TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 4, pp. 1171-1184, August 2014
DOI: 10.11650/tjm.18.2014.3917
This paper is available online at http://journal.taiwanmathsoc.org.tw

JANOWSKI STARLIKENESS IN SEVERAL COMPLEX VARIABLES AND COMPLEX HILBERT SPACES

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Abstract. In this paper, we consider two new subclasses, $S^*(a, b, B^n)$ and $\mathcal{A}S^*(a, b, B^n)$, of the class of starlike mappings on B^n ($a, b \in \mathbb{R}$, $|a-1| < b \leq a$, and B^n is the Euclidean unit ball in $\mathbb{C}^n$). The class $S^*(a, b, B)$ is the n-dimensional version of Janowski class of one variable starlike functions. We obtain sharp growth results and upper distortion estimates for these two classes of starlike mappings. We also derive sufficient conditions for normalized holomorphic mappings (expressed in terms of their coefficient bounds) to belong to one of the classes $S^*(a, b, B^n)$, respectively $\mathcal{A}S^*(a, b, B^n)$. Finally, similar notions on the unit ball in a complex Hilbert space are analogously presented.

1. INTRODUCTION

It is known that many results in univalent function theory cannot be extended without restrictions to higher dimensions. We can mention in this direction the failure of the Riemann mapping theorem, the inexistence of growth or distortion theorems or coefficients bounds for the entire class of biholomorphic mappings defined on the unit ball of $\mathbb{C}^n$. This is one of the reasons to study particular subclasses of the full class of biholomorphic mappings.

In [7], W. Janowski introduced the following class of univalent normalized functions defined on the unit disk U of the complex plane.

If $A, B \in \mathbb{R}$, $-1 \leq B < A \leq 1$, then

$$S^*[A, B] = \left\{ f \in \mathcal{H}(U),\ f(0) = 0,\ f'(0) = 1,\ \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\}.$$

Geometrically, the above subordination condition means that the image of the unit disk U by the function $\frac{zf'(z)}{f(z)}$ is in the open disk whose two diameter endpoints are

Received October 17, 2013, accepted January 6, 2014.
Communicated by Alexander Vasiliev.
2010 *Mathematics Subject Classification*: 32H02, 30C45.
Key words and phrases: Biholomorphic mapping, Locally biholomorphic mapping, Starlike mapping, Almost starlike mapping.

$\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$. Particular choices of A, B lead us to the following classes: $S^*[1,-1]$ is the class of starlike functions, $S^*[1-2\alpha,-1]$ is the class of starlike functions of order α, $0 \le \alpha < 1$.

Closely related to Janowski starlike class of functions is the following class of univalent functions [17, 18].

If $a, b \in \mathbb{R}$, $a \ge b$ then

$$S^*(a,b) = \left\{ f \in \mathcal{H}(U),\ f(0)=0,\ f'(0)=1,\ \left| \frac{zf'(z)}{f(z)} - a \right| < b \right\}.$$

In [18], the authors show that $S^*[A,B] = S^*(a,b)$ under suitable relations between the parameters.

Various results concerning the classes $S^*[A,B]$ and $S^*(a,b)$ can be found in [7, 17, 18, 1] and [15] (and the references therein).

In this paper, starting from the Janowski class of starlike functions of one complex variable, we introduce and study two subclasses of starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$ and on the unit ball in a complex Hilbert space. We obtain sharp growth results and upper distortion estimates for these two classes of starlike mappings. We also derive sufficient conditions for normalized holomorphic mappings to belong to one of the classes mentioned before.

2. Preliminaries

Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, z_2, \dots, z_n)$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The open unit ball $\{z \in \mathbb{C}^n :\ \|z\| < 1\}$ is denoted by B^n. Let $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ be the space of linear continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm, $\|A\| = \sup\{\|A(z)\| :\ \|z\| = 1\}$ and let $I_n = I$ be the identity in $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$.

We denote by $\mathcal{H}(B^n)$ the set of holomorphic mappings from B^n into $\mathbb{C}^n$. If $f \in \mathcal{H}(B^n)$ we say that f is normalized if $f(0) = 0$ and the complex Jacobian matrix of f at $z = 0$, $Df(0)$, is the identity operator I. If $f \in \mathcal{H}(B^n)$ is a normalized mapping, then f has the following series expansion: $f(z) = z + \sum_{k=2}^\infty A_k(z^k),\ z \in B^n$, where $A_k(w^k) = \frac{1}{k!} D^k f(0)(w^k)$, and $D^k(0)(w^k)$ is the k^{th} order Fréchet derivative of f at $z = 0$.

The mapping $f \in \mathcal{H}(B^n)$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B^n)$. It is known that any univalent mapping on B^n (holomorphic and injective) is biholomorphic on B^n. Let $S(B^n)$ be the set of normalized biholomorphic mappings on B^n and let $S^*(B^n)$ be the set of biholomorphic normalized starlike mappings with respect to zero.

