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FROM EQUILIBRIUM PROBLEMS AND FIXED POINTS PROBLEMS TO MINIMIZATION PROBLEMS

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Abstract. Algorithms approach to equilibrium problems and fixed points problems have been extensively studied in the literature. The purpose of this paper is devoted to consider the minimization problem of finding a point $x^\dagger$ with the property

$$x^\dagger \in \Omega \quad \text{and} \quad \|x^\dagger\|^2 = \min_{x\in\Omega} \|x\|^2,$$

where Ω is the intersection of the solution set of equilibrium problem and the fixed points set of nonexpansive mapping. For this purpose, we suggest two algorithms:

$$F(z_t, y) + \frac{1}{\lambda}\Big\langle y - z_t, z_t - \Big((1-t)I - \lambda A\Big)Sz_t\Big\rangle \geq 0,\ \forall y \in C.$$

and

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\Big\langle y - z_n, z_n - (1-\alpha_n)x_n\Big\rangle \geq 0,\ \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sz_n,\ n \geq 0. \end{cases}$$

It is shown that under some mild conditions, the net $\{z_t\}$ and the sequences $\{z_n\}$ and $\{x_n\}$ converge strongly to $\tilde{x}$ which is the unique solution of the above minimization problem. It should be point out that our suggested algorithms solve the above minimization problem without involving projection.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a nonexpansive mapping. Denote the set of fixed points of S by $F(S)$. For a nonlinear mapping

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$A : C \to H$ and a bifunction $F : C \times C \to R$, the equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by EP. If $A = 0$, then (1.1) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \forall y \in C. \tag{1.2}$$

If $F = 0$, then (1.1) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{1.3}$$

The equilibrium problem (1.2) and the variational inequality problem (1.3) have been investigated by many authors. Please see [6-18], [21-29], [40-43] and the references therein. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others. See, e.g., [1, 3-5].

For solving equilibrium problem (1.1), Moudafi [5] introduced an iterative algorithm and proved a weak convergence theorem. Further, Takahashi and Takahashi [3] introduced the following iterative algorithm for finding an element of $F(S) \cap EP$:

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n x_n + S[\alpha_n u + (1 - \beta_n)z_n], n \geq 0. \end{cases} \tag{1.4}$$

and they proved that the sequence $\{x_n\}$ converges strongly to $z = P_{F(S)\cap EP}(u)$.

Now we concern the following minimization problem of finding a point $x^\dagger$ with the property

$$x^\dagger \in F(S) \cap EP \quad \text{and} \quad \|x^\dagger\| = \min_{x \in F(S)\cap EP} \|x\|. \tag{1.5}$$

This problem is motivated by the least-squares solution to the constrained linear inverse problem. Some related works, please see [30-34].

We note that the algorithm (1.4) does not find the minimum-norm element in $F(S) \cap EP$ because S is a self-mapping on C and u is an element in C. In order to solve (1.5), we may consider the following algorithm by using projection:

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)SP_C[\alpha_n u + (1 - \alpha_n)z_n],\ n \geq 0. \end{cases} \tag{1.6}$$

where $u \in H$ is a fixed point and P_C is the metric projection from H onto C. Related works involved in metric projection, please see [34, 35] and the references therein.

Remark 1.1. Note that in order to find the minimum norm element, we have used the projection P_C. It is well-known that projection methods are used extensively in a variety of methods in optimization theory. Apart from theoretical interest, the main advantage of projection methods, which makes them successful in real-word applications, is computational. The field of projection methods is vast, see e.g., [36-39]. However, it is clear that if the set C is simple enough, so that the projection onto it is easily executed, then this method is particularly useful; but, if C is a general closed and convex set, then a minimal distance problem has to be solved in order to obtain the next iterative. This might seriously affect the efficiency of the method. Hence, it is an very interesting work of solving (1.5) without involving projection

Motivated and inspired by the results in the literature, in this paper we suggest two algorithms:

$$F(z_t, y) + \frac{1}{\lambda}\Big\langle y - z_t, z_t - \Big((1-t)I - \lambda A\Big)Sz_t\Big\rangle \geq 0,\ \forall y \in C.$$

and

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\Big\langle y - z_n, z_n - (1-\alpha_n)x_n\Big\rangle \geq 0,\ \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sz_n,\ n \geq 0. \end{cases}$$

It is shown that under some mild conditions, the net $\{z_t\}$ and the sequences $\{z_n\}$ and $\{x_n\}$ converge strongly to $\tilde{x}$ which is the unique solution of the above minimization problem (1.5). It should be point out that our suggested algorithms solve the above minimization problem (1.5) without involving projection.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

It is clear that any α-inverse-strongly monotone mapping is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. A mapping $S : C \to C$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C$.

Throughout this paper, we assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(H1) $F(x, x) = 0$ for all $x \in C$;

(H2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(H3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

We need the following lemmas for proving our main results.

Lemma 2.1. ([2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)-(H4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Further, if $T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, then the following hold:

- (i) *T_r is single-valued and T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
- (ii) *EP is closed and convex and $EP = Fix(T_r)$.*

Lemma 2.2. *Let C, H, F and $T_r x$ be as in Lemma 2.1. Then the following holds:*

$$\|T_s x - T_t x\|^2 \leq \frac{s - t}{s}\langle T_s x - T_t x, T_s x - x\rangle,$$

for all $s, t > 0$ and $x \in H$.

Lemma 2.3. ([8]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $r > 0$ be a constant. Then, we have*

$$\|(I - rA)x - (I - rA)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2, \forall x, y \in C.$$

In particular, if $0 \leq r \leq 2\alpha$, then $I - rA$ is nonexpansive.

