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PRODUCTS OF MULTIPLICATION, COMPOSITION AND DIFFERENTIATION OPERATORS FROM MIXED-NORM SPACES TO WEIGHTED-TYPE SPACES

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Abstract. Let φ be an analytic self-map of the unit disk $\mathbb{D}$, $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$ and $\psi_1, \psi_2 \in H(\mathbb{D})$. Recently Stević and co-workers defined the following operator

$$T_{\psi_1,\psi_2,\varphi} f(z) = \psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z)), \quad f \in H(\mathbb{D}).$$

The boundedness and compactness of the operators $T_{\psi_1,\psi_2,\varphi}$ from mixed-norm spaces to weighted-type spaces are investigated in this paper.

1. Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. A positive continuous function ϕ on the interval [0,1) is called normal if there exist positive numbers $a, b, 0 < a < b$ and $t_0 \in [0,1)$, such that

$\frac{\phi(t)}{(1-t^2)^a}$ is decreasing for $t_0 \le t < 1$ and $\lim\limits_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^a} = 0$;

$\frac{\phi(t)}{(1-t^2)^a}$ is increasing for $t_0 \le t < 1$ and $\lim\limits_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^a} = \infty$

(see, e.g., [20]).

For $0 < p < \infty, 0 < q < \infty$ and a normal function ϕ, the mixed-norm space $H(p,q,\phi)$ is the space of analytic functions on the unit disk $\mathbb{D}$ such that

$$\|f\|_{p,q,\phi} = \Big(\int_0^1 M_q^p(f,r) \frac{\phi^p(r)}{1-r} r\,dr \Big)^{1/p},$$

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where the integral means $M_p(f,r)$ are defined by

$$M_p(f,r) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\Big)^{1/p}, \quad 0 \le r < 1.$$

For $1 \le p < \infty$, $H(p,q,\phi)$ equipped with the norm $\|\cdot\|_{p,q,\phi}$ is a Banach space. When $0 < p < 1$, $\|\cdot\|_{p,q,\phi}$ is a quasinorm on $H(p,q,\phi)$, $H(p,q,\phi)$ is a Fréchet space but not a Banach space. If $0 < p = q < \infty$, then $H(p,p,\phi)$ is the Bergman-type space

$$H(p,p,\phi) = \{f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p \frac{\phi^p(|z|)}{1-|z|} dA(z) < \infty\},$$

where $dA(z)$ denotes the normalized Lebesgue area measure on the unit disk $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. Note that if $\phi(r) = (1-r)^{(\alpha+1)/p}$, then $H(p,p,\phi)$ is the weighted Bergman space A^p_α defined for $0 < p < \infty$ and $\alpha > -1$, as the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty$$

(see, e.g., [4]).

Let μ be a positive continuous function on $\mathbb{D}$ (weight). The weighted-type space $H^\infty_\mu(\mathbb{D}) = H^\infty_\mu$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{D}} \mu(z)|f(z)| < \infty.$$

It is known that H^∞_μ is a Banach space. Let $H^\infty_{\mu,0}$ denote the subspace of H^∞_μ consisting of those $f \in H^\infty_\mu$ such that $\sup\limits_{|z|\to 1} \mu(z)|f(z)| = 0$. This space is called the little weighted-type space. For some results on weighted-type spaces see, e.g.[6] and the related references therein.

Denote by $S(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. For $\varphi \in S(\mathbb{D})$ the composition operator C_φ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

It is interesting to provide a function theoretic characterization for φ inducing a bounded or compact composition operator on various spaces. It is well known that the composition operator is bounded on Hardy space, the Bergman space and the Bloch space. The composition operator was studied extensively by many people, see, for example, [1, 19, 31] and references therein.

For $\psi \in H(\mathbb{D})$, the multiplication operator M_ψ is defined by

$$M_\psi f = \psi \cdot f, \quad f \in H(\mathbb{D}).$$

Given $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the weighted composition operator with symbols ψ and φ is defined as the linear operator on $H(\mathbb{D})$ given by

$$(\psi C_\varphi f)(z) = \psi(z) f(\varphi(z)) = (M_\psi C_\varphi f)(z), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Special cases for $\psi(z) = 1$ and $\varphi(z) = z$, $z \in \mathbb{D}$, are the composition operator C_φ and the multiplication operator M_ψ. For some recent articles on weighted composition operators on some H^∞-type spaces, see, for example, [7, 9, 17, 21, 22, 23] and references therein.