We next review some subclasses of $S^*(B^n)$ that will be used in the next sections.

If $f \in \mathcal{H}(B^n)$, we say that f is locally biholomorphic on B^n if $\det Df(z) \ne 0$, $z \in B^n$. Let $\mathcal{L}S(B^n)$ be the set of normalized locally biholomorphic mappings on B^n.

It is well-known [16] that a locally biholomorphic mapping $f \in \mathcal{H}(B^n)$ such that $f(0) = 0$ is starlike if and only if $\mathrm{Re}\ \langle (Df(z))^{-1} f(z), z\rangle > 0,\ z \in B^n \setminus \{0\}$.

We present now the notion of starlikeness of order $\alpha \in [0,1)$ that was introduced in [2, 8].

Definition 2.1. Let $f \in \mathcal{L}S(B^n)$ and let $\alpha \in [0,1)$. The mapping f is said to be starlike of order α if

$$\mathrm{Re}\ \frac{\|z\|^2}{\langle (Df(z))^{-1} f(z), z\rangle} > \alpha,\ z \in B^n \setminus \{0\}.$$

We denote by $S^*_\alpha(B^n)$ the class of mappings that are starlike of order α on B^n.

The following notion of starlikeness was introduced in [9, 6, 4].

Definition 2.2. Let $f \in \mathcal{L}S(B^n)$ and let $\alpha \in [0,1)$. We say that f is almost starlike of order α if

$$\mathrm{Re}\ \frac{\langle (Df(z))^{-1} f(z), z\rangle}{\|z\|^2} > \alpha,\ z \in B^n \setminus \{0\}.$$

We denote by $\mathcal{A}S^*_\alpha(B^n)$ the class of mappings of order α on B^n.

It is obvious that f is starlike of order 0 on B^n if and only if f is almost starlike of order 0 on B^n. Moreover, we have that $S^*_0(B^n) = \mathcal{A}S^*_0(B^n) = S^*(B^n)$.

The following set of normalized mappings is the generalization to n-complex variables of the well-known Carathéodory class of one variable holomorphic functions with positive real part on the unit disk of complex plane.

$$\mathcal{M} = \{h \in \mathcal{H}(B^n) :\ h(0) = 0,\ Dh(0) = I,\ \mathrm{Re}\ \langle h(z), z\rangle > 0,\ z \in B^n \setminus \{0\}\}.$$

The class $\mathcal{M}$, introduced in [14] plays a fundamental role in the study of the Loewner differential equation. Also, it is closely related to certain subclasses of biholomorphic mappings on B, such as starlike mappings, mappings with parametric representation [5], etc.

We introduce now some subclasses of the class $\mathcal{M}$ [5, 10, 6].

Definition 2.3. Let $g \in \mathcal{H}(U)$ be an univalent function, such that $g(0) = 1$, $g(\overline{\zeta}) = \overline{g(\zeta)}$ for $\zeta \in U$, $\mathrm{Re}\ g(\zeta) > 0$ on U, and assume that g satisfies the following conditions for $r \in (0,1)$:

$$\begin{cases} \min\limits_{|\zeta|=r} \mathrm{Re}\ g(\zeta) = \min\{g(r), g(-r)\} \\ \max\limits_{|\zeta|=r} \mathrm{Re}\ g(\zeta) = \max\{g(r), g(-r)\}. \end{cases}$$

Let $\mathcal{M}_g$ be the class of holomorphic mappings given by

$$\mathcal{M}_g = \left\{h \in \mathcal{H}(B^n) : h(0) = 0,\ Dh(0) = I,\ \frac{1}{\|z\|^2}\langle h(z), z\rangle \in g(U),\ z \in B \setminus \{0\}\right\}.$$

It is clear that $\mathcal{M}_g \subseteq \mathcal{M}$ and for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$ it follows that $\mathcal{M}_g = \mathcal{M}$. Particular choices of the function g (that satisfies Definition 2.3) provide various subclasses of class $\mathcal{M}$.

Remark 2.4.

(a) Let $0 \le \alpha < 1$, let $g : U \to \mathbb{C}$ be the function defined by $g(\zeta) = \frac{1-\zeta}{1-(2\alpha-1)\zeta}$ and let $f \in \mathcal{L}S(B^n)$. Then $f \in S^*_\alpha(B^n)$ if and only if $[Df(z)]^{-1}f(z) \in \mathcal{M}_g$.

(b) Let $0 \le \alpha < 1$, let $g : U \to \mathbb{C}$ be the function defined by $g(\zeta) = \frac{1+(2\alpha-1)\zeta}{1+\zeta}$ and let $f \in \mathcal{L}S(B^n)$. Then $f \in \mathcal{A}S^*_\alpha(B^n)$ if and only if $[Df(z)]^{-1}f(z) \in \mathcal{M}_g$.

