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BURES DISTANCE FOR α-COMPLETELY POSITIVE MAPS AND TRANSITION PROBABILITY BETWEEN P-FUNCTIONALS

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Abstract. In this paper we discuss the Bures distance between α-CP maps on a C^*-algebra and the transition probability between P-functionals on a *-algebra. We first review the notion of α-CP maps and the representation theorem associated to α-CP maps. Using the Krein space representation and the set of intertwiners between Krein space representations, we study the Bures distance between α-CP maps. We prove that the transition probability between P-functionals can be estimated by some functionals using J-representations on Krein spaces.

1. INTRODUCTION

For two normal states μ, ν on a von Neumann algebra $\mathcal{M}$, the *Bures distance* between μ and ν is defined as $\beta(\mu,\nu) = \inf \|x_\mu - y_\nu\|$ where the infimum is taken over all vectors x_μ and y_ν representing μ and ν, respectively, in some common normal representation Hilbert space. In [6], Bures showed that $\beta : (\mu,\nu) \mapsto \beta(\mu,\nu)$ is a metric on the set of normal states of $\mathcal{M}$. This metric is a quantum generalization of the Fisher information metric in the quantum information and is identical to the Fubini-Study metric. Kretschmann, Schlingemann and Werner [16] extended the notion to completely positive linear maps from a C^*-algebra into $\mathcal{B}(\mathcal{H})$ using the Stinespring's dilation. Recently, Bhat and Sumesh [5] generalized this notion to completely positive linear maps between arbitrary C^*-algebras via Hilbert C^*-module language. They proved that the Bures distance may not be a metric when the range algebra is a general C^*-algebra and gave some examples with explicit computations of the Bures distance.

In quantum physics, the value $|\langle x, y\rangle|^2$ associated to unit vectors x, y in a Hilbert space is interpreted as a transition probability between the corresponding states. Uhlmann

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[19] introduced the *transition probability* $P(\mu,\nu)$ between two states μ, ν of a *-algebra $\mathcal{A}$, which is defined as the supremum of $|\langle x_\mu, y_\nu\rangle|^2$ where the supremum is taken over all vectors x_μ, y_ν representing μ and ν as vector states in some common *-representation of $\mathcal{A}$. The transition probability for normal states of a von Neumann algebra $\mathcal{M}$ is related to the Bures distance by the formula $P(\mu,\nu) = (1-\frac{1}{2}\beta(\mu,\nu)^2)^2$. In particular, if μ and ν are states on the matrix algebra M_n with density matrices ρ_μ and ρ_ν, respectively, then $D_{1/2}(\rho_\mu|\rho_\nu) \geq \beta(\mu,\nu)^2$ where D_t $(t \in (0,1])$ is the quantum Tsallis relative entropy [1].

The completely positive maps are used as mathematical models for quantum instruments and quantum probability and has many applications in quantum information theory. However, in some local quantum field theories, the locality is in conflict with positivity, which leads to the modification of the positivity. From the motivation in local quantum field theory, we introduced the notion of α-completely positive (α-CP) maps between (locally) C^*-algebras, which generalizes α-positivity [15] and P-functionals [17]. The α-complete positivity was studied in several papers [8, 9, 10, 11, 12, 13].

The purpose of the present paper is to study the Bures distance and the transition probability for α-completely positive maps ϕ_1 and ϕ_2 on a C^*-algebra. In Section 2, we will formally introduce the Bures distance between α-CP maps using the KSGNS representation theorem [9] for α-CP maps and study the set of intertwiners between KSGNS representations associated to α-CP maps. In Section 3, we estimate the Bures distance between α-CP maps using the set of intertwiners and prove the triangle inequality the Bures distance between α-CP maps. In Section 4, we study the transition probability between P-functionals for a *-algebra $\mathcal{A}$ using J-representations on Krein spaces and prove that the transition probability between P-functionals can be estimated by functionals which are bounded by given P-functionals and invariant under $\alpha = 2P - \mathrm{id}_{\mathcal{A}}$.

2. Preliminaries

Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and let J be a (fundamental) symmetry on $\mathcal{H}$, i.e., $J = J^* = J^{-1} \in \mathcal{B}(\mathcal{H})$. We define a sesquilinear form on $\mathcal{H}$ by $[\xi,\eta]_J = \langle\xi, J\eta\rangle$. A Hilbert space $\mathcal{H}$ equipped with $[\cdot,\cdot]_J$ for a (fundamental) symmetry J is called a *Krein space* or a *J-space*, denote by $(\mathcal{H}, J)$. For each $T \in \mathcal{B}(\mathcal{H})$ there exists an $T^J \in \mathcal{B}(\mathcal{H})$ such that

$$[T\xi,\eta]_J = [\xi, T^J\eta]_J, \quad (\xi,\eta \in \mathcal{H}). \tag{2.1}$$

Indeed, we have that $[T\xi,\eta]_J = \langle T\xi, J\eta\rangle = \langle J\xi, JT^*J\eta\rangle = [\xi, JT^*J\eta]_J$, so that $T^J = JT^*J$ satisfies the equation (2.1). The operator T^J is called a *J-adjoint* of T. Let $\mathcal{A}$ be a C^*-algebra and π a representation of $\mathcal{A}$ on $\mathcal{H}$, where a representation means an algebra (not necessarily *-) homomorphism. If there exists a symmetry J on

$\mathcal{H}$ such that

$$[\pi(a)\xi,\eta]_J = [\xi,\pi(a^*)\eta]_J \quad \text{for any } a \in \mathcal{A} \text{ and } \xi,\eta \in \mathcal{H},$$

then π is said to be a *J-representation*. It is easy to check that a representation π is a J-representation if and only if $\pi(a^*) = \pi(a)^J$ for any $a \in \mathcal{A}$. We also see that $\|\pi(a)\| = \|\pi(a^*)\|$ for all $a \in \mathcal{A}$. Indeed, we have that

$$\|\pi(a)\|^2 = \|J\pi(a^*)J\pi(a)\| \leq \|\pi(a^*)\|\|\pi(a)\|,$$

so that $\|\pi(a)\| \leq \|\pi(a^*)\|$. By symmetry, the converse inequality holds.

Let $\mathcal{B} \subset \mathcal{A}$ be unital C^*-algebras and P a conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$, that is, $P(1_{\mathcal{A}}) = 1_{\mathcal{A}}$, $P(b_1 a b_2) = b_1 P(a) b_2$ and $P(a^*) = P(a)^*$ for any $a \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. A Hermitian linear functional ρ on $\mathcal{A}$ is called a *P-functional* on $\mathcal{A}$ if the following conditions are satisfied:

(i) $\rho(P(a)) = \rho(a)$ for any $a \in \mathcal{A}$,

(ii) $2\rho\big(P(a)^*P(a)\big) \geq \rho(a^*a)$ for any $a \in \mathcal{A}$.

If $\alpha : \mathcal{A} \to \mathcal{A}$ is defined by $\alpha(a) = 2P(a) - a$, then it is easy to check that

$$\rho\big(\alpha(a_1)\alpha(a_2)\big) = \rho(a_1 a_2) \quad \text{and} \quad \rho\big(\alpha(a)^*a\big) \geq 0$$

for any $a_1, a_2, a \in \mathcal{A}$ [4].

2.1. α-CP maps on C^*-algebras and Krein space representations

Motivated by the α-positivity [15] and P-functionals in [4, 14, 17], we introduced a notion of α-completely positive maps as a generalization of completely positive maps [9]. Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a Hermitian map. Then for any $n \geq 1$ and $a_{ij} \in \mathcal{A}$ $(1 \leq i,j \leq n)$, we have $\phi_n\left[(a_{ij})^*\right] = [\phi(a_{ji}{}^*)] = [\phi(a_{ij})]^*$, so that ϕ_n is Hermitian for any $n \geq 1$.