Let D be the differentiation operator, it is defined by

$$Df = f', \quad f \in H(\mathbb{D}).$$

The differentiation operator is typically unbounded on many analytic function spaces.

Products of concrete linear operators between spaces of holomorphic functions have been the object of study for recent several years, see, e.g. [3, 5, 8, 10, 11, 12, 14, 15, 18, 25, 27, 30, 32] and the related references therein.

The products of composition operator and differentiation operator DC_φ and $C_\varphi D$ are defined respectively as follows

$$DC_\varphi f = f'(\varphi)\varphi', \quad f \in H(\mathbb{D})$$

and

$$C_\varphi D f = f' \circ \varphi, \quad f \in H(\mathbb{D}).$$

They have been recently studied, for example, in [5, 8, 10, 11, 12, 14, 18, 25, 27, 29, 30] (see also the related references therein). Ohon in [18] devoted most of the paper to finding necessary and sufficient conditions for $C_\varphi D$ to be bounded as well as for $C_\varphi D$ to be compact on the Hardy space H^2. The operator DC_φ was studied for the first time in [5], where the boundedness and compactness of DC_φ between Bergman and Hardy spaces are investigated. Li and Stević in [8, 10, 12] studied the boundedness and compactness of the operator DC_φ between Bloch type space, weighted Bergman space A^p_α and Bloch type space B^β, mixed-norm space and α-Bloch space B^α as well as the space of bounded analytic functions and the Bloch-type space. Liu and Yu in [15] studied the boundedness and compactness of the operator DC_φ from H^∞ and Bloch spaces to Zygmund spaces. Yang in [28] studied the same problems for operators $C_\varphi D$ and DC_φ from $Q_K(p,q)$ space to B_μ and $B_{\mu,0}$.

The products of differentiation operator and multiplication operator, denoted by DM_φ, is defined as follows

$$DM_\psi f = \psi' \cdot f + \psi \cdot f', \quad f \in H(\mathbb{D}).$$

Stević in [26] studied the boundedness and compactness of the products of differentiation and multiplication operators DM_ψ from mixed-norm spaces to weighted-type spaces. Liu and Yu in [14] studied the operators DM_ψ from H^∞ to Zygmund spaces. Yu and Liu in [30] investigated the same problems for operators DM_ψ from mixed-norm spaces to Bloch-type spaces.

Zhu in [32] completely characterized the boundedness and compactness of linear operators which are obtained by taking products of differentiation, composition and multiplication operators and which act from Bergman type spaces to Bers spaces. Kumar and Singh investigated the same problem for operators $DC_\varphi M_\psi$ acting on A^p_α and used the Carleson-type conditions. They also found the essential norm estimates of $M_\psi DC_\varphi$ in the spirit of the work by Čučković and Zhao [2].

The products of composition, multiplication and differentiation operators can be defined in following six ways

$$
\begin{aligned}
(M_\psi C_\varphi Df)(z) &= \psi(z)f'(\varphi(z));\\
(M_\psi DC_\varphi f)(z) &= \psi(z)\varphi'(z)f'(\varphi(z));\\
(C_\varphi M_\psi Df)(z) &= \psi(\varphi(z))f'(\varphi(z));\\
(DM_\psi C_\varphi f)(z) &= \psi'(z)f(\varphi(z)) + \psi(z)\varphi'(z)f'(\varphi(z));\\
(C_\varphi DM_\psi f)(z) &= \psi'(\varphi(z))f(\varphi(z)) + \psi(\varphi(z))f'(\varphi(z));\\
(DC_\varphi M_\psi f)(z) &= \psi'(\varphi(z))\varphi'(z)f(\varphi(z)) + \psi(\varphi(z))\varphi'(z)f'(\varphi(z));
\end{aligned}
\tag{1}
$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

It is interesting to provide a function theoretic characterization of ψ and φ when the six above operators become bounded or compact operators between spaces of analytic functions in the unit disk, the polydisk and the unit ball.

