TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 4, pp. 1283-1291, August 2014
DOI: 10.11650/tjm.18.2014.4198
This paper is available online at http://journal.taiwanmathsoc.org.tw

THE MEAN MINKOWSKI MEASURES FOR CONVEX BODIES OF CONSTANT WIDTH

HaiLin Jin and Qi Guo

Abstract. In this paper, we study the so-called mean Minkowski measures, proposed and studied by Toth in a series of papers, for convex bodies of constant width. We show that, with respect to the mean Minkowski measure, the completions of regular simplices are, as well as for many other measures, the most asymmetric ones among all convex bodies of constant width.

1. Introduction

Measures of (central) symmetry or, as we prefer, asymmetry for convex bodies have been extensively studied (see [6, 7, 8, 15, 18, 21]). For a given asymmetry measure defined on some class of convex bodies, it is important to determine the extremal bodies with maximal asymmetry measure. For instance, for many known asymmetry measures, circles are most symmetric and, among 2-dimensional convex bodies of constant width, the Reuleaux triangles are most asymmetric (see [1, 5, 11]). Recently Jin and Guo extended such a result to the general n-dimensional cases as follows.

Theorem A. ([12, 13]). *Let K be an n-dimensional convex body of constant width. Then*

$$1 \leq \mathrm{as}_\infty(K) \leq \frac{n+\sqrt{2n(n+1)}}{n+2},$$

where $\mathrm{as}_\infty(\cdot)$ denotes the Minkowski measure of asymmetry for convex bodies. Equality holds on the left-hand side if and only if K is an Euclidean ball and on the right-hand if and only if K is a completion of a regular simplex.

Received December 12, 2013, accepted January 13, 2014.
Communicated by Bang-Yen Chen.
2010 *Mathematics Subject Classification*: 52A20, 52A39.
Key words and phrases: Measure of asymmetry, Mean Minkowski measure, Convex body of constant width, Reuleaux triangle, Meissner's bodies, Completion.
This research was supported, in part, by the national NSF of China No. 11271244 and by the national NSF of China No. 11271282.

In a series of papers ([20, 21, 22, 24, 25]), Toth introduced and studied, for an n-dimensional convex body C (in this paper, we use C for a generic convex body and K for a convex body of constant width), a family of measures (functions) of symmetry $\sigma_m := \sigma_m(C, \cdot\,) : \mathrm{int}(C) \to \mathbb{R}$, $(m \geq 1)$, called the mean Minkowski measures. They enjoy many nice properties and give out some useful information about the shape of C. As expected, among all σ_m, σ_n is the most important one. For instance, Toth proved the following:

Theorem B. ([20]). *Let $C \subset \mathbb{R}^n$ be a convex body. Then for any $o \in \mathrm{int}(C)$,*

$$1 \leq \sigma_n(C, o) \leq \frac{n+1}{2}.$$

Equality holds on the left-hand side for some $o \in \mathrm{int}(C)$ if and only if C is a simplex, and equality holds on the right-hand side for some $o \in \mathrm{int}(C)$ if and only if C is a centrally symmetric convex body centered at o.

In this paper, we will study the mean Minkowski measure σ_n for convex bodies K of constant width. More precisely, we will study the possible values of $\sigma_n(K, o)$, where o (called a base point by Toth) is chosen to be the (unique) ∞-critical point of K (see below for definitions). The reason for such a choice is that the insphere and circumsphere of a convex body of constant width are concentric (not true in general) and, as shown in [12], the common center of its insphere and circumsphere is precisely its (unique) ∞-critical point. In fact, as pointed out by Toth, even for general convex bodies, the ∞-critical points are still the most important base points.

The main result in this paper is the following theorem.

Main Theorem. *Let K be an n-dimensional convex body of constant width with base point o, the unique ∞-critical point of K. Then*

$$n + 1 - \frac{\sqrt{2n(n+1)}}{2} \leq \sigma_n(K, o) \leq \frac{n+1}{2}.$$

Equality holds on the right-hand side if and only if K is an Euclidean ball and on the left-hand if and only if K is a completion of a regular simplex.

2. Preliminaries

$\mathbb{R}^n$ denotes the usual n-dimensional Euclidean space with the canonical inner product $\langle \cdot, \cdot \rangle$. A bounded closed convex set $C \subset \mathbb{R}^n$ is called a *convex body* if it has non-empty interior. The family of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. For this and other concepts and notations our standard reference is [19].

