TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 6, pp. 1827-1839, December 2014
DOI: 10.11650/tjm.18.2014.4266
This paper is available online at http://journal.taiwanmathsoc.org.tw

THE INVARIANCE OF DOMAIN FOR k-SET-PSEUDO-CONTRACTIVE OPERATORS IN BANACH SPACES

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Abstract. We introduce a new family of nonlinear operators called k-set-pseudo-contractions where several well-known mappings, such as, the condensing mappings (for $k = 1$) and the compact perturbations of k-pseudo-contractive mappings are embraced in the class of k-set-pseudo-contractions. We prove an invariance of domain theorem and (as a consequence) a fixed point theorem for a k-set-pseudo-contraction ($0 < k < 1$) which is also an L-set-contraction ($L \geq 0$). Several well known results can be deduced from our theorems.

1. Introduction

Let X be a metric space and let $\mathcal{B}(X)$ denote the family of bounded subsets of X. The *Kuratowski* [12] measure of noncompactness of $A \in \mathcal{B}(X)$ is defined by

$$\gamma(A) = \inf\{\epsilon > 0 : A \text{ can be covered by finitely many sets with diameter } \leq \epsilon\}.$$

It is clear that γ maps $\mathcal{B}(X)$ into $[0, \infty)$. The mapping γ satisfies the following basic properties (see for examples, [17, 1, 2]), which will be needed in the sequel. For all $A, A_1, A_2 \in \mathcal{B}(X)$,

(a) *Regularity*: $\gamma(A) = 0 \Leftrightarrow A$ is precompact.
(b) *Invariance under closure*: $\gamma(\overline{A}) = \gamma(A)$.
(c) *Semi-additivity*: $\gamma(A_1 \bigcup A_2) = \max\{\gamma(A_1), \gamma(A_2)\}$.

Furthermore, if X is a Banach space then γ also satisfies the following.

Received December 31, 2013, accepted March 19, 2014.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: Primary 47H08, 47H10, 47H04; Secondary 47H09, 54C60, 47H06.
Key words and phrases: k-Set-pseudo-contractions, L-Set-contractions, k-Pseudo-contractions, Measures of noncompactness.
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(d) *Semi-homogeneity*: $\gamma(tA) = |t|\gamma(A)$ for any real number t.

(e) *Algebraic sub-additivity*: $\gamma(A_1 + A_2) \leq \gamma(A_1) + \gamma(A_2)$.

(f) *Invariance on convex hull*: $\gamma(co(A)) = \gamma(A)$, (where $co(A)$ denotes the convex hull of A).

Let X^* denote the dual space of a normed space X. For each $x \in X$, the *normalized duality mapping* $J : X \longrightarrow 2^{X^*}$ is defined by

$$J(x) := \{j \in X^* : \langle x, j\rangle = \|x\|^2,\ \|j\| = \|x\|\}.$$

Definition 1.1. Let X be a Banach space. An operator $T : D \subseteq X \longrightarrow X$ is called *strongly pseudo-contractive* (see for example, [3]) if there exists $t > 1$ such that

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{1.1}$$

for all $x, y \in D$ and $r > 0$. If $t = 1$, the operator T is said to be *pseudo-contractive*.

The mapping T is said to be *strictly pseudo-contractive* (in the sense of Browder and Petryshyn [4]) if and only if for each pair $x, y \in D$ there exist $\alpha > 0,\ j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \alpha\|(I - T)x - (I - T)y\|^2.$$

By a characterization of Kato [10], T is strongly pseudo-contractive if and only if for each $x, y \in D$, there exists some $j(x - y) \in J(x - y)$ and a number $t > 1$ such that $\langle Tx - Ty, j(x - y)\rangle \leq t^{-1}\|x - y\|^2$ (where t is as in Definition 1.1), and pseudo-contractive if and only if for each $x, y \in D$, there exists some $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2$. Therefore, it is clear that the strict pseudo-contractive mappings form a subclass of the class of Lipschitz pseudo-contractive mappings.

Definition 1.2. An operator $T : D \subseteq X \longrightarrow X$ is said to be *k-pseudo-contractive* with $k > 0$ (see for instance, [13]) if for each pair $x, y \in D$ and $\lambda > k$,

$$(\lambda - k)\|x - y\| \leq \|\lambda x - Tx - (\lambda y - Ty)\|, \tag{1.2}$$

and T is called *pseudo-contractive* if $k = 1$.