Note that the operator $M_\psi C_\varphi D$ induces many known operators. If $\psi(z) = 1$, then $M_\psi C_\varphi D = C_\varphi D$. When $\psi(z) = \varphi'(z)$, then we get the operator DC_φ. If we put $\varphi(z) = z$, then $M_\psi C_\varphi D = M_\psi D$, that is, the product of differentiation operator. Also note that $M_\psi DC_\varphi = M_{\psi\varphi'} C_\varphi D$ and $C_\varphi M_\psi D = M_{\psi\circ\varphi} C_\varphi D$. Thus the corresponding characterizations of boundedness and compactness of $M_\psi DC_\varphi$ and $C_\varphi M_\psi D$ can be obtained by replacing ψ, respectively by $\psi\varphi$ and $\psi\circ\varphi$ in the results stated for $M_\psi C_\varphi D$.

Let $\psi_1, \psi_2 \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. The products of multiplication composition and differentiation operators are defined as follows

$$T_{\psi_1,\psi_2,\varphi} f(z) = \psi_1(z)f(\varphi(z)) + \psi_2(z)f'(\varphi(z)), \quad f \in H(\mathbb{D}).$$

The operator $T_{\psi_1,\psi_2,\varphi}$ was studied by Stević and co-workers for the first time in [27], where the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}$ between Bergman spaces are investigated. It is clear that all products of composition, multiplication and differentiation operators in (1.1) can be obtained from the operator $T_{\psi_1,\psi_2,\varphi}$ by fixing ψ_1 and ψ_2. More specifically we have

$$M_\psi C_\varphi D = T_{0,\psi,\varphi}, \quad M_\psi DC_\varphi = T_{0,\psi\varphi',\varphi}, \quad C_\varphi M_\psi D = T_{0,\psi\circ\varphi,\varphi},$$

$$DM_\psi C_\varphi = T_{\psi',\psi\varphi,\varphi}, \quad C_\varphi DM_\psi = T_{\psi'\circ\varphi,\psi\varphi,\varphi}, DC_\varphi M_\psi = T_{(\psi'\circ\varphi)\varphi',(\psi\circ\varphi)\varphi',\varphi}.$$

Motivated by the results [26, 27], we consider the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\varphi}$ from mixed-norm space $H(p, q, \phi)$ to weighted-type space H^∞_μ.

Throughout this article, the letter C denotes a positive constant which may vary at each occurrence but it is independent of the essential variables.

2. Some Lemmas

Lemma 2.1. [24]. *Assume that $p, q \in (0, \infty)$, ϕ is normal and $f \in H(p, q, \phi)$. Then for each $n \in N_0$, there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \le C \frac{\|f\|_{p,q,\phi}}{\phi(|z|)(1 - |z|^2)^{1/q+n}}, \quad z \in \mathbb{D}.$$

By standard arguments (see [16, 21]) the following lemmas follows.

Lemma 2.2. *Assume that $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$.Then $T_{\psi_1,\psi_2,\varphi} : H(p, q, \phi) \to H^\infty_\mu$ is compact if and only if $T_{\psi_1,\psi_2,\varphi} : H(p, q, \phi) \to H^\infty_\mu$ is bounded and for any bounded sequence f_k in $H(p, q, \phi)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\|T_{\psi_1,\psi_2,\varphi} f_k\|_{H^\infty_\mu} \to 0$ as $k \to \infty$.*

Lemma 2.3. *A closed set K in $H^\infty_{\mu,0}$ is compact if and only if K is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z)|f(z)| = 0.$$

3. Main Results and Proofs

Theorem 3.1. Assume that $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $T_{\psi_1,\psi_2,\varphi} : H(p, q, \phi) \to H^\infty_\mu$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{1/q}} < \infty, \tag{1}$$

and

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_2(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{1/q+1}} < \infty. \tag{2}$$