If g is an univalent function which satisfies the conditions in Definition 2.3 we denote by $S^*_g(B^n)$ the subset of $S^*(B^n)$ consisting of the normalized locally biholomorphic mappings f such that $(Df(z))^{-1}f(z) \in \mathcal{M}_g$.

The following result [5] is the growth theorem for the class $S^*_g(B^n)$.

Theorem 2.5. *Let $g : U \to \mathbb{C}$ be a holomorphic univalent function which satisfies the conditions in Definition 2.3.*

If $f \in S^*_g(B^n)$ then $\|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1\right] \frac{dx}{x} \le \|f(z)\| \le \|z\| \int_0^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right] \frac{dx}{x},\ z \in B.$

3. Growth and Distortion Estimates for Mappings in the Classes $S^*(a, b, B^n)$ and $\mathcal{A}S^*(a, b, B^n)$

In this section we introduce and study the following classes of biholomorphic starlike mappings on B^n.

Definition 3.1. Let $a, b \in \mathbb{R}$ such that $|a - 1| < b \le a$.

$$S^*(a, b, B^n) = \left\{ f \in \mathcal{L}S(B^n) :\ \left| \frac{\|z\|^2}{\langle (Df(z))^{-1}f(z), z\rangle} - a \right| < b,\ z \in B^n \setminus \{0\} \right\}$$

$$\mathcal{A}S^*(a, b, B^n) = \left\{ f \in \mathcal{L}S(B^n) :\ \left| \frac{\langle (Df(z))^{-1}f(z), z\rangle}{\|z\|^2} - a \right| < b,\ z \in B^n \setminus \{0\} \right\}.$$

We mention that any mapping in $S^*(a, b, B^n)$ or in $\mathcal{A}S^*(a, b, B^n)$ is starlike since the conditions satisfied by the real numbers a, b in Definition 3.1 imply that the disk of center a and radius b is a subset of the right half plane. We also mention that for $n = 1$ the class $S^*(a, b, B^n)$ reduces to the class $S^*(a, b)$.

We next analyse the relationships between the two subclasses of starlike mappings introduced before.

Remark 3.2. Let $a, b \in \mathbb{R}$ such that $|a-1| < b \leq a$.
Then the following assertions are true:
(i) $S^*(a,b,B^n) = \mathcal{A}S^*(\frac{a}{a^2-b^2}, \frac{b}{a^2-b^2}, B^n)$, if $a \neq b$;
(ii) $S^*(a,a,B^n) = \mathcal{A}S^*_{1/2a}(B^n)$;
(iii) $\mathcal{A}S^*(a,b,B^n) = S^*(\frac{a}{a^2-b^2}, \frac{b}{a^2-b^2}, B^n)$, if $a \neq b$;
(iv) $\mathcal{A}S^*(a,a,B^n) = S^*_{1/2a}(B^n)$.

Proof. The assertions (i) and (iii) easily follow by using the fact that the disk of center a and radius b is mapped by the function $1/\zeta$ onto the disk of center $\frac{a}{a^2-b^2}$ and radius $\frac{b}{a^2-b^2}$.

The assertions (ii) and (iv) immediately follow from the fact that the disk of center a and radius a is mapped by the function $1/\zeta$ onto the half-plane $\left\{\zeta \mid \operatorname{Re}\zeta > \frac{1}{2a}\right\}$.

In the next remark we present the appropriate univalent functions g such that the classes $S^*(a,b,B^n)$ and $\mathcal{A}S^*(a,b,B^n)$ can be rewritten as $S^*_g(B^n)$.

Remark 3.3. (i) Let $-1 \leq B < A \leq 1$ and let $g \in \mathcal{H}(U)$ be defined by $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$. Then:

$$S^*_g(B^n) = \mathcal{A}S^*\left(\frac{1-AB}{1-B^2}, \frac{A-B}{1-B^2}, B^n\right) \text{ for } B = -1 \text{ and} \tag{3.1}$$
$$S^*_g(B^n) = \mathcal{A}S^*_{(1-A)/2}(B^n) \text{ for } B \neq 1.$$

$$S^*_g(B^n) = S^*\left(\frac{1-AB}{1-A^2}, \frac{A-B}{1-A^2}, B^n\right) \text{ for } A \neq 1 \text{ and} \tag{3.2}$$
$$S^*_g(B^n) = S^*_{(1+B)/2}(B^n) \text{ for } A = 1.$$

(ii) Let $a, b \in \mathbb{R}$ such that $|a-1| < b \leq a$. Then

$$\mathcal{A}S^*(a,b,B^n) = S^*_g(B^n), \tag{3.3}$$

where $g(\zeta) = \frac{1+(a-a^2+b^2)/b\zeta}{1+(1-a)/b\zeta}$, $\zeta \in U$

$$S^*(a,b,B^n) = S^*_g(B^n), \tag{3.4}$$

where $g(\zeta) = \frac{1+(a-1)/b\zeta}{1+(a^2-a-b^2)/b\zeta}$, $\zeta \in U$.