For $k \leq 1$, it is easy to verify that Definitions 1.1 and 1.2 are equivalent. We use the latter formulation to establish the multi-valued version of this concept.

Definition 1.3. A *multivalued* mapping $T : D \subseteq X \longrightarrow 2^X$ is said to be k-*pseudo-contractive* with $k > 0$ (see for instance, [14]) if for each $x, y \in D, u \in Tx, v \in Ty$ and $\lambda > k$,

$$(\lambda - k)\|x - y\| \leq \|(\lambda x - u) - (\lambda y - v)\|. \tag{1.3}$$

Now, we extend the definition of a k-pseudo-contractive mapping ($k > 0$) to the more general notion of what we call a k-set-pseudo-contractive mapping. We shall show below why the new definition is definitely more general.

Definition 1.4. An operator $T : D \subseteq X \longrightarrow 2^X$ is said to be *k-set-pseudo-contractive* if for each bounded subset $A \subseteq D$ for which $T(A)$ is bounded, we have

$$(\lambda - k)\gamma(A) \leq \gamma((\lambda I - T)(A)) \quad \text{for} \quad \lambda > k. \tag{1.4}$$

If $k = 1$, then T is called a 1-set-pseudo-contractive mapping.

Using property (d) of the mapping γ, it can be shown that inequality (1.4) is equivalent to the following :

$$\gamma(A) \leq \gamma((1+r)I - rk^{-1}T)(A)) \tag{1.5}$$

for all $r > 0$. As a matter of fact, from either definition, it can be easily derived that if T is a 1-set-pseudo-contraction, then kT is a k-set-pseudo-contraction.

The fact that many properties of k-contractions have been shown to carry over to k-pseudo-contractions, opens many interesting questions concerning the extension to k-set-pseudo-contractions. As a matter of fact, we address the extension of some of these properties to this new family of operators, including non-trivial examples that justify the generality of this family of operators. The motivation stems from the remark made by Gatica and Kirk concerning Theorem 1 in their paper [8]. They claim that if an operator T satisfies the inequality (1.5) with $k = 1$, Theorem 1 would still hold true. However, this claim had the need for a new open mapping theorem that was not addressed in [8].

Determining whether or not these new type of operators have fixed points under standard additional assumptions constitutes a main objective of this paper. Attaining such a goal requires two fundamental results. One, an invariance of domain theorem for the mapping $I - T$, and, two, whether $(I - T)(C)$ is closed whenever C is a closed set. It turns out that, the first result (see Theorem 3.1) holds under an additional condition, and the second result (see Proposition 2.1) holds with no extra assumptions. As a consequence of these facts, we prove a fixed point theorem under the weaker Leray-Schauder boundary condition introduced earlier by Kirk and Morales [11] as opposed to the standard Leray-Schauder condition amply used by many authors.

We shall denote by $\overline{D}$ and ∂D the closure, and the boundary of D, respectively. Also, for $x, y \in X$, we denote by $[x, y]$ the set $\{(1-t)x + ty : t \in [0, 1]\}$.

2. Preliminaries

A continuous mapping $T : D \subseteq X \longrightarrow X$ is called *L-set-contractive* [5] (with $L \geq 0$) if for each $A \in \mathcal{B}(D)$ for which $T(A) \in \mathcal{B}(X)$, we have the inequality

$$\gamma(T(A)) \leq L\gamma(A).$$

Similarly, a continuous mapping $T : D \subseteq X \longrightarrow X$ is called *condensing (or densifying)* (see [15, 6]) if $\gamma(T(A)) < \gamma(A)$, whenever $\gamma(A) > 0$.