For $C \in \mathcal{K}^n$, its support function $h(C, \cdot\,)$ is defined as $h(C, u) := \max\{\langle x, u \rangle \mid x \in C\}$, $u \in S^{n-1}$ (the unit sphere of $\mathbb{R}^n$). Then the width $\omega(C, u)$ of C in direction

$u \in S^{n-1}$ is given by $\omega(C,u) = h(C,u) + h(C,-u)$. Geometrically, this is the distance between the two parallel supporting hyperplanes of C orthogonal to u.

A convex body K is said to be of constant width ω if $\omega(K,u) = \omega$ for all $u \in S^{n-1}$ (see [3, 10, 14, 16, 17]). The class of all convex bodies of constant width is denoted by $\mathcal{W}^n$. In $\mathbb{R}^2$, among 2-dimensional bodies of constant width, the Reuleaux triangles are the most important ones. They have the minimal area among 2-dimensional bodies sharing the same constant width (see [3]). At the same time, it is the most asymmetric bodies of constant width. In $\mathbb{R}^3$, Meissner's bodies, also called Meissner tetrahedron, are the most famous ones, since they are conjectured to solve the open problem: Which bodies have the minimal volumes among 3-dimensional bodies sharing the same constant width (see [14]).

A set $C \in \mathcal{K}^n$ is said to be complete if there is no $C' \in \mathcal{K}^n$ such that $C \subsetneq C'$ and $\mathrm{diam}(C) = \mathrm{diam}(C')$. For $C \in \mathcal{K}^n$, any complete set $C^\natural$ with $C \subset C^\natural$ and $\mathrm{diam}(C) = \mathrm{diam}(C^\natural)$ is called a completion of C. In $\mathbb{R}^n$, a convex body is complete iff it is of constant width, and every convex body has at least one completion (see [3]).

Let $C \in \mathcal{K}^n$ and $x \in \mathrm{int}(C)$. For a chord l of C through x, let $\gamma(l,x)$ be the ratio, not less than 1, in which x divides the length of l. Letting $\gamma(C,x) = \max\{\gamma(l,x) \mid l \ni x\}$, the Minkowski measure $\mathrm{as}_\infty(C)$ is defined by (see [6, 15])

$$\mathrm{as}_\infty(C) = \min_{x \in \mathrm{int}(C)} \gamma(C,x).$$

A point $x \in \mathrm{int}(C)$ satisfying $\gamma(C,x) = \mathrm{as}_\infty(C)$ is called an ∞-*critical point* of C (see [8]). The set of all ∞-critical points of C is denoted by C^*. It is known that C^* is a non-empty convex set (see [6, 9, 15]).

If $x \in C^*$, a chord l satisfying $\gamma(l,x) = \mathrm{as}_\infty(C)$ is called a *critical chord* of C.

For any $x \in \mathrm{int}(C)$, let

$$\mathrm{S}_C(x) = \{p \in \mathrm{bd}(C) \mid \text{the chord } pq \ni x \text{ and } \frac{xp}{xq} = \gamma(C,x)\},$$

where bd denotes the boundary, and pq denotes the segment with endpoints p, q or its length alternatively if no confusing arise. It is known that $\mathrm{S}_C(x) \neq \phi$ (see [15]).

The following is a list of some properties of the Minkowski measure of asymmetry (see [15] for proofs).

Property 1. If $C \in \mathcal{K}^n$, then $1 \leq \mathrm{as}_\infty(C) \leq n$. Equality holds on the left-hand side if and only if C is centrally symmetric, and on the right-hand side if and only if C is a simplex.

Property 2. For $C \in \mathcal{K}^n$, $\mathrm{as}_\infty(C) + \mathrm{dim}C^* \leq n$, where dim means dimension.

Property 3. Given $x \in \mathrm{relint}(C^*)$, the relative interior of C^*, then for any $y \in \mathrm{S}_C(x)$,

$$y + \frac{\mathrm{as}_\infty(C) + 1}{\mathrm{as}_\infty(C)}(C^* - y) \subset \mathrm{bd}(C) \text{ and } y \in \mathrm{S}_C(x') \text{ for } \forall x' \in C^*.$$

This property shows that the set $\mathrm{S}_C(x)$ stays the same as x ranges over $\mathrm{relint}(C^*)$. We denote this set by $C^\dagger$.