Proof. Suppose that $T_{\psi_1,\psi_2,\varphi} : H(p, q, \phi) \to H^\infty_\mu$ is bounded, i.e., there exists a constant C such that $\|T_{\psi_1,\psi_2,\varphi} f\|_{H^\infty_\mu} \le C\|f\|_{p,q,\phi}$. For a fixed $w \in \mathbb{D}$, set

$$f_w(z) = \frac{(1 - |w|^2)^{b+1}}{\phi(|w|)} \Big(\frac{1}{(1 - \overline{w}z)^\alpha} - \frac{2\alpha(1 - |w|^2)}{(\alpha + 1)(1 - \overline{w}z)^{\alpha+1}} + \frac{\alpha(1 - |w|^2)^2}{(\alpha + 2)(1 - \overline{w}z)^{\alpha+2}}\Big),$$

where the constant b is from the definition of the normality of the function ϕ and $\alpha = 1/q + b + 1$. A straightforward calculation show that

$$\begin{aligned} f'_w(z) &= \frac{(1-|w|^2)^{b+1}\overline{w}}{\phi(|w|)}\Big(\frac{\alpha}{(1-\overline{w}z)^{\alpha+1}} - \frac{2\alpha(1-|w|^2)}{(1-\overline{w}z)^{\alpha+2}} + \frac{\alpha(1-|w|^2)^2}{(1-\overline{w}z)^{\alpha+3}}\Big),\\ f_w(w) &= \frac{2}{(\alpha+1)(\alpha+2)\phi(|w|)(1-|w|^2)^{1/q}},\end{aligned}$$

$$f'_w(w) = 0,$$

and $\sup_{w\in\mathbb{D}} \|f_w\|_{p,q,\phi} \le C$ (see [24, 26]). Hence,

$$\begin{aligned} C &\ge \|T_{\psi_1,\psi_2,\varphi} f_{\varphi(w)}\|_{H^\infty_\mu}\\ &\ge \mu(w)|\psi_1(w) f_{\varphi(w)}(\varphi(w)) + \psi_2(w) f'_{\varphi(w)}(\varphi(w))|\\ &= \mu(w)|\psi_1(w)|\frac{2}{(\alpha+1)(\alpha+2)\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q}} \quad ,\end{aligned}$$

for every $w \in \mathbb{D}$. Therefore

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} < \infty.$$

For a fixed $w \in \mathbb{D}$. Set

$$g_w(z) = \frac{(1-|w|^2)^{t+1}}{\phi(|w|)(1-\overline{w}z)^{1/q+t+1}},$$

where the constant t is from the definition of the normality of the function ϕ. A straightforward calculation show that

$$\begin{aligned} g'_w(z) &= \frac{(t+1+1/q)(1-|w|^2)^{t+1}\overline{w}}{\phi(|w|)(1-\overline{w}z)^{1/q+t+2}},\\ g_w(w) &= \frac{1}{\phi(|w|)(1-|w|^2)^{1/q}},\end{aligned}$$

and $\sup_{w\in\mathbb{D}} \|g_w\|_{p,q,\phi} \le C$ (see [13]). Hence,

$$\begin{aligned} C &\ge \|T_{\psi_1,\psi_2,\varphi} g_{\varphi(w)}\|_{H^\infty_\mu}\\ &\ge \mu(w)|\psi_1(w) g_{\varphi(w)}(\varphi(w)) + \psi_2(w) g'_{\varphi(w)}(\varphi(w))|\\ &\ge \mu(w)|\psi_2(w)||g'_{\varphi(w)}(\varphi(w))| - \mu(w)|\psi_1(w)||g_{\varphi(w)}(\varphi(w))|\\ &= \mu(w)|\psi_2(w)|\frac{(t+1+1/q)(1-|\varphi(w)|^2)^{t+1}|\varphi(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+t+2}}\\ &\quad -\mu(w)|\psi_1(w)|\frac{1}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q}}\\ &= \frac{(t+1+1/q)\mu(w)|\psi_2(w)||\varphi(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+1}} - \frac{\mu(w)|\psi_1(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q}}\end{aligned}$$

for every $w \in \mathbb{D}$. Therefore,

$$\frac{(t+1+1/q)\mu(w)|\psi_2(w)||\varphi(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q+1}} \le C + \frac{\mu(w)|\psi_1(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/q}}.$$

From (1), we get

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|\psi_2(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}} < \infty. \tag{3}$$

