 - s}{d_1d_2-1} \\
&= \frac{1}{d_1d_2-1}\left(d_2 \sum_{\substack{v_i \sim v_1 \\ v_i \neq v_2}} 1 + d_1 \sum_{\substack{v_i \sim v_2 \\ v_i \neq v_1}} 1 + 4s_1 - 2s \right) \\
&= \frac{1}{d_1d_2-1}(d_1(d_2-1) + d_2(d_1-1) + 4s_1 - 2s).
\end{aligned}$$

Since $s \leq d_2 - 1$ and $d_1 \geq 2$, so $d_1(d_2-1) - 2s \geq 0$, that is to say, $d_1(d_2 - 1) + d_2(d_1-1) + 4s_1 - 2s > 0$.

The theorem is proved. ■

Lemma 4.2. *Let $\Gamma = (G, \sigma)$ be a connected signed graph. Then*

$$\mu_n < d_n,$$

where d_n is the minimum degree of Γ.

Proof. Let $f = (0, \ldots, 0, 1) \in R^n$. Then

$$\mu_n \leq \sum_{i \sim j} (f_i - \sigma(v_iv_j)f_j)^2 = d_n.$$

Obviously, f is not the eigenvector corresponding to μ_n, so $0 < \mu_n < d_n$. ■

Theorem 4.2. *Let $\Gamma = (G, \sigma)$ be a connected signed graph, $d_1 \geq d_2 \geq \cdots \geq d_n$ and $\mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n$ be the degree sequence and the Laplacian eigenvalues of Γ, respectively. Then*

$$\sum_{i=1}^{n-1} \mu_i > \sum_{i=1}^{n-1} d_i.$$

Proof. Since $\sum\limits_{i=1}^{n} \mu_i = \sum\limits_{i=1}^{n} d_i$, then the theorem is proved from Lemma 4.2. ■

Let A be a symmetric matrix of order $n \times n$. Suppose rows and columns of

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}$$

are partitioned according to a partition $X_1, \ldots, X_m$ of $\{1, \ldots, n\}$ with characteristic matrix $\tilde{S}$ (that is, $\tilde{S}_{ij} = 1$ if $i \in X_j$ and 0 otherwise). The quotient matrix is the matrix $\tilde{B}$ whose entries are the average row sums of the blocks of A, that is

$$\tilde{B}_{ij} = \frac{1}{|X_i|} 1^T A_{ij} 1 = \frac{1}{|X_i|} \left(\tilde{S}^T A \tilde{S}\right)_{ij}$$

(1 denotes the all-one vector). The partition is called regular (or equitable) if each block A_{ij} of A has constant row (and column) sum (or $A\tilde{S} = \tilde{S}\tilde{B}$).

Lemma 4.3. [4]. *Let A be an $n \times n$ symmetric matrix with a partition $X_1, \ldots, X_m$. Suppose $\tilde{B}$ is the $m \times m$ quotient matrix of a symmetric partitioned matrix A. Then for $1 \le i \le m$*

$$\lambda_i(A) \ge \lambda_i(\tilde{B}) \ge \lambda_{n-m+i}(A).$$

Recalling some definitions in section 3, we have for a subset $S \subseteq V(\Gamma)$,

$$E(S) = \{e = uv : u, v \in S\}, \quad E^+(S) = \{e : e \in E(S) \text{ and } \sigma(e) = +\};$$
$$E(S, \overline{S}) = \{e = uv : u \in S \text{ and } v \in \overline{S}\}, \quad E^+(S, \overline{S}) = \{e : e \in E(S, \overline{S}) \text{ and } \sigma(e) = +\}.$$

Theorem 4.3. *Let Γ be a connected signed graph with n vertices and $V_i = \{v_1, \ldots, v_i\}$ be the vertex subset $V(\Gamma)$ with $d(v_i) = d_i$ ($1 \le i \le n$). Then*