Property 4. $C^\dagger$ contains at least $\mathrm{as}_\infty(C)+1$ points.

In [12], Jin and Guo showed that if $K \in \mathcal{W}^n$ with width ω, then

$$\mathrm{as}_\infty(K) = \frac{R(K)}{\omega - R(K)} \tag{*}$$

where $R(K)$ denotes the radius of the circumsphere of K.

We now recall the concept of mean Minkowski measures ([20]). Let $C \in \mathcal{K}^n$ and (a base point) $o \in \mathrm{int}(C)$. For $p \in \mathrm{bd}(C)$, the line passing through o and p intersects $\mathrm{bd}(C)$ in another point, called the opposite of p with respect to o and denoted by p^o. Clearly, $(p^o)^o = p$.

The distortion function $\Lambda(C, \cdot) : \mathrm{bd}(C) \to \mathbb{R}$ is defined by

$$\Lambda(C,p) = \frac{op}{op^o}, \ \ p \in \mathrm{bd}(C).$$

(Observe that $\max_{p\in\mathrm{bd}(C)} \Lambda(C,p) = \gamma(C,o)$).

For $m \geq 1$, a finite set $\{p_0,...,p_m\} \subset \mathrm{bd}(C)$ is called an m-configuration (relative to o) if o is contained in the convex hull $[p_0,...,p_m]$ of $\{p_i\}_{i=0}^m$.

Let $\mathcal{C}_m(C)$ denote the set of all m-configurations of C. The *mean Minkowski measure* $\sigma_m(C,o)$ is defined by (see [20])

$$\sigma_m(C,o) = \inf_{\{p_0,\dots,p_m\}\in\mathcal{C}_m(C)} \sum_{i=0}^m \frac{1}{1+\Lambda(C,p_i)}.$$

An m-configuration $\{p_0,...,p_m\}$ is called minimal if

$$\sigma_m(C,o) = \sum_{i=0}^m \frac{1}{1+\Lambda(C,p_i)}.$$

Minimal configurations always exist since C is compact.

Since an 1-configuration of C is an opposite pair of points $\{p, p^o\} \subset \mathrm{bd}(C)$ and $\Lambda(C,p^o) = 1/\Lambda(C,p)$, we have

$$\frac{1}{1+\Lambda(C,p)} + \frac{1}{1+\Lambda(C,p^o)} = 1.$$

This gives $\sigma_1(C) = 1$.

The most important invariant is $\sigma_n(C)$. As shown in [20], for $k \geq 1$, we have

$$\sigma_{n+k}(C) = \sigma_n(C) + \frac{k}{1+\max_{\mathrm{bd}(C)}\Lambda(C,\cdot)}.$$

Equivalently, $\{\sigma_m(C)\}_{m\geq n}$ is arithmetic with difference $1/(1+\max_{\mathrm{bd}(C)}\Lambda(C,\cdot))$.

3. Proof of Main Theorem

In order to prove the Main Theorem, we need some lemmas. The first one concerns the circumscribed balls of a regular simplex and its completions. It is shown by Vrecica in [26] that, for a convex body C, there is a completion $C^\natural$ with the same circumscribed ball as that of C. As it is well-known, an n-dimensional convex body ($n \geq 3$) may have many completions, and their circumscribed balls may not be the same. However, for regular simplices, we have the following result.

Lemma 1. *Let S be an n-dimensional regular simplex, and B be its circumscribed ball. Then the circumscribed balls of all completions of S coincide with B.*

Proof. Let $e_0, ..., e_n$ be the vertices of S and $D = \cap_{i=0,...,n} B(e_i, e_0e_1)$, where $B(e_i, e_0e_1)$ denotes the ball with radius e_0e_1 and center at e_i. Let $S^\natural$ be a completion of S. Then $S^\natural$ is a convex body of constant width e_0e_1. Therefore, $S^\natural$ has the spherical intersection property: $S^\natural = \cap_{x \in S^\natural} B(x, e_0e_1)$. Thus, since $e_0, ..., e_n \in S^\natural$, we have $S^\natural \subset D$. This, together with $D \subset B$ (to be shown below) implies that $S^\natural \subset B$. Thus, B is the circumscribed ball of $S^\natural$.