Throughout this paper, $\mathcal{A}$ and $\mathcal{H}$ denote a unital C^*-algebra with unit $1_{\mathcal{A}}$ and a Hilbert space over $\mathbb{C}$, respectively, unless specified otherwise.

Definition 2.1. A Hermitian linear map ϕ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ is called *α-completely positive* (briefly, *α-CP*) if there is a bounded Hermitian linear map $\alpha : \mathcal{A} \to \mathcal{A}$ such that

(i) $\alpha^2 = \mathrm{id}_{\mathcal{A}}$ where $\mathrm{id}_{\mathcal{A}}$ is the identity map on $\mathcal{A}$,

(ii) $\alpha(1_{\mathcal{A}}) = 1_{\mathcal{A}}$,

(iii) $\phi(ab) = \phi\big(\alpha(a)\alpha(b)\big)$ for any $a, b \in \mathcal{A}$,

(iv) for any $n \geq 1$, $a_1, \cdots, a_n \in \mathcal{A}$ and $\xi_1, \cdots, \xi_n \in \mathcal{H}$,

$$\sum_{i,j=1}^{n} \big\langle \phi\big(\alpha(a_j)^*a_i\big)\xi_i, \xi_j \big\rangle \geq 0$$

(v) for each $a, a_1, \ldots, a_n \in \mathcal{A}$, there exists a constant $C(a) \geq 0$ such that

$$\Big(\phi\big(\alpha(aa_j)^*aa_i\big)\Big) \leq C(a)\Big(\phi\big(\alpha(a_j)^*a_i\big)\Big),$$

where a big parenthesis denotes an $n \times n$ matrix.

In case where $\alpha = \mathrm{id}_{\mathcal{A}}$ in Definition 2.1, the α-complete positivity implies the complete positivity. Moreover, the equality $\phi(ab) = \phi\big(\alpha(ab)\big)$ immediately follows from the condition (ii) in Definition 2.1 since $\phi(ab) = \phi(ab \cdot 1_{\mathcal{A}}) = \phi\big(\alpha(ab)\alpha(1_{\mathcal{A}})\big) = \phi\big(\alpha(ab)\big)$.

See [9] for the example of an α-CP map on the 2×2-matrix algebra which is not completely positive. It is known that in massless quantum field theory the state space will be a space with indefinite metric. Motivated by this physical fact, many people extended the GNS construction to Krein spaces. More generally, Heo-Hong-Ji in [9] provided such KSGNS representations on a Krein module for α-completely positive maps on a C^*-algebra or a *-algebra. For the reader's convenience, we begin by sketching the Stinespring type construction associated with an α-completely positive map which will be needed in this paper.

Theorem 2.2. [10]. *If $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is an α-CP map, then there exist a Krein space $(\mathcal{K}, J)$, a unital J-representation π of $\mathcal{A}$ on $\mathcal{K}$ and a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

(i) $\phi(a) = V^*\pi(a)V$, so that $\phi(a^*) = V^*\pi(a)^JV$,

(ii) $V^*\pi(a)^*\pi(b)V = V^*\pi(\alpha(a)^*b)V$,

(iii) the set $\{\pi(a)V\xi : a \in \mathcal{A},\ \xi \in \mathcal{H}\}$ is total in $\mathcal{K}$.

We call the quadruple $(\mathcal{K}, J, \pi, V)$ in Theorem 2.2 the *Krein quadruple associated with an α-CP map ϕ* if it satisfies conditions (i) and (ii). In this case, we say that *$(\mathcal{K}, J, \pi, V)$ dilates ϕ.*

Remark 2.3. Let $(\mathcal{K}, J, \pi, V)$ be the quadruple constructed in Theorem 2.2. We note that the equality $JV = V$ holds. If $(\mathcal{K}', J', \pi', V')$ is another quadruples satisfying properties (i), (ii) and (iii) in Theorem 2.2 and if the equality $J'V' = V'$ holds, then there is a unitary operator U in $\mathcal{B}(\mathcal{K}, \mathcal{K}')$ such that

$$\pi'(a) = U\pi(a)U^* \quad (a \in \mathcal{A}), \quad V' = UV \quad \text{and} \quad J' = UJU^*.$$

See Theorem 4.6 and Remark 4.7 in [10] for the proof.

If a quadruple $(\mathcal{K}', J', \pi', V')$ satisfies properties (i), (ii) and (iii) in Theorem 2.2 and if the equality $J'V' = V'$ holds, then it is called a *minimal dilation associated with ϕ*. This minimal dilation is unique up to unitary equivalence. ∎

2.2. Bures distance between α-CP maps on C^*-algebras

We denote by α-CP$(\mathcal{A},\mathcal{H})$ the set of all α-CP map ϕ of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. For any ϕ_i $(i=1,2)$ in α-CP$(\mathcal{A},\mathcal{H})$, let $(\mathcal{K}_i, J_i, \pi_i, V_i)$ be a minimal Krein quadruple associated with ϕ_i. Take $\mathcal{K}=\mathcal{K}_1\oplus\mathcal{K}_2$, $J=J_1\oplus J_2$, $\pi=\pi_1\oplus\pi_2$, $\widetilde{V_1}=V_1\oplus 0$ and $\widetilde{V_2}=0\oplus V_2$. We easily see that

$$\phi_i(a)=\widetilde{V_i}^*\pi(a)\widetilde{V_i},$$

$$\widetilde{V_i}^*\pi(\alpha(a)^*b)\widetilde{V_i}=\widetilde{V_i}^*\pi(a)^*\pi(b)\widetilde{V_i}.$$

Thus, the quadruple $(\mathcal{K},J,\pi,\widetilde{V_i})$ dilates ϕ_i $(i=1,2)$. In this case, we call the triple $(\mathcal{K},J,\pi)$ *the common representation for ϕ_1 and ϕ_2.*

Let $(\mathcal{K},J)$ be a Krein space and let $\pi:\mathcal{A}\to\mathcal{B}(\mathcal{K})$ be a J-representation. We denote by $S(\phi,\pi)$ the set of all bounded operators $V\in\mathcal{B}(\mathcal{H},\mathcal{K})$ such that $(\mathcal{K},J,\pi,V)$ dilates $\phi\in\alpha$-CP$(\mathcal{A},\mathcal{H})$. Here, we don't demand minimality for Krein quadruples.

Definition 2.4. (cf. [16]) Let ϕ_i $(i=1,2)$ be any element of α-CP$(\mathcal{A},\mathcal{H})$.

(i) The π-distance between ϕ_1 and ϕ_2 is defined as
$$\beta_\pi(\phi_1,\phi_2)=\inf\big\{\|V_1-V_2\|:V_i\in S(\phi_i,\pi)\big\}.$$

(ii) The Bures distance between ϕ_1 and ϕ_2 is the smallest π-distance
$$\beta(\phi_1,\phi_2)=\inf_\pi \beta_\pi(\phi_1,\phi_2) \tag{2.2}$$
where the infimum is taken over all representations $(\mathcal{K},J,\pi,V_i)$ which dilates ϕ_i.