Lemma 2.4. ([19]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.5. ([10]). *Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

Lemma 2.6. ([20]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that (*i*) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(*ii*) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we will prove our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions (H1)-(H4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping. Suppose that $F(S) \cap EP \neq \emptyset$. Let λ be a constant satisfying $a \leq \lambda \leq b$ where $[a, b] \subset (0, 2\alpha)$. For $t \in (0, 1-\frac{\lambda}{2\alpha})$, let $\{z_t\} \subset C$ be a net generated by the implicit manner*

$$F(z_t, y) + \frac{1}{\lambda}\Big\langle y - z_t, z_t - \Big((1-t)I - \lambda A\Big)Sz_t\Big\rangle \geq 0, \forall y \in C. \tag{3.1}$$

Then the net $\{z_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x}$ which is the minimum norm element in $F(S) \cap EP$.

Proof. First, we show the net $\{z_t\}$ is well-defined. We observe by Lemma 2.1 that we only need to show that the implicit algorithm $z_t = T_\lambda((1-t)I - \lambda A)Sz_t$ is well-defined. Now, we define a mapping $W_t := T_\lambda((1-t)I - \lambda A)S$. It is clear that W_t is a self-mapping of C according to Lemma 2.1. For $x, y \in C$, we have

$$\begin{aligned} \|W_t x - W_t y\| &= \|T_\lambda((1-t)I - \lambda A)Sx - T_\lambda((1-t)I - \lambda A)Sy\| \\ &\leq \|((1-t)I - \lambda A)Sx - ((1-t)I - \lambda A)Sy\| \\ &= (1-t)\Big\|\Big(I - \frac{\lambda}{1-t}A\Big)Sx - \Big(I - \frac{\lambda}{1-t}A\Big)Sy\Big\|. \end{aligned}$$

Since $I - \frac{\lambda}{1-t}A$ (by Lemma 2.3) and S are nonexpansive, we deduce

$$\|W_t x - W_t y\| \leq (1-t)\|x - y\|, \forall x, y \in C.$$

This indicates that W_t is a contraction on C. It has a unique fixed point, denoted by z_t, in C. That is, $z_t = T_\lambda((1-t)I - \lambda A)Sz_t$. Hence, (3.1) is well-defined.

Take any $z \in F(S) \cap EP$. It is obvious that $z = T_\lambda(z - \lambda Az)$ for all $\lambda > 0$. So, we have $z = Sz = T_\lambda(z - \lambda Az) = T_\lambda(Sz - \lambda ASz) = T_\lambda\Big(tSz + (1-t)(Sz - \frac{\lambda}{1-t}ASz)\Big)$ for all $t \in (0, 1 - \frac{\lambda}{2\alpha})$. Since T_λ is nonexpansive, we have

$$\begin{aligned} &\|z_t - z\|^2 \\ &= \Big\|T_\lambda\Big((1-t)Sz_t - \lambda ASz_t\Big) - z\Big\|^2 \\ &= \Big\|T_\lambda\Big((1-t)(Sz_t - \frac{\lambda}{1-t}ASz_t)\Big) - T_\lambda\Big(tz + (1-t)(Sz - \frac{\lambda}{1-t}ASz)\Big)\Big\|^2 \end{aligned} \tag{3.2}$$

$$\begin{aligned}&\leq \Big\|\Big((1-t)(Sz_t-\frac{\lambda}{1-t}ASz_t)\Big)-\Big(tz+(1-t)(z-\frac{\lambda}{1-t}ASz)\Big)\Big\|^2\\&=\Big\|(1-t)\Big((Sz_t-\frac{\lambda}{1-t}ASz_t)-(Sz-\frac{\lambda}{1-t}ASz)\Big)+t(-z)\Big\|^2.\end{aligned}$$

By using the convexity of $\|\cdot\|$ and the α-inverse strong monotonicity of A, we derive

$$\begin{aligned}&\Big\|(1-t)\Big((Sz_t-\frac{\lambda}{1-t}ASz_t)-(Sz-\frac{\lambda}{1-t}ASz)\Big)+t(-z)\Big\|^2\\&\leq (1-t)\Big\|(Sz_t-\frac{\lambda}{1-t}ASz_t)-(Sz-\frac{\lambda}{1-t}ASz)\Big\|^2+t\|z\|^2\\&=(1-t)\|(Sz_t-Sz)-\lambda(ASz_t-ASz)/(1-t)\|^2+t\|z\|^2\\&=(1-t)\Big(\|Sz_t-Sz\|^2-\frac{2\lambda}{1-t}\langle ASz_t-ASz,Sz_t-Sz\rangle\\&\quad+\frac{\lambda^2}{(1-t)^2}\|ASz_t-ASz\|^2\Big)+t\|z\|^2\\&\leq (1-t)\Big(\|Sz_t-Sz\|^2-\frac{2\alpha\lambda}{1-t}\|ASz_t-ASz\|^2\\&\quad+\frac{\lambda^2}{(1-t)^2}\|ASz_t-ASz\|^2\Big)+t\|z\|^2\\&=(1-t)\Big(\|Sz_t-Sz\|^2+\frac{\lambda}{(1-t)^2}(\lambda-2(1-t)\alpha)\|ASz_t-ASz\|^2\Big)+t\|z\|^2\\&\leq (1-t)\Big(\|z_t-z\|^2+\frac{\lambda}{(1-t)^2}(\lambda-2(1-t)\alpha)\|ASz_t-ASz\|^2\Big)+t\|z\|^2.\end{aligned}\tag{3.3}$$