It remains to show that $D \subset B$. Let o be the unique ∞-critical point (centroid as well) of S. Clearly, we have $B = B(o, oe_0)$ (Theorem 1 in [12]). Let H_i ($0 \leq i \leq n$) be the affine span of $\{e_j\}_{0 \leq j \leq n, j \neq i}$; H_i is an affine hyperplane. Let H_i^- be the closed half-space, not containing o, determined by H_i. Then set $S_i = \mathrm{bd}(B(e_i, e_0e_1)) \cap H_i^-$. Thus, for any $x \in \mathrm{bd}(D)$, there exists an $i \in \{0, \cdots, n\}$, say $i = 0$, such that $x \in S_0$. Write $\angle oe_0x = \alpha$ and $\angle oe_0e_1 = \beta$, then $0 \leq \alpha \leq \beta \leq \pi/2$. Therefore, by $ox^2 = oe_0^2 + e_0x^2 - 2oe_0 \cdot e_0x \cos\alpha$, $oe_1^2 = oe_0^2 + e_0e_1^2 - 2oe_0 \cdot e_0e_1 \cos\beta$ and $e_0x = e_0e_1$, we have $ox \leq oe_1 = oe_0$. Thus, $x \in B$, and we obtain $D \subset B$. ■

Next lemma determines the mean Minkowski measures of a completion of a regular simplex.

Lemma 2. *Let $S^\natural$ be a completion of an n-dimensional regular simplex S and base point o, the unique ∞-critical point of $S^\natural$. Then $\sigma_n(S^\natural) = n + 1 - \frac{\sqrt{2n(n+1)}}{2}$.*

Proof. By Lemma 1, $S^\natural$ and S have the same circumscribed ball B. By Theorem 1 of [12], o is the center of B. Also, o is the centroid of S. Denote by $e_0, ..., e_n$ the vertices of S and by R the radius of B. Set $d := \mathrm{diam}(S) = \mathrm{diam}(S^\natural)$. In a regular simplex S, $oe_i = R = d\sqrt{\frac{n}{2(n+1)}}, i = 0, ..., n$. Since $S^\natural$ is of constant width d, o is also the ∞-critical point of $S^\natural$ (see [12]). Therefore, we have $\mathrm{as}_\infty(S^\natural) = \gamma(S^\natural, o)$ and $\mathrm{as}_\infty(S^\natural) = R/(d - R)$.

Since, for $p, q \in \mathrm{bd}(S^\natural)$ with $o \in pq$, $\frac{op}{oq} = \Lambda(S^\natural, p)$ and $\gamma(C^\natural, o) = \max_{\mathrm{bd}(S^\natural)} \Lambda(S, \cdot)$, we have

$$\mathrm{S}_{S^\natural}(o) = \{p \in \mathrm{bd}(S^\natural) \mid \Lambda(S^\natural, p) = \max_{\mathrm{bd}(S^\natural)} \Lambda(S^\natural, \cdot)\}.$$

Now we claim that $e_i \in \mathrm{S}_{S^\natural}(o)$ for $i = 0, ..., n$. In fact

$$\Lambda(S^\natural, e_i) = \frac{oe_i}{oe_i^o} = \frac{oe_i}{e_ie_i^o - oe_i} \geq \frac{oe_i}{d - oe_i} = R/(d-R) = \gamma(S^\natural, o) = \max_{\mathrm{bd}(S^\natural)} \Lambda(S^\natural, \cdot).$$

Hence, $\Lambda(S^\natural, e_i) = \max_{\mathrm{bd}(S^\natural)} \Lambda(S^\natural, \cdot)$.