2.3. Intertwiners between Krein quadruples associated with α-CP maps

For each $i=1,2$, let $(\mathcal{K}_i,J_i,\pi_i,V_i)$ be a Krein quadruple associated with ϕ_i in α-CP$(\mathcal{A},\mathcal{H})$. For two maps ϕ_1,ϕ_2 in α-CP$(\mathcal{A},\mathcal{H})$ $(i=1,2)$, we define the set

$$\begin{aligned}\mathcal{I}(\phi_1,\phi_2)\,=\,\big\{V_1^*WV_2:\ &W\in\mathcal{B}(\mathcal{K}_2,\mathcal{K}_1)\text{ with }\|W\|\le 1,\\ WJ_2\,=\,J_1W\text{ and }&W\pi_2(a)=\pi_1(a)W\ \forall a\in\mathcal{A}\big\}.\end{aligned}$$

If $(\widehat{\mathcal{K}_i},\widehat{J_i},\widehat{\pi_i},\widehat{V_i})$ is a minimal Krein quadruple associated with ϕ_i, by uniqueness of a minimal Krein quadruple, there exists an isometry $U_i:\widehat{\mathcal{K}_i}\to\mathcal{K}_i$ such that

$$U_i\widehat{V_i}=V_i,\quad U_i\widehat{J_i}=J_iU_i,\quad \pi_i(a)=U_i\widehat{\pi_i}(a)U_i^*\quad(a\in\mathcal{A}). \tag{2.3}$$

Since $U_iU_i^*$ is a projection onto the closed linear span of the set $\{\pi_i(a)V_i\xi:a\in\mathcal{A},\xi\in\mathcal{H}\}$, we have that $U_iU_i^*V_i=V_i$. For any $W\in\mathcal{I}(\phi_1,\phi_2)$, we see that

$$V_1^*WV_2=V_1^*U_1U_1^*WU_2U_2^*V_2=\widehat{V_1}^*U_1^*WU_2\widehat{V_2}=\widehat{V_1}^*\widehat{W}\widehat{V_2}$$

where $\widehat{W} := U_1^* W U_2$ is a contractive operator from $\widehat{\mathcal{K}_2}$ into $\widehat{\mathcal{K}_1}$. It follows from (2.3) that

$$\widehat{W}\widehat{\pi_2}(a) = U_1^* W \pi_2(a) U_2 = \widehat{\pi_1}(a) U_1^* W U_2 = \widehat{\pi_1}(a)\widehat{W},$$

and that

$$\widehat{W}\widehat{J}_2 = U_1^* W J_2 U_2 = \widehat{J}_1 U_1^* W U_2 = \widehat{J}_1 \widehat{W}.$$

This implies that

$$\mathcal{I}(\phi_1,\phi_2) \subseteq \widehat{\mathcal{I}}(\phi_1,\phi_2) := \left\{ \widehat{V_1}^* \widehat{W}\widehat{V_2} : \widehat{W}\widehat{J}_2 = \widehat{J}_1\widehat{W},\ \widehat{W}\widehat{\pi_2}(a) = \widehat{\pi_1}(a)\widehat{W}\ \forall a \in \mathcal{A} \right\}$$

where $\big(\widehat{\mathcal{K}_i}, \widehat{J}_i, \widehat{\pi}_i, \widehat{V}_i\big)$ is a minimal Krein quadruple associated to ϕ_i $(i = 1, 2)$. By taking $W = U_1 \widehat{W} U_2^*$, we similarly get the reverse inclusion.

For two α-CP maps ϕ_i $(i = 1, 2)$ and a J-representation π of $\mathcal{A}$ on a Krein space $(\mathcal{K}, J)$, we define the sets

$$\mathcal{J}_\pi(\phi_1,\phi_2) = \big\{ V_1^* V_2 : V_i \in S(\phi_i, \pi) \big\} \subset \mathcal{B}(\mathcal{H})$$
$$\mathcal{J}(\phi_1,\phi_2) = \bigcup_\pi \mathcal{J}_\pi(\phi_1,\phi_2)$$

where the union is over all representations π of $\mathcal{A}$, admitting a common Krein space dilation for ϕ_1 and ϕ_2.

Proposition 2.5. *If $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is two α-CP maps $(i = 1, 2)$, then*

$$\mathcal{I}(\phi_1,\phi_2) = \mathcal{J}(\phi_1,\phi_2).$$

Proof. Let $(\widehat{\mathcal{K}_i}, \widehat{J}_i, \widehat{\pi}_i, \widehat{V}_i)$ be a minimal Krein quadruple associated with ϕ_i and let $\widehat{W} : \widehat{\mathcal{K}_2} \to \widehat{\mathcal{K}_1}$ with $\|\widehat{W}\| \le 1$ be such that

$$\widehat{W}\widehat{J}_2 = \widehat{J}_1\widehat{W}, \quad \text{and} \quad \widehat{W}\widehat{\pi_2}(a) = \widehat{\pi_1}(a)\widehat{W} \quad \text{for all } a \in \mathcal{A}.$$

Define bounded linear operators $V_i : \mathcal{H} \to \widehat{\mathcal{K}_1} \oplus \widehat{\mathcal{K}_2}$ by

$$V_1 = \widehat{V_1} \oplus 0 \quad \text{and} \quad V_2 = \widehat{W}\widehat{V_2} \oplus \Big(1_{\widehat{\mathcal{K}_2}} - \widehat{W}^*\widehat{W}\Big)^{1/2} \widehat{V_2}$$

Let $a \in \mathcal{A}$. We easily see that $\phi_1(a) = V_1^*(\widehat{\pi_1}(a) \oplus \widehat{\pi_2}(a))V_1$ and that

$$V_1^*\big(\widehat{\pi_1}(a) \oplus \widehat{\pi_2}(a)\big)^*\big(\widehat{\pi_1}(b) \oplus \widehat{\pi_2}(b)\big)V_1 = V_1^*\big(\widehat{\pi_1}(\alpha(a)^*b) \oplus \widehat{\pi_2}(\alpha(a)^*b)\big)V_1.$$

We also have that

$$\begin{aligned} V_2^*\big(\widehat{\pi_1}(a) \oplus \widehat{\pi_2}(a)\big)V_2 &= \widehat{V_2}^*\widehat{W}^*\widehat{\pi_1}(a)\widehat{W}\widehat{V_2} + \widehat{V_2}^*\big(1_{\widehat{\mathcal{K}_2}} - \widehat{W}^*\widehat{W}\big)\widehat{\pi_2}(a)\widehat{V_2} \\ &= \widehat{V_2}^*\widehat{\pi_2}(a)\widehat{V_2} = \phi_2(a) \end{aligned}$$

and that

$$\begin{aligned}
&V_2^*\big(\widehat{\pi_1}(a) \oplus \widehat{\pi_2}(a)\big)^*\big(\widehat{\pi_1}(b) \oplus \widehat{\pi_2}(b)\big)V_2 \\
&= \widehat{V_2}^*\widehat{W}^*\widehat{\pi_1}(a)^*\widehat{\pi_1}(b)\widehat{W}\widehat{V_2} + \widehat{V_2}^*\big(1_{\widehat{\mathcal{K}_2}} - \widehat{W}^*\widehat{W}\big)\widehat{\pi_2}(a)^*\widehat{\pi_2}(b)\widehat{V_2} \\
&= V_2^*\big(\widehat{\pi_1}(\alpha(a)^*b) \oplus \widehat{\pi_2}(\alpha(a)^*b)\big)V_2.
\end{aligned}$$

Let $J = J_1 \oplus J_2$ and $\pi = \widehat{\pi_1} \oplus \widehat{\pi_2}$. For any $a \in \mathcal{A}$, we have that

$$\pi(a)^J = (J_1 \oplus J_2)\big(\widehat{\pi_1}(a) \oplus \widehat{\pi_2}(a)\big)^*(J_1 \oplus J_2) = \pi(a^*),$$

so that $\pi = \widehat{\pi_1} \oplus \widehat{\pi_2}$ is a J-representation of $\mathcal{A}$ on $\big(\widehat{\mathcal{K}_1} \oplus \widehat{\mathcal{K}_2}, \widehat{J_1} \oplus \widehat{J_2}\big)$. Hence, the quadruple $\big(\widehat{\mathcal{K}_1} \oplus \widehat{K_2}, \widehat{J_1} \oplus \widehat{J_2}, \widehat{\pi_1} \oplus \widehat{\pi_2}, V_i\big)$ dilates each ϕ_i $(i = 1, 2)$. Moreover, we have that

$$V_1^*V_2 = \Big(\widehat{V_1} \oplus 0\Big)^* \left(\widehat{W}\widehat{V_2} \oplus \Big(1_{\widehat{\mathcal{K}_2}} - \widehat{W}^*\widehat{W}\Big)^{1/2}\widehat{V_2}\right) = \widehat{V_1}^*\widehat{W}\widehat{V_2},$$

which implies that $\mathcal{J}(\phi_1, \phi_2) \neq \emptyset$ and $\mathcal{I}(\phi_1, \phi_2) \subseteq \mathcal{J}(\phi_1, \phi_2)$.