Proposition 2.1. *Let D be a bounded subset of a Banach space X and let $T : D \to X$ be a 1-set-pseudo-contraction. Suppose $y_n = x_n - t_n T x_n$ with $x_n \in D$, such that $y_n \to y$, while $t_n \to t \in [0, 1)$. Then there exists a convergent subsequence of $\{x_n\}$. In addition, for D closed and T continuous, $(I - tT)(D)$ is closed for each $t \in [0, 1)$.*

Proof. We first observe that as a direct consequence of Definition 1.4, we have

$$(\lambda - t)\gamma(A) \leq \gamma((\lambda I - tT)(A)), \ \ (\lambda > 1)$$

for $A \subseteq D$ and $T(A) \in \mathcal{B}(X)$. Then

$$\begin{aligned}(\lambda - t)\gamma(\{x_n\}) &\leq \gamma((\lambda I - tT)(\{x_n\})) \\ &\leq (\lambda - 1)\gamma(\{x_n\}) + \gamma((t_n - t)T\{x_n\}) \\ &\leq (\lambda - 1)\gamma(\{x_n\}) + |t_n - t|\gamma(T(D)),\end{aligned}$$

which implies that $\gamma(\{x_n\}) = 0$. Thus, there exists a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x \in X$. But D closed, $x \in D$ and the continuity of T imply that $y = x - tTx \in (I - tT)(D)$, which completes the proof. ■

We state a well-known invariance of domain theorem obtained by Nussbaum [16], which will be used in an invariance of domain theorem for this new family of k-set-pseudo-contractive mappings.

Theorem N. *Let G be an open subset of a Banach space X and let $T : G \to X$ be a condensing mapping such that $I - T$ is one-to-one. Then $(I - T)(G)$ is open.*

2.1. Properties of set-pseudo-contractive mappings

The following propositions show that the new class of k-set-pseudo-contractive mappings is quite ample and includes well-known family of operators.

Proposition 2.2. *Let X be a Banach space and let $T : D \subseteq X \longrightarrow X$ be a k-set-contraction $(0 < k \leq 1)$. Then T is a k-set-pseudo-contraction.*

Proof. Let $A \in \mathcal{B}(D)$, let T be a k-set-contraction and let $\lambda > k$. Applying the properties of γ, we have that

$$\begin{aligned}\lambda\gamma(A) &= \gamma(\lambda A) = \gamma[(\lambda I - T + T)(A)] \\ &\leq \gamma[(\lambda I - T)(A) + T(A)] \leq \gamma((\lambda I - T)(A)) + \gamma(T(A)) \\ &\leq \gamma((\lambda I - T)(A)) + k\gamma(A).\end{aligned}$$

Thus $(\lambda - k)\gamma(A) \leq \gamma((\lambda I - T)(A))$. Hence, T is a k-set-pseudo-contraction. ∎

It is known that every non-expansive mapping is a 1-set-contraction. In this case, we shall show that every k-pseudo-contraction is a k-set-pseudo-contraction; in particular, we derive that every pseudo-contraction is a 1-set-pseudo-contraction. We continue our journey by proving that the sum of k_1-pseudo-contractive mapping and a k_2-set-contractive mapping is $(k_1 + k_2)$-set-pseudo-contractive. In particular, it follows that a compact perturbation of a k-pseudo-contractive mapping is also k-set-pseudo-contractive.

Proposition 2.3. *Let X be a Banach space, let $T : D \subseteq X \longrightarrow X$ be a k_1-pseudo-contractive mapping and let $f : D \longrightarrow X$ be a k_2-set-contractive mapping. Then $T + f$ is $(k_1 + k_2)$-set-pseudo-contractive.*

Proof. Let T be a k_1-pseudo-contractive mapping $(k_1 > 0)$ and suppose that $A \in \mathcal{B}(D)$ while $T(A) \in \mathcal{B}(X)$. Let $k = k_1 + k_2$ and let $\lambda > k$. Set $T_\lambda := (\lambda - k_1)^{-1}(\lambda I - T)$. Then T_λ is one-to-one and T_λ^{-1} is a non-expansive mapping on its domain. Let $C = T_\lambda(A)$. Then

$$\begin{aligned}\gamma(A) &= \gamma(T_\lambda^{-1}(C)) \leq \gamma(C)\\ &= \gamma(T_\lambda(A)) = (\lambda - k_1)^{-1}\gamma((\lambda I - T)(A)),\end{aligned}$$

and it follows that $(\lambda - k_1)\gamma(A) \leq \gamma((\lambda I - T)(A))$. Since f is k_2-set-contractive, we have

$$\begin{aligned}(\lambda - k_1)\gamma(A) &\leq \gamma((\lambda I - T)(A))\\ &= \gamma((\lambda I - T - f + f)(A))\\ &\leq \gamma((\lambda I - T - f)(A)) + \gamma(f(A))\\ &\leq \gamma((\lambda I - (T + f))(A)) + k_2\gamma(A).\end{aligned}$$

This yields that $(\lambda - k)\gamma(A) \leq \gamma((\lambda I - (T + f))(A))$. Hence, $T + f$ is a k-set-pseudo-contraction. ∎

Since compact operators are obviously 0-set-contractions, we derive from Proposition 2.3, the following.