By the assumption, we have $\lambda-2(1-t)\alpha\leq 0$ for all $t\in(0,1-\frac{\lambda}{2\alpha})$. Then, from (3.2) and (3.3), we obtain

$$\begin{aligned}&\|z_t-z\|^2\\&\leq (1-t)\Big(\|z_t-z\|^2+\frac{\lambda}{(1-t)^2}(\lambda-2(1-t)\alpha)\|ASz_t-ASz\|^2\Big)+t\|z\|^2\\&\leq (1-t)\|z_t-z\|^2+t\|z\|^2.\end{aligned}\tag{3.4}$$

It follows that

$$\|z_t-z\|\leq\|z\|.$$

Therefore, $\{z_t\}$ is bounded. Hence, $\{Sz_t\}$ and $\{ASz_t\}$ are also bounded.

From (3.4), we obtain

$$\|z_t-z\|^2\leq(1-t)\|z_t-z\|^2+\frac{\lambda}{(1-t)}(\lambda-2(1-t)\alpha)\|ASz_t-Az\|^2+t\|z\|^2.$$

So,

$$\frac{\lambda}{(1-t)}(2(1-t)\alpha-\lambda)\|ASz_t-Az\|^2\le t\|z\|^2\to 0.$$

This implies that

$$\lim_{t\to 0+}\|ASz_t-Az\|=0. \tag{3.5}$$

Next, we show $\|z_t-Sz_t\|\to 0$. By using the firm nonexpansivity of T_λ (see Lemma 2.1), we have

$$\begin{aligned}\|z_t-z\|^2 &= \Big\|T_\lambda\Big((1-t)Sz_t-\lambda ASz_t\Big)-z\Big\|^2\\ &= \Big\|T_\lambda\Big((1-t)Sz_t-\lambda ASz_t\Big)-T_\lambda\Big(Sz-\lambda ASz\Big)\Big\|^2\\ &\le \Big\langle(1-t)Sz_t-\lambda ASz_t-(Sz-\lambda ASz),z_t-z\Big\rangle\\ &= \frac{1}{2}\Big(\|(1-t)Sz_t-\lambda ASz_t-(Sz-\lambda ASz)\|^2+\|z_t-z\|^2\\ &\quad -\|(1-t)Sz_t-\lambda(ASz_t-\lambda ASz)-z_t\|^2\Big).\end{aligned}$$

By the nonexpansivity of $I-\lambda A/(1-t)$, we have

$$\begin{aligned}&\|(1-t)Sz_t-\lambda ASz_t-(Sz-\lambda ASz)\|^2\\ &=\|(1-t)((Sz_t-\lambda ASz_t/(1-t)-(Sz-\lambda ASz/(1-t)))+t(-z)\|^2\\ &\le(1-t)\|(Sz_t-\lambda ASz_t/(1-t)-(Sz-\lambda ASz/(1-t))\|^2+t\|z\|^2\\ &\le(1-t)\|Sz_t-Sz\|^2+t\|z\|^2\\ &\le(1-t)\|z_t-z\|^2+t\|z\|^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\|z_t-z\|^2\\ &\le\frac{1}{2}\Big((1-t)\|z_t-z\|^2+t\|z\|^2+\|z_t-z\|^2-\|(1-t)Sz_t-z_t-\lambda(ASz_t-ASz)\|^2\Big).\end{aligned}$$

It follows that

$$\begin{aligned}0&\le t\|z\|^2-\|(1-t)Sz_t-z_t-\lambda(ASz_t-ASz)\|^2\\ &=t\|z\|^2-\|(1-t)Sz_t-z_t\|^2+2\lambda\langle(1-t)Sz_t-z_t,ASz_t-ASz\rangle-\lambda^2\|ASz_t-ASz\|^2\\ &\le t\|z\|^2-\|(1-t)Sz_t-z_t\|^2+2\lambda\|(1-t)Sz_t-z_t\|\|ASz_t-ASz\|.\end{aligned}$$

Hence,

$$\|(1-t)Sz_t - z_t\|^2 \leq t\|z\|^2 + 2\lambda\|(1-t)Sz_t - z_t\|\|ASz_t - Az\|.$$

Since $\|ASz_t - Az\| \to 0$ by (3.5), we deduce

$$\lim_{t\to 0+} \Big\|(1-t)Sz_t - z_t\Big\| = 0.$$

Therefore,

$$\lim_{t\to 0+} \Big\|z_t - Sz_t\Big\| = 0. \tag{3.6}$$

From (3.2), we have

$$\begin{aligned}
\|z_t - z\|^2 &\leq \Big\|(1-t)\Big((Sz_t - \frac{\lambda}{1-t}ASz_t) - (Sz - \frac{\lambda}{1-t}ASz)\Big) - tz\Big\|^2 \\
&= (1-t)^2\Big\|(Sz_t - \frac{\lambda}{1-t}ASz_t) - (Sz - \frac{\lambda}{1-t}ASz)\Big\|^2 \\
&\quad -2t(1-t)\Big\langle z, (Sz_t - \frac{\lambda}{1-t}ASz_t) - (Sz - \frac{\lambda}{1-t}ASz)\Big\rangle + t^2\|z\|^2 \\
&\leq (1-t)^2\|z_t - z\|^2 - 2t(1-t)\Big\langle z, Sz_t - \frac{\lambda}{1-t}(ASz_t - Az) - z\Big\rangle + t^2\|z\|^2 \\
&= (1-2t)\|z_t - z\|^2 + 2t\Big\{ -(1-t)\Big\langle z, Sz_t - z - \frac{\lambda}{1-t}(ASz_t - Az)\Big\rangle \\
&\quad + t^2(\|z\|^2 + \|z_t - z\|^2)\Big\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_t - z\|^2 &\leq -\Big\langle z, Sz_t - z - \frac{\lambda}{1-t}(ASz_t - Az)\Big\rangle + \frac{t}{2}(\|z\|^2 + \|z_t - z\|^2) \\
&\quad + t\|z\|\Big\|Sz_t - z - \frac{\lambda}{1-t}(ASz_t - Az)\Big\| \\
&\leq \langle z, z - Sz_t\rangle + \frac{\lambda}{1-t}\|z\|\|ASz_t - Az\| + tM,
\end{aligned} \tag{3.7}$$