For the reverse inclusion, let π be any common J-representation of $\mathcal{A}$ on $(\mathcal{K}, J)$ for ϕ_1 and ϕ_2 and let $V_1^*V_2 \in \mathcal{J}_\pi(\phi_1, \phi_2)$. We denote by $\mathcal{K}_2$ the closed linear span of $\{\pi(a)V_2\xi : a \in \mathcal{A},\ \xi \in \mathcal{H}\}$. Let $W = P_{\mathcal{K}_2}$ be the projection of $\mathcal{K}$ onto $\mathcal{K}_2$. Then we see that $V_1^*V_2 = V_1^*WV_2$ and $W\pi(a) = \pi(a)W$, so that $\mathcal{J}_\pi(\phi_1, \phi_2) \subseteq \mathcal{I}(\phi_1, \phi_2)$. ■

Remark 2.6. From the proof of Proposition 2.5, we have that

$$\mathcal{J}_\pi(\phi_1, \phi_2) \subseteq \mathcal{I}(\phi_1, \phi_2) \subseteq \mathcal{J}_{\widehat{\pi_1} \oplus \widehat{\pi_2}}(\phi_1, \phi_2)$$

where π is a common J-representation and $\widehat{\pi_i}$ is a minimal $\widehat{J_i}$-representation. Hence, we have that

$$\beta_{\widehat{\pi_1} \oplus \widehat{\pi_2}}(\phi_1, \phi_2) \leq \beta_\pi(\phi_1, \phi_2),$$

which implies that the Bures distance between α-CP maps is evaluated in the direct sum of the minimal representations.

3. Estimating Bures Distance

We recall that the map $x \mapsto \mathrm{Tr}(x(\cdot))$ defines an isometric isomorphism from $\mathcal{B}(\mathcal{H})$ to normalized trace class operators $\mathcal{B}(\mathcal{H})_{*,1}$.

Theorem 3.1. *If $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ $(i = 1, 2)$ is an α-CP map, then we have*

$$\begin{aligned}
&\beta(\phi_1, \phi_2)^2 \\
&= \inf_{V_1^*V_2 \in \mathcal{J}(\phi_1, \phi_2)} \left\{ \sup_{\rho \in \mathcal{B}(\mathcal{H})_{*,1}} \big\{ \mathrm{Tr}\big(\phi_1(1_{\mathcal{A}})\rho\big) + \mathrm{Tr}\big(\phi_2(1_{\mathcal{A}})\rho\big) - 2Re\big[\mathrm{Tr}\big(V_1^*V_2\rho\big)\big] \big\} \right\}.
\end{aligned}$$

Proof. Let $(\mathcal{K}, J, \pi, V_i)$ be a Krein quadruple associated with ϕ_i in α-CP$(\mathcal{A}, \mathcal{H})$ $(i = 1, 2)$. We have that

$$\begin{aligned}\|V_1 - V_2\|^2 &= \sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \operatorname{Tr}\big((V_1 - V_2)^*(V_1 - V_2)\rho\big) \\ &= \sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \operatorname{Tr}\big(V_1^*V_1\rho + V_2^*V_2\rho - V_2^*V_1\rho - V_1^*V_2\rho\big) \\ &= \sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \big\{\operatorname{Tr}\big(\phi_1(1_{\mathcal{A}})\rho\big) + \operatorname{Tr}\big(\phi_2(1_{\mathcal{A}})\rho\big) - 2\mathrm{Re}\big[\operatorname{Tr}\big(V_1^*V_2\rho\big)\big]\big\}.\end{aligned}$$

By definition of the Bures distance, we obtain that

$$\begin{aligned}&\beta(\phi_1, \phi_2)^2 \\ &= \inf_\pi \beta_\pi(\phi_1, \phi_2)^2 = \inf_\pi \left\{\inf_{V_i\in S(\phi_i,\pi)} \|V_1 - V_2\|^2\right\} \\ &= \inf_\pi\left\{\inf_{V_i\in S(\phi_i,\pi)} \sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \big\{\operatorname{Tr}\big(\phi_1(1_{\mathcal{A}})\rho\big) + \operatorname{Tr}\big(\phi_2(1_{\mathcal{A}})\rho\big) - 2\mathrm{Re}\big[\operatorname{Tr}\big(V_1^*V_2\rho\big)\big]\big\}\right\} \\ &= \inf_{V_1^*V_2\in\mathcal{J}(\phi_1,\phi_2)}\left\{\sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \big\{\operatorname{Tr}\big(\phi_1(1_{\mathcal{A}})\rho\big) + \operatorname{Tr}\big(\phi_2(1_{\mathcal{A}})\rho\big) - 2\mathrm{Re}\big[\operatorname{Tr}\big(V_1^*V_2\rho\big)\big]\big\}\right\}. \quad \blacksquare\end{aligned}$$

Remark 3.2. Let ϕ_i $(i = 1, 2)$ be α-CP maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ and let $(\widehat{\mathcal{K}_i}, \widehat{J}_i, \widehat{\pi}_i, \widehat{V}_i)$ be a minimal Krein quadruple associated with ϕ_i. Assume that $(\mathcal{K}, J, \pi, V_i)$ is any Krein quadruple associated with ϕ_i. It follows from the proof of Proposition 2.5 that

$$\mathcal{J}_\pi(\phi_1, \phi_2) \subseteq \mathcal{I}(\phi_1, \phi_2) \subseteq \mathcal{J}_{\widehat{\pi_1}\oplus\widehat{\pi_2}}(\phi_1, \phi_2),$$

so that $\beta_{\widehat{\pi_1}\oplus\widehat{\pi_2}}(\phi_1, \phi_2) \le \beta_\pi(\phi_1, \phi_2)$. Hence, we have that $\beta(\phi_1, \phi_2) = \beta_{\widehat{\pi_1}\oplus\widehat{\pi_2}}(\phi_1, \phi_2)$. This implies that the Bures distance $\beta(\phi_1, \phi_2)$ can be evaluated in the direct sum of the minimal $\widehat{J}_i$-representations. $\blacksquare$

Let $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ $(i = 1, 2)$ be α-CP maps with Krein space dilations $(\mathcal{K}_i, J_i, \pi_i, V_i)$. Then we have that

$$\begin{aligned}&\beta(\phi_1, \phi_2)^2 \\ &= \inf_{V_1^*V_2\in\mathcal{J}(\phi_1,\phi_2)}\left\{\sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \big\{\operatorname{Tr}\big(\phi_1(1_{\mathcal{A}})\rho\big) + \operatorname{Tr}\big(\phi_2(1_{\mathcal{A}})\rho\big) - 2\mathrm{Re}\big[\operatorname{Tr}\big(V_1^*V_2\rho\big)\big]\big\}\right\} \\ &= \inf_{V_1^*WV_2\in\mathcal{I}(\phi_1,\phi_2)}\left\{\sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \big\{\operatorname{Tr}\big(\phi_1(1_{\mathcal{A}})\rho\big) + \operatorname{Tr}\big(\phi_2(1_{\mathcal{A}})\rho\big) - 2\mathrm{Re}\big[\operatorname{Tr}\big(V_1^*WV_2\rho\big)\big]\big\}\right\} \\ &= \inf_{\substack{\|W\|\le 1 \\ WJ_2 = J_1W \\ W\pi_2(\cdot) = \pi_1(\cdot)W}}\left\{\sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \big\{\operatorname{Tr}\big(\phi_1(1_{\mathcal{A}})\rho\big) + \operatorname{Tr}\big(\phi_2(1_{\mathcal{A}})\rho\big) - 2\mathrm{Re}\big[\operatorname{Tr}\big(V_1^*WV_2\rho\big)\big]\big\}\right\}.\end{aligned}$$

Since $\rho \in \mathcal{B}(\mathcal{H})_{*,1}$ is of trace-class, the operator $V_2\rho V_1^*$ is also of trace-class. The functional $\mathrm{Tr}\big(V_2\rho V_1^*(\cdot)\big)$ is weakly continuous in $\mathcal{W} = \{W : \|W\| \le 1,\ WJ_2 = J_1W,\ W\pi_2(\cdot) = \pi_1(\cdot)W\}$, so that the infimum is attained by compactness of the set $\mathcal{W}$.