Corollary 2.4. *Let X be a Banach space and let $T : D \subseteq X \longrightarrow X$ be a k-pseudo-contractive mapping $(k > 0)$ and let $f : D \longrightarrow X$ be a compact mapping. Then $T + f$ is a k-set-pseudo-contraction.*

Corollary 2.5. *Let X be a Banach space and let $T : D \subseteq X \longrightarrow X$ be a k-pseudo-contractive mapping $(k > 0)$. Then T is a k-set-pseudo-contraction.*

Corollary 2.6. *Let X be a Banach space and let $T : D \subseteq X \longrightarrow X$ be a strictly pseudo-contractive mapping. Then T is a 1-set-pseudo-contraction which is also an L-set-contraction.*

Proof. The L-set-contraction follows from the fact that every strictly pseudo-contractive mapping is Lipschitz. ∎

The next proposition shows that the class of k-set-pseudo-contractions is larger than expected. It shows that every multivalued k-pseudo-contractive mapping (as in Definition 1.3) is a k-set-pseudo-contraction.

Proposition 2.7. *Let X be a Banach space and let $T : D \subseteq X \longrightarrow 2^X$ be a multivalued k-pseudo-contractive mapping $(k > 0)$ and let $f : D \longrightarrow X$ be a compact mapping. Then $T + f$ is a k-set-pseudo-contraction.*

Proof. We shall first prove that a multivalued k-pseudo-contractive mapping is a k-set-pseudo-contraction. Let $\lambda > k$. Let K be a bounded subset of D and suppose that $(\lambda I - T)(K)$ is bounded. Given $\epsilon > 0$, let $\{V_i\}_{i\in I} \subset 2^X$, where I is some finite index set, such that

$$(\lambda I - T)(K) \subseteq \bigcup_{i\in I} V_i \text{ and } \operatorname{diam} V_i < \gamma((\lambda I - T)(K)) + \epsilon \text{ for all } i \in I.$$

Then

$$\begin{aligned}
&\gamma((\lambda I - T)(K)) + \epsilon \\
&> \max_{i\in I} \operatorname{diam} V_i \\
&\geq \max_{i\in I} \operatorname{diam} [V_i \cap (\lambda I - T)(K)] \\
&= \max_{i\in I} \sup\{\|(\lambda x - u) - (\lambda y - v)\| : \\
&\quad u \in Tx, v \in Ty; \lambda x - u, \lambda y - v \in V_i \cap (\lambda I - T)(K)\} \\
&\geq \max_{i\in I} \sup\{(\lambda - k)\|x - y\| : \\
&\quad u \in Tx, v \in Ty; \lambda x - u, \lambda y - v \in V_i \cap (\lambda I - T)(K)\} \\
&= (\lambda - k) \max_{i\in I} \operatorname{diam} [K \cap (\lambda I - T)^{-1}(V_i)] \\
&\geq (\lambda - k) \max_{i\in I} \gamma[K \cap (\lambda I - T)^{-1}(V_i)] \\
&= (\lambda - k)\gamma[K \cap \Big(\bigcup_{i\in I} (\lambda I - T)^{-1}(V_i)\Big)] \\
&= (\lambda - k)\gamma(K) \quad \text{(since } K \subseteq \bigcup_{i\in I} (\lambda I - T)^{-1}(V_i)).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$(\lambda - k)\gamma(K) \leq \gamma((\lambda I - T)(K)),$$

proving that a multivalued k-pseudo-contractive mapping (defined as in Definition 1.3) is a k-set-pseudo-contraction. Next, we prove that a compact perturbation of a multivalued k-pseudo-contractive mapping is a k-set-pseudo-contraction.