where M is some constant such that

$$\sup\Big\{\|z\|^2 + \|z_t - z\|^2 + \|z\|\|Sz_t - z - \frac{\lambda}{1-t}(ASz_t - Az)\|, t \in (0, 1 - \frac{\lambda}{2\alpha})\Big\} \leq M.$$

Next we show that $\{z_t\}$ is relatively norm-compact as $t \to 0+$. Assume $\{t_n\} \subset (0,1)$ is such that $t_n \to 0+$ as $n \to \infty$. Put $z_n := z_{t_n}$. From (3.7), we have

$$\|z_n - z\|^2 \leq \langle z, z - Sz_n\rangle + \tfrac{\lambda}{1-t_n}\|z\|\|ASz_n - Az\| + t_nM, \ z \in F(S) \cap EP. \tag{3.8}$$

Since $\{z_n\}$ is bounded, without loss of generality, we may assume that $z_n \rightharpoonup \tilde{x} \in C$. From (3.6), we have

$$\lim_{n\to\infty} \|z_n - Sz_n\| = 0. \tag{3.9}$$

We can use Lemma 2.5 to (3.9) to deduce $\tilde{x} \in F(S)$. Further, we show that $\tilde{x}$ is also in EP. Since $z_n = T_\lambda((1-t_n)Sz_n - \lambda ASz_n)$ for any $y \in C$, we have

$$F(z_n, y) + \langle ASz_n, y - z_n\rangle + \frac{1}{\lambda}\langle y - z_n, z_n - (1-t_n)Sz_n\rangle \geq 0.$$

From (H2), we have

$$\langle ASz_n, y - z_n\rangle + \frac{1}{\lambda}\langle y - z_n, z_n - (1-t_n)Sz_n\rangle \geq F(y, z_n). \tag{3.10}$$

Put $x_t = ty + (1-t)\tilde{x}$ for all $t \in (0, 1 - \frac{\lambda}{2\alpha})$ and $y \in C$. Then, we have $x_t \in C$. So, from (3.10), we have

$$\begin{aligned}
\langle x_t - z_n, Ax_t\rangle \geq\ & \langle x_t - z_n, Ax_t\rangle - \langle x_t - z_n, ASz_n\rangle \\
& -\frac{1}{\lambda}\langle x_t - z_n, z_n - (1-t_n)Sz_n\rangle + F(x_t, z_n) \\
=\ & \langle x_t - z_n, Ax_t - Az_n\rangle + \langle x_t - z_n, Az_n - ASz_n\rangle \\
& -\frac{1}{\lambda}\langle x_t - z_n, z_n - (1-t_n)Sz_n\rangle + F(x_t, z_n).
\end{aligned}$$

Since $\|z_n - Sz_n\| \to 0$, we have $\|Az_n - ASz_n\| \to 0$. Further, from monotonicity of A, we have $\langle x_t - z_n, Ax_t - Az_n\rangle \geq 0$. So, from (H4), we have

$$\langle x_t - \tilde{x}, Ax_t\rangle \geq F(x_t, \tilde{x}), \text{as } n \to \infty. \tag{3.11}$$

From (H1), (H4) and (3.11), we also have

$$\begin{aligned}
0 =\ & F(x_t, x_t) \\
\leq\ & tF(x_t, y) + (1-t)F(x_t, \tilde{x}) \\
\leq\ & tF(x_t, y) + (1-t)\langle x_t - \tilde{x}, Ax_t\rangle \\
=\ & tF(x_t, y) + (1-t)t\langle y - \tilde{x}, Ax_t\rangle
\end{aligned}$$

and hence

$$0 \leq F(x_t, y) + (1-t)\langle y - \tilde{x}, Ax_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq F(\tilde{x}, y) + \langle y - \tilde{x}, A\tilde{x}\rangle.$$

This implies $\tilde{x} \in EP$. Therefore we can substitute $\tilde{x}$ for z in (3.8) to get

$$\|z_n - \tilde{x}\|^2 \le \langle \tilde{x}, \tilde{x} - Sz_n\rangle + \frac{\lambda}{1-t_n}\|\tilde{x}\|\|ASz_n - A\tilde{x}\| + t_nM,\ z \in F(S)\cap EP.$$

By (3.5), we know that $\|ASz_n - Az\| \to 0$ for any $z \in F(S)\cap EP$. Then, we get $\|ASz_n - A\tilde{x}\| \to 0$. Consequently, the weak convergence of $\{z_n\}$ (and $\{Sz_n\}$) to $\tilde{x}$ actually implies that $z_n \to \tilde{x}$. This has proved the relative norm-compactness of the net $\{z_t\}$ as $t \to 0+$.