By convexity of the set $\mathcal{W}$, the order of inf and sup can be interchanged in the equation. Hence, we obtain that

$$
\begin{aligned}
&\beta(\phi_1,\phi_2)^2\\
&= \inf_{\substack{\|W\|\le 1\\ WJ_2=J_1W\\ W\pi_2(\cdot)=\pi_1(\cdot)W}} \left\{ \sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \left\{\mathrm{Tr}\big(\phi_1(1_{\mathcal{A}})\rho\big) + \mathrm{Tr}\big(\phi_2(1_{\mathcal{A}})\rho\big) - 2\mathrm{Re}\big[\mathrm{Tr}\big(V_1^*WV_2\rho\big)\big]\right\}\right\}\\
&= \sup_{\rho\in\mathcal{B}(\mathcal{H})_{*,1}} \left\{ \inf_{\substack{\|W\|\le 1\\ WJ_2=J_1W\\ W\pi_2(\cdot)=\pi_1(\cdot)W}} \left\{\mathrm{Tr}\big(\phi_1(1_{\mathcal{A}})\rho\big) + \mathrm{Tr}\big(\phi_2(1_{\mathcal{A}})\rho\big) - 2\mathrm{Re}\big[\mathrm{Tr}\big(V_1^*WV_2\rho\big)\big]\right\}\right\}\\
&= \sup_{\substack{\xi\in\mathcal{H}\otimes\mathcal{H}\\ \|\xi\|\le 1}} \left\{ \inf_{\substack{\|W\|\le 1\\ WJ_2=J_1W\\ W\pi_2(\cdot)=\pi_1(\cdot)W}} \left\{\big\langle \xi, (\phi_1(1_{\mathcal{A}})\otimes 1_{\mathcal{H}})\xi\big\rangle + \big\langle \xi, (\phi_2(1_{\mathcal{A}})\otimes 1_{\mathcal{H}})\xi\big\rangle \right.\right.\\
&\qquad\qquad \left.\left. - 2\mathrm{Re}\big[\big\langle \xi, (V_1^*\otimes 1_{\mathcal{H}})(W\otimes 1_{\mathcal{H}})(V_2\otimes 1_{\mathcal{H}})\xi\big\rangle\big]\right\}\right\}.
\end{aligned}
$$

We denote by $\omega_{\xi,\xi}$ the vector functional by ξ and take $a \in \mathcal{A}$ and $x \in \mathcal{B}(\mathcal{H})$. Then we have that

$$
\big[\omega_{\xi,\xi}\circ(\phi_i\otimes 1_{\mathcal{B}(\mathcal{H})})\big](a\otimes x) = \big\langle \xi, (V_i^*\otimes 1_{\mathcal{H}})(\pi_i(a)\otimes x)(V_i\otimes 1_{\mathcal{H}})\xi\big\rangle.
$$

Let $\widetilde{W} = W\otimes 1_{\mathcal{B}(\mathcal{H})}$ be an operator from $\mathcal{K}_2\otimes\mathcal{H}$ into $\mathcal{K}_1\otimes\mathcal{H}$ where $W : \mathcal{K}_2 \to \mathcal{K}_1$ is such that $WJ_2 = J_1W$ and $W\pi_2(a) = \pi_1(a)W$ for all $a \in \mathcal{A}$. We immediately see that

$$
\widetilde{W}\widetilde{J}_2 = \widetilde{J}_1\widetilde{W} \quad\text{and}\quad \widetilde{W}\big(\pi_2(a)\otimes 1_{\mathcal{B}(\mathcal{H})}\big) = \big(\pi_1(a)\otimes 1_{\mathcal{B}(\mathcal{H})}\big)\widetilde{W}, \quad (a\in\mathcal{A})
$$

where $\widetilde{J}_i = J_i\otimes 1_{\mathcal{H}}$.

Theorem 3.3. *Let ϕ_i $(i = 1,2,3)$ be α-CP maps from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$. The triangle inequality holds; $\beta(\phi_1,\phi_3) \le \beta(\phi_1,\phi_2) + \beta(\phi_2,\phi_3)$.*

Proof. Let $(\mathcal{K}, J, \pi)$ be a common representation for ϕ_1 and ϕ_2, i.e., for $i = 1,2$

$$
\phi_i(a) = V_i^*\pi(a)V_i \quad\text{and}\quad V_i^*\pi(a)^*\pi(b)V_i = V_i^*\pi\big(\alpha(a)^*b\big)V_i
$$

where $a, b \in \mathcal{A}$ and let $\big(\widetilde{\mathcal{K}}, \widetilde{J}, \widetilde{\pi}\big)$ be a common representation for ϕ_2 and ϕ_3 with $\widetilde{V_1}$ and $\widetilde{V_2}$, respectively. We denote by $\big(\widehat{\mathcal{K}_i}, \widehat{J}_i, \widehat{\pi}_i, \widehat{V}_i\big)$ the minimal Krein quadruple associated with ϕ_i $(i=1,2,3)$. Let $U_i : \widehat{\mathcal{K}_i} \to \mathcal{K}$ $(i=1,2)$ and $\widetilde{U_j} : \widehat{\mathcal{K}_j} \to \widetilde{\mathcal{K}}$ $(j=2,3)$ be isometries satisfying equations (2.3).

We define three maps $\overline{V_i} : \mathcal{H} \to \widehat{\mathcal{K}_1} \oplus \widehat{\mathcal{K}_2} \oplus \widehat{\mathcal{K}_3}$ $(i=1,2,3)$ by

$$\begin{aligned}
\overline{V_1} &= \left(1_{\widehat{\mathcal{K}_1}} - U_1^* U_2 U_2^* U_1\right)^{\frac{1}{2}} \widehat{V_1} \oplus U_2^* V_1 \oplus 0, \\
\overline{V_2} &= 0 \oplus \widehat{V_2} \oplus 0, \\
\overline{V_3} &= 0 \oplus \widetilde{U_2}^* \widetilde{V_3} \oplus \left(1_{\widehat{\mathcal{K}_3}} - \widetilde{U_3}^* \widetilde{U_2} \widetilde{U_2}^* \widetilde{U_3}\right)^{\frac{1}{2}} \widehat{V_3}.
\end{aligned}$$