It is obvious that $\{e_0, ..., e_n\} \in \mathcal{C}_n(S^\natural)$. Now we claim that $\{e_0, ..., e_n\}$ is a minimal n-configuration of $S^\natural$. In fact,

$$\begin{aligned} \sigma_n(S^\natural) &= \inf_{\{p_0,...,p_n\}\in\mathcal{C}_n(S^\natural)} \sum_{i=0}^{n} \frac{1}{1+\Lambda(S^\natural, p_i)} \\ &\leq \sum_{i=0}^{n} \frac{1}{1+\Lambda(S^\natural, e_i)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma_n(S^\natural) &= \inf_{\{p_0,...,p_n\}\in\mathcal{C}_n(S^\natural)} \sum_{i=0}^{n} \frac{1}{1+\Lambda(S^\natural, p_i)} \\ &\geq \sum_{i=0}^{n} \frac{1}{1+\max_{\mathrm{bd}(S^\natural)} \Lambda(S^\natural, \cdot)} \\ &= \sum_{i=0}^{n} \frac{1}{1+\Lambda(S^\natural, e_i)}. \end{aligned}$$

Hence, $\sigma_n(S^\natural) = \sum_{i=0}^{n} \frac{1}{1+\Lambda(S^\natural, e_i)} = \frac{n+1}{1+R/(d-R)} = n + 1 - \frac{\sqrt{2n(n+1)}}{2}$. ■

Remark. From the proof of Lemma 2, we see that $e_0, e_1, \cdots, e_n \in \mathrm{S}_{S^\natural}(o)$. This implies that $(S^\natural)^\dagger$ contains at least $n+1$ points. This is a significant improement of Klee's result of four points (Property 4 in Section 2) (noticing that, by Theorem A, $\mathrm{as}_\infty(S^\natural) \leq \frac{n+\sqrt{2n(n+1)}}{n+2} \leq 1 + \sqrt{2}$). In general, observing that $S^\natural$ is of constant width, we have the following conjecture (Grünbaum had a general conjecture in the early 1960's).

Conjecture. For any convex body K of constant width, $K^\dagger$ contains at least $n+1$ points.

Now we give the proof of Main Theorem.

Proof of the Main Theorem. First, Theorem B implies that $\sigma_n(K, o) \leq \frac{n+1}{2}$, and that equality holds if and only if K is a centrally symmetric convex bodies centered at o. In addition, the centrally symmetric convex bodies of constant width are precisely Euclidean balls, so that the statement for the upper bound follows.

For the inequality on the left-hand side, we have, by the definition of σ_n and Theorem A,

$$\begin{aligned}\sigma_n(K) &= \inf_{\{p_0,\ldots,p_n\}\in\mathcal{C}_n(K)} \sum_{i=0}^{n} \frac{1}{1+\Lambda(K,p_i)} \\ &\geq \sum_{i=0}^{n} \frac{1}{1+\max_{\mathrm{bd}(K)} \Lambda(K,\cdot)} \\ &= \frac{n+1}{1+\gamma(K,o)} = \frac{n+1}{1+\mathrm{as}_\infty(K)} \\ &\geq n+1-\frac{\sqrt{2n(n+1)}}{2}.\end{aligned}$$

If equality holds, then $\mathrm{as}_\infty(K) = \frac{n+\sqrt{2n(n+1)}}{n+2}$. By Theorem A again, K is the completion of a regular simplex. Conversely, if K is a completion of a regular simplex, then $\sigma_n(K) = n+1-\frac{\sqrt{2n(n+1)}}{2}$ by Lemma 2.

Since a Reuleaux triangle is the unique completion of a regular triangle, we have the following

Corollary 1. *Let K be a 2-dimensional convex body of constant width and the base point o, the unique ∞-critical point of K. Then*

$$3-\sqrt{3} \leq \sigma_2(K,o) \leq \frac{3}{2}.$$

Equality holds on the right-hand side if and only if K is an Euclidean circle and on the left-hand if and only if K is a Reuleaux triangle.

Meissner's bodies are the completions of a regular tetrahedron. However, arbitrary tetrahedron has infinite completions (see [4]). We have the following

Corollary 2. *Let K be a 3-dimensional convex body of constant width and the base point o, the unique ∞-critical point of K. Then*

$$4-\sqrt{6} \leq \sigma_3(K,o) \leq 2.$$

Equality holds on the right-hand side if and only if K is an Euclidean circle and on the left-hand side if K is a Meissner's body.

ACKNOWLEDGMENTS

The authors would like to give sincere thanks to referees for their valuable suggestions.