Now we return to (3.8) and take the limit as $n \to \infty$ to get

$$\|\tilde{x} - z\|^2 \le \langle z, z - \tilde{x}\rangle,\ z \in F(S)\cap EP.$$

Equivalently,

$$\|\tilde{x}\|^2 \le \langle \tilde{x}, z\rangle, \quad z \in F(S)\cap EP.$$

This clearly implies that

$$\|\tilde{x}\| \le \|z\|, \quad z \in F(S)\cap EP.$$

Therefore, $\tilde{x}$ is the minimum-norm element in $F(S)\cap EP$. This completes the proof. ■

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions (H1)-(H4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping. Suppose that $EP \ne \emptyset$. Let λ be a constant satisfying $a \le \lambda \le b$ where $[a, b] \subset (0, 2\alpha)$. For $t \in (0, 1 - \frac{\lambda}{2\alpha})$, let $\{z_t\} \subset C$ be a net generated by the implicit manner*

$$F(z_t, y) + \frac{1}{\lambda}\Big\langle y - z_t, z_t - \Big((1-t)I - \lambda A\Big)z_t\Big\rangle \ge 0, \forall y \in C.$$

Then the net $\{z_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x}$ which is the minimum norm element in EP.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions (H1)-(H4). Suppose that $EP \ne \emptyset$. Let λ be a constant satisfying $a \le \lambda \le b$ where $[a, b] \subset (0, 2\alpha)$. For $t \in (0, 1 - \frac{\lambda}{2\alpha})$, let $\{z_t\} \subset C$ be a net generated by the implicit manner*

$$F(z_t, y) + \frac{1}{\lambda}\Big\langle y - z_t, z_t - (1 - t - \lambda)z_t\Big\rangle \ge 0, \forall y \in C.$$

Then the net $\{z_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x}$ which is the minimum norm element in EP.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions (H1)-(H4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping. Suppose that $F(S) \cap EP \neq \emptyset$. Let $x_0 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \dfrac{1}{\lambda_n}\langle y - z_n, z_n - (1-\alpha_n)x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Sz_n, n \geq 0, \end{cases} \tag{3.12}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$*;*

(ii) $0 < c \leq \beta_n \leq d < 1$*;*

(iii) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$ *where* $[a,b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.12) converges strongly to the minimum norm element x^* in $F(S) \cap EP$.

Proof. Note that z_n can be rewritten as $z_n = T_{\lambda_n}((1-\alpha_n)x_n - \lambda_n Ax_n)$ for each n. Take $z \in F(S) \cap EP$. It is obvious that $z = T_{\lambda_n}(z - \lambda_n Az) = T_{\lambda_n}\Big(\alpha_n z + (1-\alpha_n)(z - \frac{\lambda_n Az}{1-\alpha_n})\Big)$ for all $n \geq 0$. By using the nonexpansivity of T_{λ_n} and the convexity of $\|\cdot\|$, we derive

$$\begin{aligned} \|z_n - z\|^2 &= \Big\|T_{\lambda_n}\Big((1-\alpha_n)x_n - \lambda_n Ax_n\Big) - T_{\lambda_n}(z - \lambda_n Az)\Big\|^2 \\ &= \Big\|T_{\lambda_n}\Big((1-\alpha_n)(x_n - \frac{\lambda_n Ax_n}{1-\alpha_n})\Big) - T_{\lambda_n}\Big(\alpha_n z + (1-\alpha_n)(z - \frac{\lambda_n Az}{1-\alpha_n})\Big)\Big\|^2 \\ &\leq \Big\|\Big((1-\alpha_n)(x_n - \frac{\lambda_n Ax_n}{1-\alpha_n})\Big) - \Big(\alpha_n z + (1-\alpha_n)(z - \frac{\lambda_n Az}{1-\alpha_n})\Big)\Big\|^2 \\ &= \Big\|(1-\alpha_n)\Big((x_n - \frac{\lambda_n Ax_n}{1-\alpha_n}) - (z - \frac{\lambda_n Az}{1-\alpha_n})\Big) + \alpha_n(-z)\Big\|^2 \\ &\leq (1-\alpha_n)\Big\|(x_n - \frac{\lambda_n Ax_n}{1-\alpha_n}) - (z - \frac{\lambda_n Az}{1-\alpha_n})\Big\|^2 + \alpha_n\|z\|^2. \end{aligned}$$

Since A is α-inverse strongly monotone, we know from Lemma 2.3 that

$$\Big\|(x_n - \frac{\lambda_n Ax_n}{1-\alpha_n}) - (z - \frac{\lambda_n Az}{1-\alpha_n})\Big\|^2 \leq \|x_n - z\|^2 + \frac{\lambda_n(\lambda_n - 2(1-\alpha_n)\alpha)}{(1-\alpha_n)^2}\|Ax_n - Az\|^2.$$

It follows that

$$\begin{aligned} &\|z_n - z\|^2 \\ &\leq (1-\alpha_n)\Big(\|x_n - z\|^2 + \frac{\lambda_n(\lambda_n - 2(1-\alpha_n)\alpha)}{(1-\alpha_n)^2}\|Ax_n - Az\|^2\Big) + \alpha_n\|z\|^2 \\ &\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|z\|^2. \end{aligned} \tag{3.13}$$

So, we have that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\beta_n(x_n-z)+(1-\beta_n)(Sz_n-z)\|^2\\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|z_n-z\|^2\\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)((1-\alpha_n)\|x_n-z\|^2+\alpha_n\|z\|^2)\\ &= [1-(1-\beta_n)\alpha_n]\|x_n-z\|^2+(1-\beta_n)\alpha_n\|z\|^2\\ &\le \max\{\|x_n-z\|^2,\|z\|^2\}.\end{aligned}$$

By induction, we have

$$\|x_{n+1}-z\|^2 \le \max\{\|x_0-z\|^2,\|z\|^2\}.$$

Therefore, $\{x_n\}$ is bounded. Hence, $\{Ax_n\}$, $\{z_n\}$, $\{Sz_n\}$ are also bounded.