Let $f : D \longrightarrow X$ be a compact map. If $\lambda x - u \in (\lambda I - T)(K)$ with $u \in Tx$, then

$$\lambda x - u = (\lambda x - u - f(x)) + f(x) \in (\lambda I - T - f)(K) + f(K),$$

so that

$$(\lambda I - T)(K) \subseteq (\lambda I - T - f)(K) + f(K).$$

Therefore,

$$\begin{aligned}(\lambda - k)\gamma(K) &\leq \gamma((\lambda I - T)(K)) \leq \gamma[(\lambda I - T - f)(K) + f(K)]\\ &\leq \gamma[(\lambda I - T - f)(K)] + \gamma(f(K))\\ &= \gamma[(\lambda I - T - f)(K)].\end{aligned}$$

Hence, $T + f$ is also a k-set-pseudo-contraction. ■

3. Main Results

Let X be a Banach space and let D be a subset of X with $0 \in D$. Following [11], we define $\mathcal{E}_D := \{\lambda > 1 : Tx = \lambda x \text{ for some } x \in D\}$. We prove an invariance of domain theorem and a fixed point theorem for k-set-pseudo-contractive mappings (where $0 < k < 1$), and derive as corollaries corresponding results for 1-set-pseudo-contractive mappings

In the sequel, we shall assume that $L \geq 1$, otherwise the results obtained in this work would be well-known.

3.1. An Invariance of domain theorem

Theorem 3.1. *Let G be an open subset of a Banach space X and let $T : G \longrightarrow X$ be a k-set-pseudo-contractive and L-set-contractive mapping such that $I - tT$ is one-to-one for all $t \in [0, 1]$. Then $(I - T)(G)$ is open.*

Proof. Let $T_t := I - tT$ and let $S = \{t \in [0, 1] : T_t(G) \text{ is open}\}$. Due to Theorem N, $[0, L^{-1}) \subset S$. We shall show now that $1 \in S$. To see this, we first prove that S is open in $[0, 1]$. Let $t \in S$ and denote by R_t the inverse of T_t.

Let $\tilde{A} \subseteq T_t(G)$ such that $R_t(\tilde{A})$ is bounded. Then there exists $A \subseteq G$ such that $\tilde{A} = T_t(A)$. As observed in the proof of Proposition 2.1, $(\lambda - kt)\gamma(A) \leq \gamma((\lambda I - tT)(A))$, for all $\lambda > k$. By choosing $\lambda > 1$, we have that

$$\begin{aligned}(\lambda - kt)\gamma(R_t(\tilde{A})) &= (\lambda - kt)\gamma(A) \le \gamma((\lambda I - tT)(A)) \\ &\le \gamma(T_t(A)) + (\lambda - 1)\gamma(A) \\ &= \gamma(\tilde{A}) + (\lambda - 1)\gamma(R_t(\tilde{A})).\end{aligned}$$

which implies that $(1 - kt)\gamma(R_t(\tilde{A})) \le \gamma(\tilde{A})$. Therefore R_t is a $(1 - kt)^{-1}$-set-contraction on $T_t(G)$. Next, choose $0 < \delta < \frac{1-k}{L}$ and let $s > 0$ such that $|t - s| < \delta$. Then for $w \in T_t(G)$, we have

$$T_s(R_t(w)) = T_t(R_t(w)) + (T_s - T_t)(R_t(w)) = w + U_s(w), \tag{3.1}$$

where $U_s := (T_s - T_t) \circ R_t$. Then $I + U_s$ is clearly a one-to-one mapping. In addition, since $T_s - T_t = (t-s)T$ is δL-set-contraction, and due to the choice of δ, we conclude that U_s is a condensing mapping on $T_t(G)$. Once again, by Theorem N, we derive that $I + U_s$ is an open mapping on $T_t(G)$. Thus, $T_s(G) = (I + U_s)(T_t(G))$ is an open set in X, and hence $[t, t+\delta) \subset J$, which implies that S is open.

To complete the proof, let $[0, t_0)$ be the largest interval contained in S. Then by the above argument, $t_0 \in S$, which completes the proof. Consequently $(I - T)(G)$ is open. ∎

We now derive the following as a consequence of Theorem 3.1.