By letting $\overline{J} := \widehat{J_1} \oplus \widehat{J_2} \oplus \widehat{J_3}$, we see that $\overline{\pi} := \widehat{\pi_1} \oplus \widehat{\pi_2} \oplus \widehat{\pi_3}$ is a $\overline{J}$-representation of $\mathcal{A}$ on $\overline{\mathcal{K}} := \widehat{\mathcal{K}_1} \oplus \widehat{\mathcal{K}_2} \oplus \widehat{\mathcal{K}_3}$, i.e., for all $a, b \in \mathcal{A}$,

$$\overline{\pi}(ab) = \overline{\pi}(a)\overline{\pi}(b), \quad \overline{\pi}(a^*) = \overline{\pi}(a)^{\overline{J}}$$

where $\overline{\pi}(a)^{\overline{J}} := \widehat{\pi_1}(a)^{\widehat{J_1}} \oplus \widehat{\pi_2}(a)^{\widehat{J_2}} \oplus \widehat{\pi_3}(a)^{\widehat{J_3}}$. It immediately follows from definition that $\overline{V_2} \in S(\phi_2, \overline{\pi})$, i.e., for all $a, b \in \mathcal{A}$,

$$\phi_2(a) = \overline{V_2}^* \overline{\pi}(a) \overline{V_2} \quad \text{and} \quad \overline{V_2}^* \overline{\pi}(a)^* \overline{\pi}(b) \overline{V_2} = \overline{V_2}^* \overline{\pi}(\alpha(a)^* b) \overline{V_2}.$$

For any $a, b \in \mathcal{A}$, we have that

$$\begin{aligned}
&\overline{V_1}^* \overline{\pi}(a) \overline{V_1} \\
&= \widehat{V_1}^* \left(1_{\widehat{\mathcal{K}_1}} - U_1^* U_2 U_2^* U_1\right)^{\frac{1}{2}} \widehat{\pi_1}(a) \left(1_{\widehat{\mathcal{K}_1}} - U_1^* U_2 U_2^* U_1\right)^{\frac{1}{2}} \widehat{V_1} + V_1^* U_2 \widehat{\pi_2}(a) U_2^* V_1 \\
&= V_1^* U_1 U_1^* \pi(a) \left(1_{\widehat{\mathcal{K}_1}} - U_2 U_2^*\right) U_1 U_1^* V_1 + V_1^* \widehat{\pi_1}(a) U_2 U_2^* V_1 \\
&= \phi_1(a)
\end{aligned}$$

and that

$$\begin{aligned}
\overline{V_1}^* \overline{\pi}(a)^* \overline{\pi}(b) \overline{V_1} &= V_1^* U_1 U_1^* \pi(a)^* \pi(b) \left(1_{\widehat{\mathcal{K}_1}} - U_2 U_2^*\right) U_1 U_1^* V_1 + V_1^* \pi(a)^* \pi(b) U_2 U_2^* V_1 \\
&= \phi_1(\alpha(a)^* b) = \overline{V_1}^* \overline{\pi}(\alpha(a)^* b) \overline{V_1}.
\end{aligned}$$

Hence, $\overline{V_1} \in S(\phi_1, \overline{\pi})$. We similarly get $\overline{V_3} \in S(\phi_3, \overline{\pi})$. Therefore, $(\overline{\mathcal{K}}, \overline{J}, \overline{\pi})$ is a common Krein space representation for ϕ_1, ϕ_2 and ϕ_3.

Moreover, we have that

$$\overline{V_2}^* \overline{V_1} = \left(0 \oplus \widehat{V_2}^* \oplus 0\right) \left(\left(1_{\widehat{\mathcal{K}_1}} - U_1^* U_2 U_2^* U_1\right)^{\frac{1}{2}} \widehat{V_1} \oplus U_2^* V_1 \oplus 0\right) = \widehat{V_2}^* U_2^* V_1 = V_2^* V_1,$$

$$\overline{V_2}^* \overline{V_3} = \left(0 \oplus \widehat{V_2}^* \oplus 0\right) \left(0 \oplus \widetilde{U_2}^* \widetilde{V_3} \oplus \left(1_{\widehat{\mathcal{K}_3}} - \widetilde{U_3}^* \widetilde{U_2} \widetilde{U_2}^* \widetilde{U_3}\right)^{\frac{1}{2}} \widehat{V_3}\right) = \widehat{V_2}^* \widetilde{U_2}^* \widetilde{V_3} = \widetilde{V_2}^* \widetilde{V_3}.$$

Suppose that $(\mathcal{K}, J, \pi)$ and $(\widetilde{\mathcal{K}}, \widetilde{J}, \widetilde{\pi})$ are chosen as in Remark 3.2, i.e.,

$$\beta(\phi_1, \phi_2) = \beta_\pi(\phi_1, \phi_2) \quad \text{and} \quad \beta(\phi_2, \phi_3) = \beta_{\widetilde{\pi}}(\phi_2, \phi_3).$$

Thus, we have that

$$\|V_1 - V_2\| = \beta_\pi(\phi_1, \phi_2) = \beta(\phi_1, \phi_2) \quad \text{and} \quad \|\widetilde{V_2} - \widetilde{V_3}\| = \beta_{\widetilde{\pi}}(\phi_2, \phi_3) = \beta(\phi_2, \phi_3).$$

By the triangle inequality of operator norms, we obtain that

$$\beta(\phi_1, \phi_3) \leq \|\overline{V_1} - \overline{V_3}\| \leq \|V_1 - V_2\| + \|\widetilde{V_2} - \widetilde{V_3}\| = \beta(\phi_1, \phi_2) + \beta(\phi_2, \phi_3),$$

which completes the proof. ∎

Proposition 3.4. *If* $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ $(i = 1, 2)$ *are* α*-CP maps with Krein space dilations* $(\mathcal{K}_i, J_i, \pi_i, V_i)$*, then we have that for any* $0 \leq t \leq 1$*,*

$$\bigl|\beta(\phi_1, \phi_2) - \beta(\phi_1, t\phi_1 + (1-t)\phi_2)\bigr| \leq t^{1/2}\bigl(\|V_1\| + \|V_2\|\bigr).$$

Proof. Putting $\widetilde{V_1} = V_1 \oplus 0$ and $\pi = \pi_1 \oplus \pi_2$, we clearly have that $\widetilde{V_1} \in S(\phi_1 \oplus \phi_2, \pi_1 \oplus \pi_2)$. We also see that $\widetilde{V_2} = V_1 \oplus V_2 \in S(\phi_1 \oplus \phi_2, \pi_1 \oplus \pi_2)$. Indeed, for any $a, b \in \mathcal{A}$ we have that

$$(\phi_1 \oplus \phi_2)(a) = (V_1 \oplus V_2)^*(\pi_1 \oplus \pi_2)(a)(V_1 \oplus V_2) = \widetilde{V_2}^*\pi(a)\widetilde{V_2}$$

and that

$$\widetilde{V_2}^*\pi(a)^*\pi(b)\widetilde{V_2} = V_1^*\pi_1(\alpha(a)^*b)V_1 \oplus V_2^*\pi_2(\alpha(a)^*b)V_2 = \widetilde{V_2}^*\pi(\alpha(a)^*b)\widetilde{V_2}.$$

Moreover, we also see that $\widetilde{V_1} \in S(\phi_1 \oplus \phi_2, \pi_1 \oplus \pi_2)$. Hence, we have that

$$\beta(\phi_1, \phi_1 + \phi_2) \leq \|\widetilde{V_1} - \widetilde{V_2}\| = \|V_2\|.$$

Let $0 \leq t \leq 1$. By the triangle inequality of the Bures distance, we have that

$$\begin{aligned}
&\bigl|\beta(\phi_1, \phi_2) - \beta(\phi_1, t\phi_1 + (1-t)\phi_2)\bigr| \\
&\leq \beta\bigl(\phi_2, t\phi_1 + (1-t)\phi_2\bigr) \\
&\leq \beta\bigl(\phi_2, (1-t)\phi_2\bigr) + \beta\bigl((1-t)\phi_2, t\phi_1 + (1-t)\phi_2\bigr) \\
&\leq t^{1/2}\bigl(\|V_1\| + \|V_2\|\bigr),
\end{aligned}$$

which completes the proof. ∎

4. Unbounded Representations of *-Algebras in Krein Spaces

In this section, we assume that $\mathcal{A}$ is a unital *-algebra with unit $1_{\mathcal{A}}$. A representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a linear mapping of $\mathcal{A}$ into the algebra of all closable linear operators defined on a common dense subspace $\mathcal{D}(\pi)$ such that

(i) $\mathcal{D}(\pi)$ is globally invariant under $\pi(a)$ $(a \in \mathcal{A})$,

(ii) $\pi(1_{\mathcal{A}}) = 1$, the unit operator,

(iii) $\pi(a)\pi(b)\xi = \pi(ab)\xi$ for all $a, b \in \mathcal{A}$, $\xi \in \mathcal{D}(\pi)$.