Putting $u_n=(1-\alpha_n)x_n-\lambda_n Ax_n$ for all n, we have

$$z_{n+1}-z_n = T_{\lambda_{n+1}}u_{n+1}-T_{\lambda_{n+1}}u_n+T_{\lambda_{n+1}}u_n-T_{\lambda_n}u_n.$$

It follows that

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|T_{\lambda_{n+1}}u_{n+1}-T_{\lambda_{n+1}}u_n\|+\|T_{\lambda_{n+1}}u_n-T_{\lambda_n}u_n\|\\ &\le \|u_{n+1}-u_n\|+\|T_{\lambda_{n+1}}u_n-T_{\lambda_n}u_n\|.\end{aligned} \tag{3.14}$$

From Lemma 2.3, we know that $I-\lambda A$ is nonexpansive for all $\lambda\in(0,2\alpha)$. Thus, we have $I-\lambda_{n+1}A/(1-\alpha_{n+1})$ is nonexpansive for all n due to the fact that $\lambda_{n+1}/(1-\alpha_{n+1})\in(0,2\alpha)$. Then, we get

$$\begin{aligned}&\|u_{n+1}-u_n\|\\ &= \|(1-\alpha_{n+1})x_{n+1}-\lambda_{n+1}Ax_{n+1}-((1-\alpha_n)x_n-\lambda_nAx_n)\|\\ &= \|(1-\alpha_{n+1})\Big(x_{n+1}-\frac{\lambda_{n+1}}{1-\alpha_{n+1}}Ax_{n+1}\Big)-(1-\alpha_n)\Big(x_n-\frac{\lambda_n}{1-\alpha_n}Ax_n\Big)\|\\ &\le (1-\alpha_{n+1})\Big\|\Big(I-\frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\Big)x_{n+1}-\Big(I-\frac{\lambda_{n+1}}{1-\alpha_{n+1}}A\Big)x_n\Big\|\\ &\quad+\Big\|(1-\alpha_{n+1})\Big(x_n-\frac{\lambda_{n+1}}{1-\alpha_{n+1}}Ax_n\Big)-(1-\alpha_n)\Big(x_n-\frac{\lambda_n}{1-\alpha_n}Ax_n\Big)\Big\|\\ &\le \|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|.\end{aligned} \tag{3.15}$$

By Lemma 2.2, we have

$$\|T_{\lambda_{n+1}}u_n-T_{\lambda_n}u_n\| \le \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|T_{\lambda_{n+1}}u_n-u_n\|. \tag{3.16}$$

From (3.14)-(3.16), we obtain

$$\|z_{n+1}-z_n\| \le \|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\| + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|T_{\lambda_{n+1}}u_n-u_n\|.$$

Then,

$$\begin{aligned}\|Sz_{n+1}-Sz_n\| &\le \|z_{n+1}-z_n\| \\ &\le \|x_{n+1}-x_n\|+|\alpha_{n+1}-\alpha_n|\|x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\| \\ &\quad + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|T_{\lambda_{n+1}}u_n-u_n\|.\end{aligned}$$

Therefore,

$$\|Sz_{n+1}-Sz_n\|-\|x_{n+1}-x_n\| \le |\alpha_{n+1}-\alpha_n|\|x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\| + \frac{|\lambda_{n+1}-\lambda_n|}{\lambda_{n+1}}\|T_{\lambda_{n+1}}u_n-u_n\|.$$

Since $\alpha_n \to 0$, $\lambda_{n+1}-\lambda_n \to 0$ and $\liminf_{n\to\infty}\lambda_n > 0$, we obtain

$$\limsup_{n\to\infty}(\|Sz_{n+1}-Sz_n\|-\|x_{n+1}-x_n\|) \le 0.$$

From Lemma 2.4, we get

$$\lim_{n\to\infty}\|Sz_n-x_n\|=0.$$

Consequently, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|Sz_n-x_n\|=0.$$

From (3.12) and (3.13), we have

$$\begin{aligned}&\|x_{n+1}-z\|^2 \\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)\Big\|ST_{\lambda_n}\Big((1-\alpha_n)x_n-\lambda_nAx_n\Big)-z\Big\|^2 \\ &\le (1-\beta_n)\Big\{(1-\alpha_n)\Big(\|x_n-z\|^2+\frac{\lambda_n}{(1-\alpha_n)^2}(\lambda_n-2(1-\alpha_n)\alpha)\|Ax_n-Az\|^2\Big) \\ &\quad +\alpha_n\|z\|^2\Big\}+\beta_n\|x_n-z\|^2 \\ &= [1-(1-\beta_n)\alpha_n]\|x_n-z\|^2+\frac{(1-\beta_n)\lambda_n}{(1-\alpha_n)}(\lambda_n-2(1-\alpha_n)\alpha)\|Ax_n-Az\|^2 \\ &\quad +(1-\beta_n)\alpha_n\|z\|^2 \\ &\le \|x_n-z\|^2+\frac{(1-\beta_n)\lambda_n}{(1-\alpha_n)}(\lambda_n-2(1-\alpha_n)\alpha)\|Ax_n-Az\|^2+(1-\beta_n)\alpha_n\|z\|^2.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\frac{(1-\beta_n)\lambda_n}{(1-\alpha_n)}(2(1-\alpha_n)\alpha-\lambda_n)\|Ax_n-Az\|^2\\ \le\ &\|x_n-z\|^2-\|x_{n+1}-z\|^2+(1-\beta_n)\alpha_n\|z\|^2\\ \le\ &(\|x_n-z\|-\|x_{n+1}-z\|)\|x_{n+1}-x_n\|+(1-\beta_n)\alpha_n\|z\|^2.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$ and $\liminf_{n\to\infty}\frac{(1-\beta_n)\lambda_n}{(1-\alpha_n)}(2(1-\alpha_n)\alpha-\lambda_n)>0$, we have