Proof. The equalities in (3.1) easily follows from the fact that the unit disk is mapped by the function $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ onto the disk centered at $a = \frac{1-AB}{1-B^2}$ with radius $b = \frac{A-B}{1-B^2}$ in the case when $B \neq -1$, respectively onto the half-plane $\left\{\zeta \mid \operatorname{Re}\zeta > \frac{1-A}{2}\right\}$ in the case when $B = -1$.

The equalities in (3.2) can be analogously justified, eventually by using the relationships between the two subclasses presented in Remark 3.2.

To prove (3.3) we have to find A, B $(-1 < B < A \leq 1)$ such that $a = \frac{1-AB}{1-B^2}$ and $b = \frac{A-B}{1-B^2}$. Elementary computations lead to the following equation $bB^3 + (a-1)B^2 - bB + 1 - a = 0$, whose solutions are $B = 1$, $B = -1$ and $B = \frac{1-a}{b}$. Since $-1 < B < 1$ ($B = -1$ would imply that $g(U)$ is a half-plane, which is not the case here) we conclude that $B = \frac{1-a}{b}$ and hence $A = \frac{a-a^2+b^2}{b}$.

The equality in (3.4) is an immediate consequence of (3.3) and assertion (iii) from Remark 3.2.

Next, we will give the growth theorems for the mappings in the classes $\mathcal{A}S^*(a, b, B^n)$ and $S^*(a, b, B^n)$. Due to the relationships among the classes of mappings presented in Remark 3.3, it is sufficient to obtain the growth theorem for the mappings in the class $S_g^*(B^n)$ (where $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ and $-1 \leq B < A \leq 1$). To this end we will apply Theorem 2.5 the function g to be $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$, where $-1 \leq B < A \leq 1$.

Lemma 3.4. *Let* $-1 \leq B < A \leq 1$, $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$, *and let* $f \in S_g^*(B^n)$. *If* $A \neq 0$ *then:*

$$\frac{\|z\|}{(1+A\|z\|)^{\frac{A-B}{A}}} \leq \|f(z)\| \leq \frac{\|z\|}{(1-A\|z\|)^{\frac{A-B}{A}}}, \; z \in B^n. \tag{3.5}$$

If $A = 0$ *then*

$$\|z\| \exp(B\|z\|) \leq \|f(z)\| \leq \|z\| \exp(-B\|z\|), \; z \in B^n. \tag{3.6}$$

The estimations in (3.5) and (3.6) are sharp.

Proof. It is clear that the function g satisfies the conditions in Definition 2.3. We also have that $\min_{|\zeta|=r} g(\zeta) = g(-r) = \frac{1-Ar}{1-Br}$ and $\max_{|\zeta|=r} g(\zeta) = g(r) = \frac{1+Ar}{1+Br}$.

In the case when $A \neq 0$, elementary computations give us: $\|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right] \frac{dx}{x} = \|z\| \int_0^{\|z\|} \frac{A-B}{1-Ax} dx = \frac{\|z\|}{(1-A\|z\|)^{(A-B)/A}}$ and $\|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1\right] \frac{dx}{x} = \|z\| \int_0^{\|z\|} \frac{B-A}{1+Ax} dx = \frac{\|z\|}{(1+A\|z\|)^{(A-B)/A}}$.

By applying now Theorem 2.5 we get that the inequalities (3.5) are fulfilled.

In the case $A = 0$, analogously we get $\|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\min\{g(x), g(-x)\}} - 1\right] \frac{dx}{x}$ $= \|z\| \exp(-B\|z\|)$ and $\|z\| \exp \int_0^{\|z\|} \left[\frac{1}{\max\{g(x), g(-x)\}} - 1\right] \frac{dx}{x} = \|z\| \exp(B\|z\|)$, hence inequalities (3.6) are satisfied.

These estimations are sharp with the extremal functions $f(z) = \left(\frac{z_1}{(1+Az_1)^{\frac{A-B}{A}}}, z_2, \ldots, z_n\right)$, $z = (z_1, \ldots, z_n) \in B^n (A \neq 0)$, respectively $f(z) = (z_1 \exp(Bz_1), z_2, \ldots, z_n)$, $z = (z_1, \ldots, z_n) \in B^n (A = 0)$.

Now, by applying Lemma 3.4 we derive the growth estimates for the mappings in the classes $S^*(a,b,B^n)$ and $\mathcal{A}S^*(a,b,B^n)$.

Theorem 3.5. *Let* $a,b \in \mathbb{R}$ *such that* $|a-1| < b \le a$.