(1) *If Γ is balanced, then for $1 \le k \le n-1$ we have*

$$\sum_{i=1}^{k} \mu_i \ge \sum_{i=1}^{k} d_i + \frac{\left|E\left(V_k, \overline{V_k}\right)\right|}{n-k};$$

(2) *If Γ is non-balanced, we have*

$$\sum_{i=1}^{k} \mu_i \ge \sum_{i=1}^{k} d_i.$$

In particular, if $d_{k+1} < \dfrac{\left|E\left(V_k, \overline{V_k}\right)\right| + 4\left|E^-\left(\overline{V_k}\right)\right|}{n-k}$ ($1 \le k \le n-2$), then we have

$$\sum_{i=1}^{k+1} \mu_i > \sum_{i=1}^{k+1} d_i.$$

Proof. Consider the partition of the vertex set $V(\Gamma)$ into $k+1$ parts: $V_i = \{v_1, \ldots, v_i\}$ ($1 \leq i \leq k$). Then the corresponding quotient matrix is

$$B = \begin{pmatrix} L' & * \\ * & b_{k+1,k+1} \end{pmatrix} \tag{4.2}$$

where L' is the principal submatrix of $L(\Gamma)$ with rows and columns indexed by the vertices $v_1, \ldots, v_k$. Let $\theta_1 \geq \cdots \geq \theta_{k+1}$ and $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of B and $L(\Gamma)$, respectively.

(1) If Γ is balanced, we have $\Gamma \sim (G, +)$ from Lemma 2.2. Then all rows sum of B are 0, that is $\theta_{k+1} = 0$ and

$$b_{k+1,k+1} = \frac{\left|E^+\left(V_k, \overline{V_k}\right)\right| - \left|E^-\left(V_k, \overline{V_k}\right)\right|}{n-k}.$$

Since $\left|E^-\left(V_k, \overline{V_k}\right)\right| = 0$ from Lemma 2.2, we have

$$\begin{aligned}\sum_{i=1}^{k} \mu_i &\geq \sum_{i=1}^{k} \theta_i = \sum_{i=1}^{k} d_i + b_{k+1,k+1} \\ &\geq \sum_{i=1}^{k} d_i + \frac{\left|E^+\left(V_k, \overline{V_k}\right)\right|}{n-k} = \sum_{i=1}^{k} d_i + \frac{\left|E\left(V_k, \overline{V_k}\right)\right|}{n-k}.\end{aligned}$$

(2) If Γ is non-balanced, we know

$$\sum_{i=1}^{k} \mu_i \geq \sum_{i=1}^{k} \theta_i = \sum_{i=1}^{k} d_i \text{ for } 1 \leq k \leq n-1.$$

In particular, since Γ is not banlanced, then $\theta_{k+1} > 0$ and

$$b_{k+1,k+1} = \frac{\sum_{i=k+1}^{n} d_i + 2|E^-\left(\overline{V_k}\right)| - 2|E^+(\overline{V_k})\ |}{n-k} = \frac{\left|E\left(V_k, \overline{V_k}\right)\right| + 4\left|E^-\left(\overline{V_k}\right)\right|}{n-k}.$$

Then

$$\begin{aligned}\sum_{i=1}^{k+1} \mu_i &\geq \sum_{i=1}^{k+1} \theta_i \\ &\geq \sum_{i=1}^{k} d_i + b_{k+1,k+1} \\ &\geq \sum_{i=1}^{k+1} d_i + \frac{\left|E\left(V_k, \overline{V_k}\right)\right| + 4\left|E^-\left(\overline{V_k}\right)\right| - (n-k)d_{k+1}}{n-k} \\ &> \sum_{i=1}^{k+1} d_i\end{aligned}$$

$$\text{if } d_{k+1} < \frac{\left|E\left(V_k, \overline{V_k}\right)\right| + 4\left|E^-\left(\overline{V_k}\right)\right|}{n-k} \text{ for } 1 \leq k \leq n-2.$$

■

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