Corollary 3.2. *Let G be an open subset of a Banach space X, $T : G \longrightarrow X$ be a 1-set-pseudo-contractive mapping which is also an L-set-contractive mapping such that $I - tT$ is one-to-one for all $t \in [0, 1)$. Then $I - tT$ is an open mapping.*

Proof. Since T is a 1-set-pseudo-contraction, tT is a t-set-pseudo-contraction, and hence by Theorem 3.1, $I - tT$ is an open mapping for $t \in [0, 1)$. ∎

Next, we show a new example that reflects that the class of k-set-pseudo-contractive mappings is a much larger class than the k-pseudo-contractive mappings.

Example 3.3. Let B denote the closed unit ball of a Banach space X, let a be a real constant satisfying $0 < a < 1$ and let $f : B \longrightarrow 2^B$ be a compact multivalued map. Define $T : B \longrightarrow 2^X$ by $Tx = f(x) + \left(1 - \dfrac{1}{\|x\| + a}\right)x$ for all $x \in B$. Then T is a k_0-set-pseudo-contractive mapping, where $k_0 = \frac{1+a+a^2}{(1+a)^2}$.

Proof. Let $\lambda > k_0$ and let K be a subset of B such that $\gamma(K) > 0$. Suppose that $(\lambda I - T)(K)$ is bounded. Given $\epsilon > 0$, let $\{V_i\}_{i \in I}$ be a finite collection of subsets of X such that

$$(\lambda I - T + f)(K) \subseteq \bigcup_{i \in I} V_i \text{ and } \operatorname{diam} V_i < \gamma((\lambda I - T + f)(K)) + \epsilon \text{ for all } i \in I.$$

$$
\begin{aligned}
&\gamma((\lambda I-T)(K))+\epsilon \geq \gamma((\lambda I-T)(K)+f(K))+\epsilon \geq \gamma((\lambda I-T+f)(K))+\epsilon \\
&> \max_{i\in I} \operatorname{diam} V_i \geq \max_{i\in I} \operatorname{diam}\, (V_i \cap [(\lambda I - T + f)(K)]) \\
&= \max_{i\in I} \sup\{\Big\|(\lambda-1)(x-y) + \frac{x}{\|x\|+a} - \frac{y}{\|y\|+a}\Big\| : \\
&\quad \lambda x - Tx + f(x), \lambda y - Ty + f(y) \subset V_i \cap (\lambda I - T + f)(K)\} \\
&= \max_{i\in I} \sup\{\Big\|\Big(\lambda - 1 + \frac{a}{(\|x\|+a)(\|y\|+a)}\Big)(x-y) + \frac{\|y\|x - \|x\|y}{(\|x\|+a)(\|y\|+a)}\Big\| : \\
&\quad \lambda x - Tx + f(x), \lambda y - Ty + f(y) \subset V_i \cap (\lambda I - T + f)(K)\} \\
&= \max_{i\in I} \sup\{\Big\|\Big(\lambda - 1 + \frac{a}{(\|x\|+a)(\|y\|+a)}\Big)(x-y) + \frac{\|y\|(x-y) + (\|y\|-\|x\|)y}{(\|x\|+a)(\|y\|+a)}\Big\| : \\
&\quad \lambda x - Tx + f(x), \lambda y - Ty + f(y) \subset V_i \cap (\lambda I - T + f)(K)\} \\
&\geq \max_{i\in I} \sup\{\Big(\lambda - 1 + \frac{\|y\|+a}{(\|x\|+a)(\|y\|+a)}\Big)\|x-y\| \\
&\quad -\frac{\|y\|}{(\|x\|+a)(\|y\|+a)} |\, \|y\| - \|x\|\, | : \\
&\quad \lambda x - Tx + f(x), \lambda y - Ty + f(y) \subset V_i \cap (\lambda I - T + f)(K)\} \\
&\geq \max_{i\in I} \sup\{\Big(\lambda - 1 + \frac{a}{(\|x\|+a)(\|y\|+a)}\Big)\|x-y\| : \\
&\quad \lambda x - Tx + f(x), \lambda y - Ty + f(y) \subset V_i \cap (\lambda I - T + f)(K)\} \\
&\geq \max_{i\in I} \sup\{\Big(\lambda - 1 + \frac{a}{(1+a)^2}\Big)\|x-y\| : \\
&\quad \lambda x - Tx + f(x), \lambda y - Ty + f(y) \subset V_i \cap (\lambda I - T + f)(K)\} \\
&= \Big(\lambda - k_0\Big) \max_{i\in I} \sup\{\|x-y\| : \\
&\quad \lambda x - Tx + f(x), \lambda y - Ty + f(y) \subset V_i \cap (\lambda I - T + f)(K)\} \\
&= \Big(\lambda - k_0\Big) \max_{i\in I} \operatorname{diam}\, [K \cap (\lambda I - T + f)^{-1}(V_i)] \\
&\geq \Big(\lambda - k_0\Big) \max_{i\in I} \gamma([K \cap (\lambda I - T + f)^{-1}(V_i)]) \\
&= \Big(\lambda - k_0\Big)\gamma(\bigcup_{i\in I}[K \cap (\lambda I - T + f)^{-1}(V_i)]) \\
&= \Big(\lambda - k_0\Big)\gamma(K \cap [\bigcup_{i\in I}(\lambda I - T + f)^{-1}(V_i)]) \\
&= \Big(\lambda - k_0\Big)\gamma(K) \text{ since } K \subseteq \bigcup_{i\in I}(\lambda I - T + f)^{-1}(V_i).
\end{aligned}
$$