(i) *Let* $f \in S^*(a,b,B^n)$. *If* $a \neq 1$ *then*

$$\frac{\|z\|}{(1-\frac{1-a}{b}\|z\|)^{\frac{(1-a)^2-b^2}{1-a}}} \le \|f(z)\| \le \frac{\|z\|}{(1+\frac{1-a}{b}\|z\|)^{\frac{(1-a)^2-b^2}{1-a}}}, z \in B^n.$$

If $a = 1$ *then*

$$\|z\| \exp\{-b\|z\|\} \le \|f(z)\| \le \|z\| \exp\{b\|z\|\},\ z \in B^n.$$

(ii) *Let* $f \in \mathcal{A}S^*(a,b,B^n)$. *If* $a \neq \frac{1+\sqrt{1+4b^2}}{2}$ *then*

$$\frac{\|z\|}{(1+\frac{a-a^2+b^2}{b}\|z\|)^{\frac{b^2-(1-a)^2}{b^2-a^2+a}}} \le \|f(z)\| \le \frac{\|z\|}{(1-\frac{a-a^2+b^2}{b}\|z\|)^{\frac{b^2-(1-a)^2}{b^2-a^2+a}}},\ z \in B^n.$$

If $a = \frac{1+\sqrt{1+4b^2}}{2}$ *then*

$$\|z\| \exp(\frac{1-\sqrt{1+4b^2}}{2b}\|z\|) \le \|f(z)\| \le \|z\| \exp(\frac{\sqrt{1+4b^2}-1}{2b}\|z\|), z \in B^n.$$

All the previous estimations are sharp.

Proof. The above estimations and extremal functions are immediate consequences of Remark 3.3 and Lemma 3.4.

From Theorem 3.5 we obtain the following corollary [2], [8]. This result is obtained by taking $a = b = \frac{1}{2\alpha}$, $0 < \alpha < 1$, in Theorem 3.5.

Corollary 3.6. *Let* $\alpha \in \mathbb{R}$, $0 < \alpha < 1$.

(i) *Let* $f \in S^*_\alpha(B^n)$. *Then*

$$\frac{\|z\|}{(1+\|z\|)^{2(1-\alpha)}} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^{2(1-\alpha)}},\ z \in B^n.$$

(ii) *Let* $f \in \mathcal{A}S^*_\alpha(B^n)$. *If* $\alpha \neq \frac{1}{2}$ *then*

$$\frac{\|z\|}{(1+(1-2\alpha)\|z\|)^{\frac{2(1-\alpha)}{1-2\alpha}}} \le \|f(z)\| \le \frac{\|z\|}{(1-(1-2\alpha)\|z\|)^{\frac{2(1-\alpha)}{1-2\alpha}}},\ z \in B^n.$$

If $\alpha = \frac{1}{2}$ *then*

$$\|z\| \exp(-\|z\|) \le \|f(z)\| \le \|z\| \exp\|z\|,\ z \in B^n.$$

We give now an upper bound estimate of distortion theorem for the mappings in the classes $S^*(a,b,B^n)$ and $\mathcal{A}S^*(a,b,B^n)$, along an unit direction in $\mathbb{C}^n$.

In order to obtain the desired bounds, we shall derive first the following result.

Lemma 3.7. *Let* $-1 \le B < A \le 1$ *and let* $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$.
If $f \in S_g^*(B^n)$ *then:*

$$\|(Df(z))^{-1}f(z)\| \ge \|z\|\frac{1-A\|z\|}{1-B\|z\|},\ z \in B^n \setminus \{0\}.$$

Proof. Fix $z \in B^n \setminus \{0\}$. Let $h : U \to \mathbb{C}$ be the holomorphic function defined by $h(0) = 1$ and $h(\zeta) = \frac{1}{\zeta}\left\langle [Df(\zeta\frac{z}{\|z\|})]^{-1}f(\zeta\frac{z}{\|z\|}), \frac{z}{\|z\|}\right\rangle, \zeta \neq 0$. From the fact that $f \in S_g^*(B^n)$, the function defined by $\frac{2(A-B)}{1-B}\left(\frac{1}{(1-B)h(\zeta)-(1-A)} - \frac{1+B}{2(A-B)}\right)$, $\zeta \in U$ has positive real part on the unit disk of the complex plane and the estimate $\left|\frac{2(A-B)}{1-B}\cdot\frac{1}{(1-B)h(\zeta)-(1-A)} - \frac{1+B}{1-B} - \frac{1+|\zeta|^2}{1-|\zeta|^2}\right| \le \frac{2|\zeta|}{1-|\zeta|^2}$ holds for every $\zeta \in U$. By using the above inequality we easily obtain that $\frac{1-A|\zeta|}{1-B|\zeta|} \le \mathrm{Re}\ h(\zeta) \le \frac{1+A|\zeta|}{1+B|\zeta|}$, $\zeta \in U$. If in the previous inequalities we take $\zeta = \|z\|$ we get $\frac{1-A\|z\|}{1-B\|z\|} \le \frac{1}{\|z\|^2}\mathrm{Re}\ \langle (Df(z))^{-1}f(z), z\rangle \le \frac{1+A\|z\|}{1+B\|z\|}$, $z \in B^n$. Hence $\|z\|^2\frac{1-A\|z\|}{1-B\|z\|} \le \mathrm{Re}\ \langle (Df(z))^{-1}f(z), z\rangle \le \|z\| \cdot \|(Df(z))^{-1} f(z)\|$, as desired.