$$\lim_{n\to\infty}\|Ax_n-Az\|=0. \tag{3.17}$$

Next, we show $\|x_n-T_{\lambda_n}u_n\|\to 0$. By using the firm nonexpansivity of T_{λ_n}, we have

$$\begin{aligned}\|T_{\lambda_n}u_n-z\|^2 &= \|T_{\lambda_n}u_n-T_{\lambda_n}\Big(z-\lambda_nAz\Big)\|^2\\ &\le \Big\langle u_n-(z-\lambda_nAz),T_{\lambda_n}u_n-z\Big\rangle\\ &= \frac{1}{2}\Big(\|u_n-(z-\lambda_nAz)\|^2+\|T_{\lambda_n}u_n-z\|^2\\ &\quad -\|(1-\alpha_n)x_n-\lambda_n(Ax_n-\lambda_nAz)-T_{\lambda_n}u_n\|^2\Big).\end{aligned}$$

We note that

$$\|u_n-(z-\lambda_nAz)\|^2\le(1-\alpha_n)\|x_n-z\|^2+\alpha_n\|z\|^2.$$

Thus,

$$\begin{aligned}\|T_{\lambda_n}u_n-z\|^2 \le\ &\frac{1}{2}\Big((1-\alpha_n)\|x_n-z\|^2+\alpha_n\|z\|^2+\|T_{\lambda_n}u_n-z\|^2\\ &-\|(1-\alpha_n)x_n-T_{\lambda_n}u_n-\lambda_n(Ax_n-\lambda_nAz)\|^2\Big).\end{aligned}$$

That is,

$$\begin{aligned}&\|T_{\lambda_n}u_n-z\|^2\\ \le\ &(1-\alpha_n)\|x_n-z\|^2+\alpha_n\|z\|^2-\|(1-\alpha_n)x_n-T_{\lambda_n}u_n-\lambda_n(Ax_n-\lambda_nAz)\|^2\\ =\ &(1-\alpha_n)\|x_n-z\|^2+\alpha_n\|z\|^2-\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|^2\\ &+2\lambda_n\Big\langle(1-\alpha_n)x_n-T_{\lambda_n}u_n,Ax_n-Az\Big\rangle-\lambda_n^2\|Ax_n-Az\|^2\\ \le\ &(1-\alpha_n)\|x_n-z\|^2+\alpha_n\|z\|^2-\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|^2\\ &+2\lambda_n\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|\|Ax_n-Az\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-z\|^2 \le\ & \beta_n\|x_n-z\|^2+(1-\beta_n)(1-\alpha_n)\|x_n-z\|^2+(1-\beta_n)\alpha_n\|z\|^2\\ & -(1-\beta_n)\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|^2\\ & +2\lambda_n(1-\beta_n)\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|\|Ax_n-Az\|\\ =\ & [1-(1-\beta_n)\alpha_n]\|x_n-z\|^2\\ & +(1-\beta_n)\alpha_n\|z\|^2-(1-\beta_n)\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|^2\\ & +2\lambda_n(1-\beta_n)\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|\|Ax_n-Az\|.\end{aligned}$$

Hence,

$$\begin{aligned}& (1-\beta_n)\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|^2\\ \le\ & \|x_n-z\|^2-\|x_{n+1}-z\|^2-(1-\beta_n)\alpha_n\|x_n-z\|^2\\ & +(1-\beta_n)\alpha_n\|z\|^2+2\lambda_n(1-\beta_n)\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|\|Ax_n-Az\|\\ \le\ & (\|x_n-z\|+\|x_{n+1}-z\|)\|x_{n+1}-x_n\|+(1-\beta_n)\alpha_n\|z\|^2\\ & +2\lambda_n(1-\beta_n)\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|\|Ax_n-Az\|.\end{aligned}$$

Since $\limsup_{n\to\infty}\beta_n<1$, $\|x_{n+1}-x_n\|\to 0$, $\alpha_n\to 0$ and $\|Ax_n-Az\|\to 0$, we deduce

$$\lim_{n\to\infty}\|(1-\alpha_n)x_n-T_{\lambda_n}u_n\|=0.$$

This implies that

$$\lim_{n\to\infty}\|x_n-T_{\lambda_n}u_n\|=0. \tag{3.18}$$

Put $\tilde{x}=P_{F(S)\cap EP}(0)$ (i.e, $\tilde{x}$ is the minimum norm element in $F(S)\cap EP$). We will finally show that $x_n\to\tilde{x}$.