Thus, $\gamma((\lambda I - T)(K)) \geq \Big(\lambda - k_0\Big)\gamma(K)$ since $\epsilon > 0$ is arbitrary. This proves that T is k_0-set-pseudo-contractive. ∎

Remark 3.4. We observe that the mapping T in Example 3.3 is neither compact nor k_0-pseudo-contractive.

We first show that T is not a compact mapping. Assume (by contradiction) that T were compact. Let $S = \{x \in B : \|x\| = 1\}$. Then $\gamma(S) = \gamma(\overline{co}S) = \gamma(B) > 0$. Now

$$\Big(1 - \frac{1}{1+a}\Big)S = \{\Big(1 - \frac{1}{\|x\|+a}\Big)x : x \in S\} = \bigcup_{x\in S}(Tx - f(x))$$

and

$$\begin{aligned} 0 < \Big(1 - \frac{1}{1+a}\Big)\gamma(S) &= \gamma(\bigcup_{x\in S}(Tx - f(x))) \\ &= \gamma((T-f)(S)) \leq \gamma(T(S)) + \gamma(f(S)) = 0, \end{aligned}$$

which is a contradiction. Hence T is not a compact mapping.

To see that T is not k_0-pseudo-contractive it suffices to show that T is not pseudo-contractive. To this end, fix $x_0 \in B$ with $\|x_0\| = \frac{1}{2}$, let $f : B \longrightarrow 2^B$ be a compact multivalued map defined by $f(x) = \Big\{\frac{3}{2}x_0, -\frac{3}{2}x_0\Big\}$ for all $x \in B$. Then the mapping T defined in Example 3.3 is given by

$$Tx = \Big\{\frac{3}{2}x_0, -\frac{3}{2}x_0\Big\} + \Big(1 - \frac{1}{\|x\|+a}\Big)x \text{ for all } x \in B.$$

Select $u = \frac{3}{2}x_0 + \Big(1 - \frac{1}{\|x_0\|+a}\Big)x_0 \in Tx_0,\ v = -\frac{3}{2}x_0 - \Big(1 - \frac{1}{\|x_0\|+a}\Big)x_0 \in T(-x_0)$. If we let $a = \frac{1}{3}$ and $\lambda = \frac{3}{2}$ then we have that

$$\|\lambda x_0 - u - (\lambda(-x_0) - v)\| = \frac{1}{5}\|x_0 - (-x_0)\| < \frac{1}{2}\|x_0 - (-x_0)\| = (\lambda - 1)\|x_0 - (-x_0)\|.$$

It follows from Definition 1.3 that T is not pseudo-contractive and thus it cannot be k_0-pseudo-contractive.

3.2. A fixed point theorem for k-set-pseudo-contractions

Theorem 3.5. *Let X be a Banach space, G be a bounded open subset of X with $0 \in G$. Let $T : \overline{G} \longrightarrow X$ be a k-set-pseudo-contractive and L-set-contractive mapping satisfying*

(*i*) $\lambda \in \mathcal{E}_{\partial G} \Rightarrow \mathcal{E}_{\overline{G}} \bigcap [1, \lambda) \neq \emptyset$;

(*ii*) $I - tT$ *is one-to-one for all* $t \in [0, 1)$.