Theorem 3.8. *Let* $-1 \le B < A \le 1$ *and let* $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$.
If $f \in S_g^*(B^n)$ *then for every* $z \in B^n \setminus \{0\}$, *there exist an unit vector* $v(z) = \frac{(Df(z))^{-1}f(z)}{\|(Df(z))^{-1}f(z)\|}$ *such that*

$$\|Df(z)v(z)\| \le \frac{1-B\|z\|}{(1-A\|z\|)^{2-\frac{B}{A}}},\ \textit{for } A \neq 0$$

and

$$\|Df(z)v(z)\| \le \frac{1-B\|z\|}{1-A\|z\|}\exp(-B\|z\|),\ \textit{for } A = 0.$$

Proof. Fix $z \in B^n \setminus \{0\}$ and let $v(z) = \frac{(Df(z))^{-1}f(z)}{\|(Df(z))^{-1}f(z)\|}$. Then $f(z) = (Df(z))(Df(z))^{-1}f(z) = \|(Df(z))^{-1}f(z)\|Df(z)v(z)$.

Let $A \neq 0$. By applying now Lemma 3.7 and the upper estimate in (3.5) we deduce $\|(Df(z))v(z)\| = \frac{\|f(z)\|}{\|(Df(z))^{-1}f(z)\|} \le \frac{\frac{\|z\|}{(1-A\|z\|)^{(A-B)/A}}}{\|z\|\frac{1-A\|z\|}{1-B\|z\|}} = \frac{1-B\|z\|}{(1-A\|z\|)^{2-B/A}}$, as needed.

Let $A = 0$. Analogously, by applying Lemma 3.7 and the upper estimate in (3.6) we deduce $\|Df(z)v(z)\| \le \frac{\|z\|\exp(-B\|z\|)}{\|z\|\frac{1-A\|z\|}{1-B\|z\|}} = \frac{1-B\|z\|}{1-A\|z\|}\exp(-B\|z\|)$. This completes the proof. ∎

Now, by applying Theorem 3.8, we immediately deduce the upper distortion estimates (along an unit direction in $\mathbb{C}^n$) for the mappings in the classes $S^*(a,b,B^n)$ and $\mathcal{A}S^*(a,b,B^n)$.

Theorem 3.9. *Let $a,b\in\mathbb{R}$ be such that $|a-1|<b\le a$.*

(i) *If $f\in S^*(a,b,B^n)$ then for every $z\in B^n\setminus\{0\}$ there exist an unit vector $v(z)=\frac{(Df(z))^{-1}f(z)}{\|(Df(z))^{-1}f(z)\|}$ such that*

$$\|Df(z)v(z)\|\le\frac{1+\frac{b^2-a^2+a}{b}\|z\|}{\left(1-\frac{1-a}{b}\|z\|\right)^{2-\frac{b^2-a^2+a}{1-a}}},\ \textit{for } a\ne 1$$

and

$$\|Df(z)v(z)\|\le(1+\frac{1}{b}\|z\|)\exp(\|z\|),\ \textit{for } a=1.$$

(ii) *If $f\in\mathcal{A}S^*(a,b,B^n)$ then for every $z\in B^n\setminus\{0\}$, there exists an unit vector $v(z)=\frac{(Df(z))^{-1}f(z)}{\|(Df(z))^{-1}f(z)\|}$ such that*

$$\|Df(z)v(z)\|\le\frac{1-\frac{1-a}{b}\|z\|}{(1-\frac{b^2-a^2+a}{b}\|z\|)^{2-\frac{1-a}{b^2-a^2+a}}},\ \textit{for } a\ne\frac{1+\sqrt{1+4b^2}}{2}$$

and

$$\|Df(z)v(z)\|\le(1-\frac{1-\sqrt{1+4b^2}}{2b}\|z\|)\exp\frac{\sqrt{1+4b^2}-1}{2b}\|z\|,$$

for $a=\frac{1+\sqrt{1+4b^2}}{2}$.

From Theorem 3.8 we obtain the following corollary [11], [13]. This result can be obtained by taking $a=b=\frac{1}{2\alpha}$, $0<\alpha<1$, in Theorem 3.8.