Let $\mathcal{D}(\pi^*) := \cap_{a\in\mathcal{A}}\mathcal{D}(\pi(a)^*)$ and $\pi^*(a)\xi := \pi(a^*)^*\xi$ $(a \in \mathcal{A}, \xi \in \mathcal{D}(\pi^*))$. Then we see that π^* is a representation of $\mathcal{A}$ on a Hilbert space $\overline{\mathcal{D}(\pi^*)}$.

If J is a (fundamental) symmetry on $\mathcal{H}$, i.e., $J = J^* = J^{-1} \in \mathcal{B}(\mathcal{H})$, then we define a sesquilinear form on $\mathcal{H}$ by

$$[\xi, \eta]_J = \langle J\xi, \eta\rangle.$$

The pair $(\mathcal{H}, J)$ is called a Krein space. If a representation π of $\mathcal{A}$ on $\mathcal{H}$ satisfies

$$[\pi(a)\xi, \eta]_J = [\xi, \pi(a^*)\eta]_J, \quad (a \in \mathcal{A}, \xi, \eta \in \mathcal{H}),$$

then π is called a J-representation of $\mathcal{A}$ on $(\mathcal{H}, J)$. Let T be a densely defined linear operator in $(\mathcal{H}, J)$ and let T^J be a J-adjoint of T defined by

$$[T\xi, \eta]_J = [\xi, T^J\eta]_J, \quad (\xi \in \mathcal{D}(T), \eta \in \mathcal{D}(T^J)).$$

We easily see that $T^J = JT^*J$. We define a J-adjoint of π by

$$\mathcal{D}(\pi^J) = \bigcap_{a\in\mathcal{A}} \mathcal{D}(\pi(a)^J), \quad \pi^J(a) = \pi(a^*)^J|_{\mathcal{D}(\pi^J)}.$$

Then π is a J-representation of $\mathcal{A}$ if and only if $\mathcal{D}(\pi) \subseteq \mathcal{D}(\pi^J)$ and $\pi(a)\xi = \pi^J(a)\xi$ for all $a \in \mathcal{A}$ and $\xi \in \mathcal{D}(\pi)$.

Let $\mathcal{B}$ be a unital *-subalgebra of $\mathcal{A}$. Suppose there exists an abstract conditional expectation P of $\mathcal{A}$ onto $\mathcal{B}$ and let ϕ be a P-functional on $\mathcal{A}$ (see [17] for definition). By [17, Theorem 3], there exists a J-representation π of $\mathcal{A}$ on a Krein space $(\mathcal{K}, J)$ with a cyclic vector ξ in $\mathcal{D}(\pi)$. If we define α by

$$\alpha(a) = 2P(a) - a \quad \text{for all } a \in \mathcal{A},$$

then ϕ is an α-completely positive linear functional.

Let J-Rep$(\mathcal{A})$ be the set of J-representations of $\mathcal{A}$ and let P-Ftnal$(\mathcal{A})$ be the set of P-functionals on $\mathcal{A}$. For $\phi \in P$-Ftnal$(\mathcal{A})$ and $\pi \in J$-Rep$(\mathcal{A})$, we denote by $S(\phi, \pi)$ the set of unit vectors $\xi \in \mathcal{D}(\pi)$ such that $\phi(a) = \langle\pi(a)\xi, \xi\rangle$. For any P-functionals ϕ_1 and ϕ_2 in P-Ftnal$(\mathcal{A})$, let $\mathfrak{F}(\phi_1, \phi_2)$ be the set of all linear functionals F on $\mathcal{A}$ such that

(1) $F\big(\alpha(a)\alpha(b)\big) = F\big(\alpha(ab)\big)$,

(2) $\big|F(\alpha(a^*)b)\big|^2 \le \phi_2\big(\alpha(a^*)a\big)\phi_1\big(\alpha(b^*)b\big)$,

where α is a bounded Hermitian map on A with $\alpha^2 = \mathrm{id}_{\mathcal{A}}$. We call any linear functional in $\mathfrak{F}(\phi_1, \phi_2)$ a *transition form* from ϕ_1 to ϕ_2.

Definition 4.1. Let ϕ_i $(i = 1, 2)$ be any elements of P-Ftnal$(\mathcal{A})$. We define the *transition probability* between ϕ_1 and ϕ_2 by

$$\mathcal{P}(\phi_1, \phi_2) = \sup_{\pi \in J\text{-Rep}(\mathcal{A})} \sup_{\xi_i \in S(\phi_i, \pi)} \big|\langle \xi_1, \xi_2 \rangle\big|^2.$$

In the case of von Neumann algebras and normal states, the definition of the transition probability appears in [6]. In general case of *-algebras and states, the definition and explicit formulae were given by Uhlmann [19]. The *Bures distance* between P-functionals ϕ_1 and ϕ_2 is given by $\beta(\phi_1, \phi_2) = \inf\{\|\xi_1 - \xi_2\| : \xi_i \in S(\phi_i, \pi),\ \pi \in J\text{-Rep}(\mathcal{A})\}$. As in the case of normal states, the Bures distance between P-functionals is also related to the transition probability by the formula $\beta(\phi_1, \phi_2)^2 = 2\big(1 - \mathcal{P}(\phi_1, \phi_2)^{1/2}\big)$.

For any $\xi_1 \in S(\phi_1, \pi)$ and $\xi_2 \in S(\phi_2, \pi)$, we define a functional F_{ξ_1,ξ_2} on $\mathcal{A}$

$$F_{\xi_1,\xi_2}(a) = \langle \pi(a)\xi_1, \xi_2 \rangle.$$

Let $a, b \in \mathcal{A}$. Then we have that

$$\begin{aligned} F_{\xi_1,\xi_2}(\alpha(ab)) &= \langle \pi(\alpha(ab))\xi_1, \xi_2 \rangle = \overline{\langle \pi(b^*)\pi(a^*)\xi_2, \xi_1 \rangle} \\ &= \langle \pi(\alpha(a))\pi(\alpha(b))\xi_1, \xi_2 \rangle = F_{\xi_1,\xi_2}(\alpha(a)\alpha(b)) \end{aligned}$$

and that

$$|F_{\xi_1,\xi_2}(\alpha(a^*)b)|^2 \le \langle \pi(\alpha(a^*)a)\xi_2, \xi_2 \rangle \langle \pi(\alpha(b^*)b)\xi_1, \xi_1 \rangle = \phi_2(\alpha(a^*)a)\phi_1(\alpha(b^*)b).$$

Hence, we see that $F_{\xi_1,\xi_2} \in \mathfrak{F}(\phi_1, \phi_2)$.

The following theorem was proved by Alberti [2] for states on C^*-algebras and by Uhlmann [20] for states for *-algebras.