From (3), we have

$$\sup_{|\varphi(z)|>\frac{1}{2}} \frac{\mu(z)|\psi_2(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}} \le \sup_{|\varphi(z)|>\frac{1}{2}} \frac{2\mu(z)|\psi_2(z)||\varphi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}} < \infty. \tag{4}$$

Since $f(z)=1, g(z)=z \in H(p,q,\phi)$, it follows that

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)| \le \|T_{\psi_1,\psi_2,\varphi} f\|_{H^\infty_\mu} \le C$$

and

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)\varphi(z)+\psi_2(z)| \le \|T_{\psi_1,\psi_2,\varphi} g\|_{H^\infty_\mu} \le C.$$

It is easy to see that

$$\mu(w)|\psi_2(w)| \le \|T_{\psi_1,\psi_2,\varphi} g\|_{H^\infty_\mu} + \mu(w)|\psi_1(w)\varphi(w)| \le C$$

for every $w \in \mathbb{D}$. From this and the fact ϕ is normal we obtain

$$\sup_{|\varphi(z)|\le\frac{1}{2}} \frac{\mu(z)|\psi_2(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}} \le C \sup_{|\varphi(z)|\le\frac{1}{2}} \mu(z)|\psi_2(z)| < \infty. \tag{5}$$

Combining (4) and (5), we get (2) as desired.

For the converse, suppose that (1) and (2) hold. For any $f \in H(p,q,\phi)$, by Lemma 2.1, we have

$$\begin{aligned} &\mu(z)|\psi_1(z)f(\varphi(z))+\psi_2(z)f'(\varphi(z))| \\ &\le \frac{\mu(z)|\psi_1(z)|\|f\|_{p,q,\phi}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} + \frac{\mu(z)|\psi_2(z)|\|f\|_{p,q,\phi}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}}. \end{aligned}$$

Therefore, $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded. The proof of the theorem is complete.

Theorem 3.2. Assume that $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is compact if and only if $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded,

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} = 0, \tag{6}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_2(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}} = 0. \tag{7}$$

Proof. Suppose that $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_\mu^\infty$ is compact. Then let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. We can use the test functions in Theorem 3.2. Let

$$f_k(z) = f_{\varphi(z_k)}(z).$$

We have

$$f_k(\varphi(z_k)) = \frac{2}{(\alpha+1)(\alpha+2)} \frac{1}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q}},$$

$f_k'(\varphi(z_k)) = 0$ and $\sup\limits_{k\in\mathbb{N}} \|f_k\|_{p,q,\phi} \le C$. For $|z| = r < 1$, using the fact that ϕ is normal, we have

$$|f_k(z)| \le \frac{C}{(1-r)^{1/q+1}}(1-|\varphi(z_k)|) \to 0 \quad (k\to\infty),$$

that is, f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, using the compactness of $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_\mu^\infty$ and Lemma 2.2, we obtain

$$\mu(z_k)|\psi_1(z_k)| \frac{2}{(\alpha+1)(\alpha+2)} \frac{1}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q}} \le \|T_{\psi_1,\psi_2,\varphi} f_k\|_{H_\mu^\infty} \to 0,$$

as $k \to \infty$. From this, and $|\varphi(z_k)| \to 1$, it follows that

$$\lim_{k\to\infty} \frac{\mu(z_k)|\psi_1(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q}} = 0,$$

and consequently (6) holds.

In order to prove (7), choose

$$g_k(z) = g_{\varphi(z_k)}(z).$$

We have

$$\begin{aligned} g_k(\varphi(z_k)) &= \frac{1}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q}}, \\ g_k'(\varphi(z_k)) &= \frac{(t+1+1/q)(1-|\varphi(z_k)|^2)^{t+1}\overline{\varphi(z_k)}}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{2+t+1/q}} \end{aligned}$$

and

$$\sup_{k\in\mathbb{N}} \|g_k\|_{p,q,\phi} \le C,$$

and g_k converges to 0 uniformly on compact subsets of $\mathbb{D}$. The lemma 2.2 implies that

$$\lim_{k\to\infty} \|T_{\psi_1,\psi_2,\varphi} g_k\|_{H^\infty_\mu} = 0.$$

It follows that

$$\begin{aligned} &\frac{(t+1+1/q)\mu(z_k)|\psi_2(z_k)||\varphi(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q+1}} \\ \le\ & \|T_{\psi_1,\psi_2,\varphi} g_k\|_{H^\infty_\mu} + \frac{\mu(z_k)|\psi_1(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q}} \\ \to\ & 0, \end{aligned}$$

as $k\to\infty$. From this, and $|\varphi(z_k)|\to 1$, it follows that

$$\lim_{k\to\infty} \frac{\mu(z_k)|\psi_2(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{1/q+1}} = 0,$$

and consequently (7) holds.