Corollary 3.10. *Let $\alpha\in\mathbb{R}$, $0<\alpha<1$.*

(i) *If $f\in S^*_\alpha(B)$ then for every $z\in B^n\setminus\{0\}$, there exists an unit vector $v(z)=\frac{(Df(z))^{-1}f(z)}{\|(Df(z))^{-1}f(z)\|}$ such that*

$$\|Df(z)v(z)\|\le\frac{1+(1-2\alpha)\|z\|}{(1-\|z\|)^{2(1-\alpha)+1}}.$$

(ii) *If $f\in\mathcal{A}S^*_\alpha(B)$ then for every $z\in B^n\setminus\{0\}$. there exists an unit vector $v(z)=\frac{(Df(z))^{-1}f(z)}{\|(Df(z))^{-1}f(z)\|}$ such that*

$$\|Df(z)v(z)\|\le\frac{1+\|z\|}{(1-(1-2\alpha)\|z\|)^{\frac{2(\alpha-1)}{2\alpha-1}+1}},\ \textit{for } \alpha\ne\frac{1}{2}$$

and

$$\|Df(z)v(z)\|\le(1+\|z\|)\exp\|z\|,\ \textit{for } \alpha=\frac{1}{2}.$$

We end this section by mentioning that when $n = 1$ the previous results reduce to the upper bound of the classical distortion theorem on normalized starlike, respectively normalized almost starlike mappings of order α, $0 < \alpha < 1$.

4. Sufficient Conditions for $S^*(a, b, B^n)$ and $\mathcal{A}S^*(a, b, B^n)$ Membership

In this section we obtain sufficient conditions for a normalized holomorphic mapping to belong to one of the classes $S^*(a, b, B^n)$, respectively $\mathcal{A}S^*(a, b, B^n)$, where $a, b \in \mathbb{R}$, $|a - 1| < b \leq a$. These sufficient conditions are expressed in terms of coefficients bounds of the considered mapping. As before, we deduce first a sufficient condition for the class $S^*_g(B^n)$, where $g : U \to \mathbb{C}$ is defined by $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$, and $-1 < B < A \leq 1$.

Theorem 4.1. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B^n and let $g : U \to \mathbb{C}$ be the univalent function defined by $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$, $-1 < B < A \leq 1$.*

If

$$\sum_{k=2}^{\infty} \left[k - \frac{1+B}{1+A}\right] \|A_k\| \leq \frac{(A-B)(1-|B|)}{(1+A)(1-B)} \tag{4.1}$$

then $f \in S^*_g(B^n)$.

Proof. From the inequality (4.1) it follows that $\sum_{k=2}^{\infty} k\|A_k\| \leq \frac{2(1+A)}{2A-B+1} \sum_{k=2}^{\infty} \left[k - \frac{1+B}{1+A}\right] \|A_k\| \leq \frac{2(A-B)(1-|B|)}{(2A+1-B)(1-B)} < 1$. By direct computation of Fréchet derivatives of f we obtain $\|Df(z) - I\| = \left\|\sum_{k=2}^{\infty} kA_k(z^{k-1}, \cdot)\right\| \leq \sum_{k=2}^{\infty} k\|A_k\| < 1,\ z \in B^n$. Hence we obtain that $Df(z) = I - (I - Df(z))$ is an invertible linear operator and

$$\|(Df(z))^{-1}\| \leq \frac{1}{1 - \|I - Df(z)\|} \leq \frac{1}{1 - \sum_{k=2}^{\infty} k\|A_k\|},\ z \in B^n. \tag{4.2}$$

For every $z \in B^n \setminus \{0\}$, we have $\|\frac{1-B^2}{A-B} f(z) - \frac{1-AB}{A-B} Df(z)(z)\| = \|Bz + \sum_{k=2}^{\infty} \left(\frac{1-B^2}{A-B} - \frac{1-AB}{A-B} k\right) A_k(z^k)\| < \|z\|(|B| + \frac{1}{A-B} \sum_{k=2}^{\infty} [k(1-AB) - (1-B^2)]\|A_k\|)$.

By using the inequality (4.2) and the previous inequality we obtain $\frac{1}{\|z\|^2} \left| \left\langle \frac{1-B^2}{A-B} (Df(z))^{-1} f(z), z \right\rangle - \frac{1-AB}{A-B} \|z\|^2 \right| = \frac{1}{\|z\|^2} \left| \left\langle (Df(z))^{-1} \left(\frac{1-B^2}{A-B} f(z) - \frac{1-AB}{A-B} Df(z)(z) \right), z \right\rangle \right| \leq \frac{1}{\|z\|} \|(Df(z))^{-1}\| \cdot \left\| \frac{1-B^2}{A-B} f(z) - \frac{1-AB}{A-B} Df(z)(z) \right\| < \frac{|B| + \frac{1}{A-B} \sum_{k=2}^{\infty} [k(1-AB) - (1-B^2)]\|A_k\|}{1 - \sum_{k=2}^{\infty} \|A_k\| k} \leq 1$, where for the last inequality we have used the relation (4.1).

In conclusion

$$\left|\left\langle \frac{(Df(z))^{-1}f(z)}{\|z\|^2}, z\right\rangle - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2},\ z \in B^n \setminus \{0\},$$

which implies that $f \in S_g^*(B^n)$.