Setting $v_n=x_n-\frac{\lambda_n}{1-\alpha_n}(Ax_n-A\tilde{x})$ for all n. Taking $z=\tilde{x}$ in (3.17) to get $\|Ax_n-A\tilde{x}\|\to 0$. First, we prove $\limsup_{n\to\infty}\langle\tilde{x},v_n-\tilde{x}\rangle\ge 0$. We take a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle\tilde{x},v_n-\tilde{x}\rangle=\lim_{i\to\infty}\langle\tilde{x},v_{n_i}-\tilde{x}\rangle.$$

It is clear that $\{v_{n_i}\}$ is bounded due to the boundedness of $\{x_n\}$ and $\|Ax_n-A\tilde{x}\|\to 0$. Then, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to some point $w\in C$. Hence, $\{x_{n_{i_j}}\}$ also converges weakly to w. At the same time, from (3.18) and $\|z_{n_{i_j}}-x_{n_{i_j}}\|=\|ST_{\lambda_{n_{i_j}}}((1-\alpha_{n_{i_j}})x_{n_{i_j}}-\lambda_{n_{i_j}}Ax_{n_{i_j}})-x_{n_{i_j}}\|\to 0$, we have

$$\lim_{j\to\infty}\|x_{n_{i_j}}-Sx_{n_{i_j}}\|=0. \tag{3.19}$$

By the demi-closedness principle of the nonexpansive mapping (see Lemma 2.5) and (3.19), we deduce $w \in F(S)$. Furthermore, by the similar argument as that of Theorem 3.1, we can show that w is also in EP. Hence, we have $w \in F(S) \cap EP$. This implies that

$$\limsup_{n\to\infty} \langle \tilde{x}, v_n - \tilde{x}\rangle = \lim_{j\to\infty} \langle \tilde{x}, v_{n_{i_j}} - \tilde{x}\rangle = \langle \tilde{x}, w - \tilde{x}\rangle.$$

Note that $\tilde{x} = P_{F(S)\cap EP}(0)$. Then, $\langle \tilde{x}, w - \tilde{x}\rangle \geq 0, w \in F(S) \cap EP$. Therefore,

$$\limsup_{n\to\infty} \langle \tilde{x}, v_n - \tilde{x}\rangle \geq 0.$$

From (3.12), we have

$$\begin{aligned}
&\|x_{n+1} - \tilde{x}\|^2 \\
\leq\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\|ST_{\lambda_n}u_n - \tilde{x}\|^2 \\
\leq\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\|T_{\lambda_n}u_n - \tilde{x}\|^2 \\
=\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\|T_{\lambda_n}u_n - T_{\lambda_n}(\tilde{x} - \lambda_n A\tilde{x})\|^2 \\
\leq\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\|u_n - (\tilde{x} - \lambda_n A\tilde{x})\|^2 \\
=\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\|(1-\alpha_n)x_n - \lambda_n Ax_n - (\tilde{x} - \lambda_n A\tilde{x})\|^2 \\
=\ & (1-\beta_n)\Big\|(1-\alpha_n)\Big((x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) - (\tilde{x} - \frac{\lambda_n}{1-\alpha_n}A\tilde{x})\Big) - \alpha_n\tilde{x}\Big\|^2 \\
& +\beta_n\|x_n - \tilde{x}\|^2 \\
=\ & (1-\beta_n)\Big((1-\alpha_n)^2\Big\|(x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) - (\tilde{x} - \frac{\lambda_n}{1-\alpha_n}A\tilde{x})\Big\|^2 \\
& -2\alpha_n(1-\alpha_n)\Big\langle \tilde{x}, (x_n - \frac{\lambda_n}{1-\alpha_n}Ax_n) - (\tilde{x} - \frac{\lambda_n}{1-\alpha_n}A\tilde{x})\Big\rangle + \alpha_n^2\|\tilde{x}\|^2\Big) \\
& +\beta_n\|x_n - \tilde{x}\|^2 \\
\leq\ & \beta_n\|x_n - \tilde{x}\|^2 + (1-\beta_n)\Big((1-\alpha_n)^2\|x_n - \tilde{x}\|^2 \\
& -2\alpha_n(1-\alpha_n)\Big\langle \tilde{x}, x_n - \frac{\lambda_n}{1-\alpha_n}(Ax_n - A\tilde{x}) - \tilde{x}\Big\rangle + \alpha_n^2\|\tilde{x}\|^2\Big) \\
\leq\ & [1-(1-\beta_n)\alpha_n]\|x_n - \tilde{x}\|^2 \\
& +(1-\beta_n)\alpha_n\Big\{-2(1-\alpha_n)\langle \tilde{x}, v_n - \tilde{x}\rangle + \alpha_n\|\tilde{x}\|^2\Big\}.
\end{aligned}$$

It is clear that $\sum_n (1-\beta_n)\alpha_n = \infty$ and $\limsup_{n\to\infty}(-2(1-\alpha_n)\langle \tilde{x}, v_n - \tilde{x}\rangle + \alpha_n\|\tilde{x}\|^2) \leq 0$. We can therefore apply Lemma 2.6 to conclude that $x_n \to \tilde{x}$. This completes the proof. ■

Corollary 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions (H1)-(H4). Let*

$A : C \to H$ be an α-inverse-strongly monotone mapping. Suppose that $EP \neq \emptyset$. Let $x_0 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - (1-\alpha_n)x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(ii) $0 < c \leq \beta_n \leq d < 1$;

(iii) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$ *where* $[a, b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to the minimum norm element x^* in EP.

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to R$ be a bifunction satisfying conditions (H1)-(H4). Suppose that $EP \neq \emptyset$. Let $x_0 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - (1-\alpha_n-\lambda_n)x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(ii) $0 < c \leq \beta_n \leq d < 1$;

(iii) $a(1-\alpha_n) \leq \lambda_n \leq b(1-\alpha_n)$ *where* $[a, b] \subset (0, 2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to the minimum norm element x^* in EP.

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