Conversely, assume that $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H^\infty_\mu$ is bounded and the conditions (6) and (7) hold. For any bounded sequence $\{f_k\}$ in $H(p,q,\phi)$ with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. To establish the assertion, it suffices, in view of Lemma 2.2, to show that $\|T_{\psi_1,\psi_2,\varphi} f_k\|_{H^\infty_\mu} \to 0$, as $k\to\infty$. We assume that $\|f_k\|_{p,q,\phi} \le 1$. From (6) and (7), there exists a $\delta\in(0,1)$, when $\delta < |\varphi(z)| < 1$, we have

$$\frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} + \frac{\mu(z)|\psi_2(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}} < \varepsilon. \tag{8}$$

From the proof of Theorem 3.1, we see that

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)| \le C.$$

and

$$\sup_{z\in\mathbb{D}} \mu(z)|\psi_2(z)| \le C.$$

Since $f_k\to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate gives that f'_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, there exists a $K_0\in\mathbb{N}$ such that $k > K_0$ implies that

$$\sup_{|\varphi(z)|\le\delta} \mu(z)|\psi_1(z) f_k(\varphi(z))| + \sup_{|\varphi(z)|\le\delta} \mu(z)|\psi_2(z) f'_k(\varphi(z))| < C\varepsilon. \tag{9}$$

From (8), (9) and Lemma 2.1, we have

$$\begin{aligned} \|T_{\psi_1,\psi_2,\varphi} f_k\|_{H^\infty_\mu} &= \sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z) f_k(\varphi(z)) + \psi_2(z) f'_k(\varphi(z))| \\ &\le \sup_{|\varphi(z)|\le\delta} \mu(z)|\psi_1(z) f_k(\varphi(z))| + \sup_{|\varphi(z)|\le\delta} \mu(z)|\psi_2(z) f'_k(\varphi(z))| \\ &\quad + \sup_{|\varphi(z)|>\delta} \Big(\frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} + \frac{\mu(z)|\psi_2(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}}\Big) \\ &< (C+1)\varepsilon, \end{aligned}$$

when $k > K_0$. By using Lemma 3.2, it follows that the operator $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_\mu^\infty$ is compact.

Theorem 3.3. Assume that $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is bounded if and only if $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_\mu^\infty$ is bounded,

$$\lim_{|z|\to 1} \mu(z)|\psi_1(z)| = 0, \tag{10}$$

and

$$\lim_{|z|\to 1} \mu(z)|\psi_2(z)| = 0, \tag{11}$$

Proof. Suppose that $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is bounded. Then it is clear that $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_\mu^\infty$ is bounded.

Taking the functions $f(z) = 1$ and $f(z) = z$, respectively, we obtain

$$\lim_{|z|\to 1} \mu(z)|\psi_1(z)| = 0$$

and

$$\lim_{|z|\to 1} \mu(z)|\psi_1(z)\varphi(z) + \psi_2(z)| = 0.$$

Since

$$\begin{aligned} \mu(z)|\psi_1(z)\varphi(z) + \psi_2(z)| &\geq \mu(z)|\psi_2(z)| - \mu(z)|\psi_1(z)\varphi(z)|, \\ \mu(z)|\psi_2(z)| &\leq \mu(z)|\psi_1(z)\varphi(z)| + \mu(z)|\psi_1(z)\varphi(z) + \psi_2(z)|, \end{aligned}$$

we get

$$\lim_{|z|\to 1} \mu(z)|\psi_2(z)| = 0.$$

Conversely, assume that $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_\mu^\infty$ is bounded and the conditions (10), (11) hold. For each polynomial $p(z)$, we get

$$\mu(z)|(T_{\psi_1,\psi_2,\varphi})p(z)| = \mu(z)|\psi_1(z)p(\varphi(z)) + \psi_2(z)p'(\varphi(z))|. \tag{12}$$