By taking $A = \frac{a-a^2+b^2}{b}$ and $B = \frac{1-a}{b}$ in Theorem 4.1 and by using Remark 3.3 we obtain the following sufficient condition for a holomorphic mapping to belong to $\mathcal{A}S^*(a,b,B^n)$.

Theorem 4.2. *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ *be a holomorphic mapping on* B^n *and let* $a, b \in \mathbb{R}$ *such that* $|a-1| < b \leq a$.

If

$$\sum_{k=2}^{\infty}\left[k - \frac{1}{a+b}\right]\|A_k\| \leq \frac{b-|1-a|}{a+b}$$

then $f \in \mathcal{A}S^*(a,b,B)$.

If in the previous theorem we take $a = b = \frac{1}{2\alpha}$, $0 < \alpha < 1$, we get the next result [12].

Corollary 4.3. *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ *be a locally biholomorphic mapping on* B^n *and let* $\alpha \in \mathbb{R}$ *be such that* $0 < \alpha < 1$.

If

$$\sum_{k=2}^{\infty}(k-\alpha)\|A_k\| \leq \frac{1-|2\alpha-1|}{2}$$

then $f \in S_\alpha^*(B^n)$.

If we take $A = \frac{a-1}{b}$ and $B = \frac{a^2-a-b^2}{b}$ in Theorem 4.1 we get the following sufficient condition for a holomorphic mapping to be in $S^*(a,b,B^n)$.

Theorem 4.4. *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ *be a holomorphic mapping on* B^n *and let* $a, b \in \mathbb{R}$ *such that* $|a-1| < b < a$.

If

$$\sum_{k=2}^{\infty}[k-(a-b)]\|A_k\| \leq \frac{b-|b^2-a^2+a|}{a+b}$$

then $f \in S^*(a,b,B^n)$.

5. Subclasses $S^*(a,b,B)$ and $\mathcal{A}S^*(a,b,B)$ in Complex Hilbert Spaces

Suppose that X is a complex Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. Let $B = \{z \in X :\ \|z\| < 1\}$ be the unit ball in X.

Let $a, b \in \mathbb{R}$, $|a-1| < b \le a$.

A mapping f is in $S^*(a,b,B)$, if and only if f is a locally biholomorphic mapping on B and satisfies

$$\left|\frac{\|z\|^2}{\langle (Df(z))^{-1}f(z), z\rangle} - a\right| < b,\ z \in B \setminus \{0\}.$$

A mapping f is in $\mathcal{A}S^*(a,b,B)$ if and only if f is a locally biholomorphic mapping on B and satisfies

$$\left|\frac{\langle (Df(z))^{-1}f(z), z\rangle}{\|z\|^2} - z\right| < b,\ z \in B \setminus \{0\}.$$

We mention that all results presented in the previous sections remain valid, with similar proofs, for holomorphic mappings defined on the unit ball of a Hilbert space.

We will present here only the main results related to the class $S^*(a,b,B)$ (similar estimations concerning the class $\mathcal{A}S^*(a,b,B)$ may be obtained analogously to those presented in the previous sections).

Theorem 5.1. *Let $a, b \in \mathbb{R}$ be such that $|a-1| < b \le a$ and let $f \in S^*(a,b,B)$.*
(i) If $a \neq 1$ then

$$\frac{\|z\|}{\left(1 - \frac{1-a}{b}\|z\|\right)^{\frac{(1-a)^2-b^2}{1-a}}} \le \|f(z)\| \le \frac{\|z\|}{\left(1 + \frac{1-a}{b}\|z\|\right)^{\frac{(1-a)^2-b^2}{1-a}}},\ z \in B.$$

If $a = 1$ then

$$\|z\| \exp\{-b\|z\|\} \le \|f(z)\| \le \|z\| \exp\{b\|z\|\},\ z \in B.$$

(ii) For every $z \in B \setminus \{0\}$ there exists an unit vector $v(z) = \frac{(Df(z))^{-1}f(z)}{\|(Df(z))^{-1}f(z)\|}$ such that

$$\|Df(z)v(z)\| \le \frac{1 + \frac{b^2-a^2+a}{b}\|z\|}{\left(1 + \frac{1-a}{b}\|z\|\right)^{2-\frac{b^2-a^2+a}{1-a}}},\ \textit{for } a \neq 1$$

and

$$\|Df(z)v(z)\| \le (1 + \frac{1}{b}\|z\|) \exp\|z\|,\ \textit{for } a = 1.$$

Theorem 5.2. *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on B and let $a, b \in \mathbb{R}$ such that $|a-1| < b < a$.*

If

$$\sum_{k=2}^{\infty}[k - (a-b)]\|A_k\| \le \frac{b - |b^2 - a^2 + a|}{a+b}$$

then $f \in S^*(a,b,B)$.

ACKNOWLEDGMENTS

This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0899.

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