REFERENCES

1. A. S. Besicovitch, Measures of asymmetry for convex curves, II. Curves of constant width, *J. London Math. Soc.*, **26** (1951), 81-93.
2. T. Bonnesen and W. Fenchel, *Theorie der Konvexen Körper*, Springer, Berlin, 1934; Engl. transl, *Theory of Convex Bodies*, BCS Associates, Moscow, USA, 1987.
3. G. D. Charkerian and H. Groemer, Convex bodies of constant width, in: *Convexity and Its Applications*, 1983, pp. 49-96.
4. H. Groemer, On complete convex bodies, *Geom. Dedicata*, **20** (1986), 319-334.
5. H. Groemer and L. J. Wallen, A measure of asymmetry for domains of constant width, *Beitr. Algebra Geom.*, **42** (2001), 517-521.
6. B. Grünbaum, Measures of symmetry for convex sets, in: *Convexity, Proc. Symposia Pure Math.*, V. Klee, et al., (eds.), Vol. **7**, American Math. Society, Providence, 1963, pp. 233-270.
7. Q. Guo, Stability of the Minkowski measure of asymmetry for convex bodies, *Discrete Comput. Geom.*, **34** (2005), 351-362.
8. Q. Guo, On p-measures of asymmetry for convex bodies, *Adv. Geom.*, **12(2)** (2012), 287-301.
9. P. C. Hammer and A. Sobczyk, The critical points of a convex body, *Bull. Amer. Math. Soc.*, **57** (1951), 227.
10. E. Heil and H. Martini, Special convex bodies, in: *Handbook of Convex Geometry*, (P. M. Gruber and J. M. Wills Eds.), North-Holland, Amsterdam, 1993, pp. 347-385.
11. H. L. Jin and Q. Guo, On the asymmetry for convex domains of constant width, *Commun. Math. Res.*, **26(2)** (2010), 176-182.
12. H. L. Jin and Q. Guo, Asymmetry of convex bodies of constant width, *Discrete Comput. Geom.*, **47** (2012), 415-423.
13. H. L. Jin and Q. Guo, A note on the extremal bodies of constant width for the Minkowski measure, *Geom. Dedicata*, **164** (2013), 227-229.
14. B. Kawohl and C. Weber, Meissner's mysterious bodies, *Math. Intelligencer*, **33(3)** (2011), 94-101.
15. V. L. Klee, Jr., The critical set of a convex set, *Amer. J. Math.*, **75** (1953), 178-188.
16. T. Lachand-Robert and E. Oudet, Bodies of constant width in arbitrary dimension, *Math. Nachr.*, **280(7)** (2007), 740-750.
17. H. Martini and K. J. Swanepoel, The geometry of Minkowski spaces - a survey, Part II, *Expositiones Math.*, **22** (2004), 93-144.
18. M. Meyer, C. Schutt and E. M. Werner, New affine measures of symmetry for convex bodies, *Adv. Math.*, **228** (2011), 2920-2942.

19. R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, 1993.

20. G. Toth, Simplicial intersections of a convex set and moduli for spherical minimal immersions, *Michigan Math. J.*, **52** (2004), 341-359.

21. G. Toth, On the shape of the moduli of shperical minimal immersions, *Trans. Amer. Math. Soc.*, **358(6)** (2006), 2425-2446.

22. G. Toth, On the structure of convex sets with symmetries, *Geom. Dedicata*, **143** (2009), 69-80.

23. G. Toth, Asymmetry of convex sets with isolated extreme points, *Proc. Amer. Math. Soc.*, **137** (2009), 287-295.

24. G. Toth, Fine structure of convex sets from asymmetric viewpoint, *Beitr. Algebra Geom.*, **52** (2011), 171-189.

25. G. Toth, A measure of symmetry for the moduli of spherical minimal immersions, *Geom. Dedicata*, **160** (2012), 1-14.

26. S. Vrecica, A note on the sets of constant width, *Publ. Inst. Math.* (*Beograd*) (*N. S.*), **29(43)** (1981), 289-291.

HaiLin Jin
Department of Mathematics
Suzhou University of Technology and Science
Suzhou 215009
P. R. China
and
Department of Mathematics
Shanghai University
Shanghai 200444
P. R. China
E-mail: jinhailin17@163.com

Qi Guo
Department of Mathematics
Suzhou University of Technology and Science
Suzhou 215009
P. R. China
E-mail: guoqi@mail.usts.edu.cn