Then T has a fixed point in $\overline{G}$.

Proof. Let $G_t = (I - tT)(G)$ for each $t \in [0,1]$ and let $S = \{t \in [0,1] : 0 \in (I - tT)(\overline{G})\}$. Then $S \neq \emptyset$ since $0 \in S$. In addition, due to Proposition 1 of [11], $[0, \frac{1}{L}) \subset S$. Therefore, $\alpha = \sup S > 0$. We prove now that $\alpha \in S$. Let $\{\alpha_n\} \subset S$ such that $\alpha_n \to \alpha$ as $n \to \infty$. Then, for each $n \in \mathbb{N}$, there exists $x_n \in \overline{G}$ such that $x_n - \alpha_n T x_n = 0$. Consequently, by Proposition 2.1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x \in \overline{G}$. Since T is continuous, it follows that $x - \alpha T x = 0$ and thus $\alpha \in S$. Suppose $\alpha < 1$ and $x \in \partial G$. Then, there exist $\mu \in \mathcal{E}_{\overline{G}} \bigcap [1, \alpha^{-1})$ and $x^* \in \overline{G}$ so that $Tx^* = \mu x^*$. This would imply that $\mu^{-1} \in S$, which is a contradiction! Therefore $x \in G$.

To complete the proof, let $t_n \in (\alpha, 1)$ such that $t_n \to \alpha$ as $n \to \infty$. Then clearly $0 \notin (I - t_n T)(\overline{G})$ while $x - t_n T x \in G_{t_n}$. Since, by Theorems 3.1 and Proposition 2.1, we know that $\partial G_{t_n} = (I - t_n T)(\partial G)$, we may choose $z_n \in [0, x - t_n T x] \bigcap \partial G_{t_n}$ for each $n \in \mathbb{N}$. This and the fact $x - \alpha T x = 0$, imply

$$\begin{aligned} \|z_n\| &\leq \|x - t_n T x\| = \|x - t_n T x - x + \alpha T x\| \\ &\leq (t_n - \alpha)\|Tx\| \to 0, \end{aligned}$$

as $n \to \infty$ and $z_n = u_n - t_n T u_n$ for some $u_n \in \partial G$. Once again, by Proposition 2.1, we may extract a convergent subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which converges to some $u \in \partial G$. The continuity of T, leads to $u - \alpha T u = 0$. However, by the Boundary Condition (i), there exist $\mu \in \mathcal{E}_{\overline{G}} \bigcap [1, \alpha^{-1})$ and $u^* \in \overline{G}$ such that $Tu^* = \mu u^*$, which would imply that $\alpha < \mu^{-1} \in S$, a contradiction!, since $\alpha = \sup S$. Hence, $\alpha = 1$ and the proof is complete. ∎

Corollary 3.6. *Let X be a Banach space and let K be a bounded convex closed subset of X with $0 \in int(K)$. Let $T : K \longrightarrow K$ be a k-set-pseudo-contractive and L-set-contractive mapping satisfying*

$$I - tT \text{ is one-to-one for all } t \in [0, 1).$$

Then T has a fixed point in K.

We should observe that Condition (i) of Theorem 3.5 is the weaker Leray-Schauder boundary condition introduced by Kirk and Morales [11]. However, the classical Leray-Schauder boundary condition (see [9]),

$$Tx \neq \lambda x \text{ for } x \in \partial G, \ \lambda > 1,$$

is equivalent to the vacuous case of condition (i). It is shown in Theorem 1 of [13] that if $T : \overline{G} \longrightarrow X$ is a continuous strongly pseudo-contractive mapping and $0 \in G$, then Condition (i) of Theorem 3.5 is sufficient to guarantee a fixed point of T in $\overline{G}$. Also observe that Condition (ii) of Theorem 3.5 holds trivially for k-pseudo-contractive mappings.

ACKNOWLEDGMENTS

The second author is grateful to the Department of Mathematics/Statistics, Akwa-Ibom State University, Nigeria for the invitation extended him.

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