Since $\sup_{z\in\mathbb{D}} p(\varphi(z)) < \infty$ and $\sup_{z\in\mathbb{D}} p'(\varphi(z)) < \infty$, from (12) it follows that $T_{\psi_1,\psi_2,\varphi} p \in H_{\mu,0}^\infty$. From the set of all polynomials is dense in $H(p,q,\phi)$, we have that for every $f \in H(p,q,\phi)$, there is a sequence of polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that $\|f - p_k\|_{p,q,\phi} \to 0$ as $k \to \infty$. Hence

$$\|T_{\psi_1,\psi_2,\varphi} f - T_{\psi_1,\psi_2,\varphi} p_k\|_{H_\mu^\infty} \leq \|T_{\psi_1,\psi_2,\varphi}\| \cdot \|f - p_k\|_{p,q,\phi} \to 0$$

as $k \to \infty$, by using the boundedness of the operator $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_\mu^\infty$. Since $H_{\mu,0}^\infty$ is a closed subset of H_μ^∞, we obtain $T_{\psi_1,\psi_2,\varphi}(H(p,q,\phi)) \subset H_{\mu,0}^\infty$. Therefore $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is bounded.

Theorem 3.4. Assume that $\psi_1, \psi_2 \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is compact if and only if

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} = 0, \tag{13}$$

and

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}} = 0. \tag{14}$$

Proof. Assume that conditions (13) and (14) hold. Then it is clear that (1) and (2) hold. Hence $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_\mu^\infty$ is bounded by Theorem 3.1. Since

$$\begin{aligned} &\mu(z)|T_{\psi_1,\psi_2,\varphi} f(z)| \\ =\ &\mu(z)|\psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z))| \\ \le\ &\frac{\mu(z)|\psi_1(z)|\|f\|_{p,q,\phi}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} + \frac{\mu(z)|\psi_2(z)|\|f\|_{p,q,\phi}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+1}}, \end{aligned}$$

Taking the supremum in above inequality over all $f \in H(p,q,\phi)$ such that $\|f\|_{p,q,\phi} \le 1$ and letting $|z| \to 1$, yields

$$\lim_{|z|\to 1} \sup_{\|f\|_{p,q,\phi}\le 1} \mu(z)|T_{\psi_1,\psi_2,\varphi} f(z)| = 0.$$

Hence, by Lemma 2.3 we see that the operator $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is compact.

Now assume that $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is compact. Then $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is bounded, and by taking the function $f(z) = 1$, it follows that

$$\sup_{|z|\to 1} \mu(z)|\psi_1(z)| = 0. \tag{16}$$

Since $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_{\mu,0}^\infty$ is compact, then $T_{\psi_1,\psi_2,\varphi} : H(p,q,\phi) \to H_\mu^\infty$ is compact, by Theorem 3.2 we have

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} = 0.$$

It follows that for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} < \varepsilon, \tag{17}$$

when $\delta < |\varphi(z)| < 1$. Using (16) we see that there exists $\tau \in (0,1)$ such that

$$\mu(z)|\psi_1(z)| < \varepsilon \inf_{t\in[0,\delta]} \phi(t)(1-t^2)^{1/q}, \tag{18}$$

when $\tau < |z| < 1$.

Therefore, when $\tau < |z| < 1$ and $\delta < |\varphi(z)| < 1$, by (17) we have

$$\frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} < \varepsilon, \tag{19}$$

On the other hand, when $\delta < |\varphi(z)| < 1$ and $|\varphi(z)| \le \delta$, by (18) we obtain

$$\frac{\mu(z)|\psi_1(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q}} \le \frac{\mu(z)|\psi_1(z)|}{\inf\limits_{t\in[0,\delta]} \phi(t)(1-t^2)^{1/q}} < \varepsilon. \tag{20}$$

From (19) and (20), we obtain (13), as desired. Similarly, the result (14) holds. This completes the proof of the theorem.

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