Theorem 4.2. *If ϕ_1 and ϕ_2 are P-functional on $\mathcal{A}$, then*

$$\mathcal{P}(\phi_1, \phi_2) = \sup_{F \in \mathfrak{F}(\phi_1, \phi_2)} |F(1_{\mathcal{A}})|^2.$$

Proof. Assume that π is a J-representation of $\mathcal{A}$ on a Krein space $(\mathcal{K}, J)$ such that $S(\phi_1, \pi)$ and $S(\phi_2, \pi)$ are non-empty. Take $\xi_1 \in S(\phi_1, \pi)$ and $\xi_2 \in S(\phi_2, \pi)$ and we define two Hilbert spaces

$$\mathcal{K}_1 = \overline{\{\pi(a)\xi_1 : a \in \mathcal{A}\}} \quad \text{and} \quad \mathcal{K}_2 = \overline{\{\pi(a)\xi_2 : a \in \mathcal{A}\}}.$$

Let $P_1 : \mathcal{K} \to \mathcal{K}_1$ and $P_2 : \mathcal{K} \to \mathcal{K}_2$ be orthogonal projections. Then we see that P_1 and P_2 belong to the commutant $\pi(\mathcal{A})'$.

Let $F \in \mathfrak{F}(\phi_1, \phi_2)$. We have that

$$\big|F(\alpha(a^*)b)\big|^2 \leq \phi_2\big(\alpha(a^*)a\big)\phi_1\big(\alpha(b^*)b\big) = \big\|\pi(a)\xi_2\big\|^2\big\|\pi(b)\xi_1\big\|^2$$

for all $a, b \in \mathcal{A}$, so that the map $(\pi(b)\xi_1, \pi(a)\xi_2) \mapsto F(\alpha(a^*)b)$ is a densely defined bounded sesquilinear form on $\mathcal{K}_1 \times \mathcal{K}_2$. Therefore, there exist a unique bounded operator T from $\mathcal{K}_1$ into $\mathcal{K}_2$ such that

$$F(\alpha(a^*)b) = \big\langle T\pi(b)\xi_1, \pi(a)\xi_2\big\rangle.$$

It is obvious that $\|T\| \leq 1$. Putting $K := P_2TP_1 \in \mathcal{B}(\mathcal{K})$, we have that $\|T\| \leq 1$ and that

$$\begin{aligned}\big\langle K\pi(a)\pi(c)\xi_1, \pi(c)\xi_2\big\rangle &= \big\langle T\pi(ab)\xi_1, \pi(b)\xi_2\big\rangle \\ &= F\big(\alpha(b^*)ac\big) \\ &= \big\langle T\pi(c)\xi_1, \pi(\alpha(a^*)b)\xi_2\big\rangle \\ &= \big\langle \pi(a)K\pi(c)\xi_1, \pi(b)\xi_2\big\rangle,\end{aligned}$$

which implies that $K \in \pi(\mathcal{A})'$.

It follows from Theorem of Russo and Dye that for any $\epsilon > 0$, there are nonnegative real numbers $r_1, \ldots, r_n$ with $\sum_i r_i = 1$ and unitaries $U_1, \ldots, U_n \in \pi(\mathcal{A})'$ such that

$$\left\|K - \sum_{i=1}^{n} r_i U_i\right\| < \epsilon.$$

Hence, $\big|F(1_{\mathcal{A}}) - \sum_{i=1}^n r_i\langle U_i\xi_1, \xi_2\rangle\big| < \epsilon$, so that $\big|F(1_{\mathcal{A}})\big| < \epsilon + \sum_{i=1}^n r_i\big|\langle U_i\xi_1, \xi_2\rangle\big|$. Since $U_i \in \pi(\mathcal{A})'$ implies $U_i\xi_1 \in S(\phi_2, \pi)$, we have that

$$\big|F(1_{\mathcal{A}})\big| < \epsilon + \mathcal{P}(\phi_1, \phi_2)^{1/2}.$$

Since $\epsilon > 0$ was arbitrary, $\big|F(1_{\mathcal{A}})\big| \leq \mathcal{P}(\phi_1, \phi_2)^{1/2}$.

Let $\epsilon > 0$ be given. Assume that $\xi_1 \in S(\phi_1, \pi)$ and $\xi_2 \in S(\phi_2, \pi)$ are such that

$$|\langle \xi_1, \xi_2\rangle|^2 \geq \mathcal{P}(\phi_1, \phi_2) - \epsilon.$$

By defining $F(a) = \langle \pi(a)\xi_1, \xi_2\rangle$, we have that $F \in \mathfrak{F}(\phi_1, \phi_2)$ and $\big|F(1_{\mathcal{A}})\big|^2 \geq \mathcal{P}(\phi_1, \phi_2) - \epsilon$. Since $\epsilon > 0$ was arbitrary, we obtain that

$$\sup_{F \in \mathfrak{F}(\phi_1, \phi_2)} \big|F(1_{\mathcal{A}})\big|^2 \geq \mathcal{P}(\phi_1, \phi_2),$$

which completes the proof. ∎

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REFERENCES

1. S. Abe, Nonadditive generalization of the quantum Kullback-Leibler divergence for measuring the degree of purification, *Phys. Rev. A*, **68** (2003), 032302.
2. P. Alberti, A note on the transition probability over C^*-algebras, *Lett. Math. Phys.*, **7** (1983), 25-32.
3. P. Alberti, Playing with fidelities, *Rep. Math. Phys.*, **51** (2002), 87-125.
4. J. P. Antoine and S. Ôta, Unbounded GNS representations of *-algebra in a Krein space, *Lett. Math. Phys.*, **18** (1989), 267-274.
5. B. V. Rajarama Bhat and K. Sumesh, *Bures Distance for Completely Positive Maps*, arXiv preprint arXiv:1305.0111v1, 2013.
6. D. Bures, An extension of Kakutani's theorem on infinite product measures to the tensor product of semifinite W^*-algebras, *Trans. Amer. Math. Soc.*, **135** (1969), 199-212.
7. E. B. Davies, *Quantum Theory of Open Systems*, Academic Press, London, 1976.
8. J. Heo, α-Completely positive maps of group systems and Krein module representations, *J. Math. Anal. Appl.*, **409** (2014), 544-555.
9. J. Heo, J. P. Hong and U. C. Ji, On KSGNS representations on Krein C^*-modules, *J. Math. Phys.*, **51** (2010), 053504.
10. J. Heo and U. C. Ji, Radon-Nikodým type theorem for α-completely positive maps, *J. Math. Phys.*, **51** (2010), 103505.
11. J. Heo, U. C. Ji and Y. Kim, α-Completely positive maps on locally C^*-algebras, Krein modules and Radon-Nikodým theorem, *J. Korean Math. Soc.*, **50** (2013), 61-80.
12. J. Heo, U. C. Ji and Y. Kim, Projective covariant representation of locally C^*-dynamical systems, *Taiwanese J. Math.*, **17** (2013), 529-544.
13. J. Heo, U. C. Ji and Y. Kim, Covariant representations on Krein C^*-modules associated to pairs of two maps, *J. Math. Anal. Appl.*, **398** (2013), 35-45.
14. G. Hofmann, An explicit realization of a GNS representation in a Krein-space, *Publ. RIMS, Kyoto Univ.*, **29** (1993), 267-287.
15. L. Jakóbczyk, Borchers algebra formulation of an indefinite inner product quantum field theory, *J. Math. Phys.*, **25** (1984), 617-622.

16. D. Kretschmann, D. Schlingemann and R. Werner, A continuity theorem for Stinespring's dilation, *J. Funct. Anal.*, **255** (2008), 1889-1904.

17. S. Ôta, Unbounded GNS representations of a *-algebra in a Krein space, *Letters in Math. Phys.*, **18** (1989), 267-274.

18. F. Strocchi, Spontaneous symmetry breaking in local gauge quantum field theory; the Higgs mechanism, *Comm. Math. Phys.*, **56** (1977), 57-78.

19. A. Uhlmann, The "transition probability" in the state space of a *-algebra, *Rep. Mathematical Phys.*, **9** (1976), 273-279.

20. A. Uhlmann, The transition probability for states of *-algebras, *Ann. Phys.*, **42** (1985), 524-532.

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