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THE EQUITABLE CHROMATIC THRESHOLD OF THE CARTESIAN PRODUCT OF BIPARTITE GRAPHS IS AT MOST 4

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Abstract. A graph G is equitably k-colorable if its vertex set can be partitioned into k independent sets, any two of which differ in size by at most 1. We prove a conjecture of Lin and Chang which asserts that for any bipartite graphs G and H, their Cartesian product $G\square H$ is equitably k-colorable whenever $k \geq 4$.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, simple and non-trivial. We assume that all variables present positive integers. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *k-coloring* of G is a mapping $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever $xy \in E(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G admits a k-coloring. We call the set $f^{-1}(i) = \{x \in V(G) \colon f(x) = i\}$ a *color class* for each $i = 1, 2, \ldots, k$. Notice that each color class is an independent set, i.e., a pairwise non-adjacent subset of $V(G)$, and hence a k-coloring is a partition of $V(G)$ into k independent sets. An *equitable k-coloring* of G is a k-coloring for which any two color classes differ in size by at most one, or equivalently, each color class is of size $\lfloor |V(G)|/k \rfloor$ or $\lceil |V(G)|/k \rceil$. A graph is *equitably k-colorable* if it admits an equitable k-coloring. The *equitable chromatic number* of G, denoted by $\chi_=(G)$, is the smallest integer k such that G is equitably k-colorable. The concept of equitable colorability was first introduced by Meyer [5]. Unlike the ordinary colorability, an equitably k-colorable graph may admit no equitable k'-coloring for some $k' > k$. A typical example is the complete bipartite graph $K_{n,n}$ where $n \geq 3$ is odd, which is clearly equitably 2-colorable but not equitably n-colorable. This phenomena suggests the concept of equitable chromatic threshold.

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The *equitable chromatic threshold* of G, denoted by $\chi^*_=(G)$, is the smallest integer k such that G is equitably k'-colorable for all $k' \geq k$. The notion of equitable coloring has received a lot of attention and we refer to [3] for a good survey.

For two graphs G and H, the *Cartesian product* $G\Box H$ of G and H is the graph with vertex set $\{(x,y)\colon x \in V(G), y \in V(H)\}$ and edge set $\{(x,y)(x',y')\colon (x = x'$ and $yy' \in E(H))$ or $(xx' \in E(G)$ and $y = y')\}$. Sabidussi [6] showed that $\chi(G\Box H) = \max\{\chi(G), \chi(H)\}$. However, the analogous question for the equitable colorability is less satisfactory. The following result due to Chen, Lih and Yan [1] gives a partial answer.

Theorem 1. ([1, Theorem 4]). *If G and H are equitably k-colorable, then so is $G\Box H$.*

Theorem 1 immediately implies the following upper bounds on $\chi_=(G\Box H)$ and $\chi^*_=(G\Box H)$.

Corollary 2. $\chi_=(G\Box H) \leq \min\{k\colon$ *both G and H are equitably k-colorable*$\}$.

Corollary 3. $\chi^*_=(G\Box H) \leq \max\{\chi^*_=(G), \chi^*_=(H)\}$.

Chen et al. [1] also gave the exact values of $\chi_=(G\Box H)$ and $\chi^*_=(G\Box H)$ when G and H are both complete graphs or both cycles, and Furmańczyk [2] gave $\chi(G\Box H) = \chi_=(G\Box H) = \chi^*_=(G\Box H) = \max\{\chi(G), \chi(H)\}$ when G and H are cycles, paths, hypercubes, or complete graphs, and $\chi_=(K_{1,m+2}\Box P_{2n+1}) = 3$. Lin and Chang [4] gave the following result for more classes of graphs.

Corollary 4. ([4, Corollary 3]). *If G and H are graphs with $\chi(G) = \chi^*_=(G)$ and $\chi(H) = \chi^*_=(H)$, then $\chi(G\Box H) = \chi_=(G\Box H) = \chi^*_=(G\Box H) = \max\{\chi(G), \chi(H)\}$.*

Note that the equitable chromatic number and threshold of bipartite graphs can be arbitrarily large but the chromatic number is just 2. For example, $\chi_=(K_{1,n}) = \chi^*_=(K_{1,n}) = \lceil \frac{n}{2} \rceil + 1$. Lin and Chang [4] gave the following two results to indicate that the bounds given in Corollaries 2 and 3 may be far from the exact values for bipartite graphs.

Theorem 5. ([4, Theorem 11]). *$K_{m,n}\Box K_{m',n'}$ is equitably 4-colorable.*

Theorem 6. ([4, Theorem 14]). *If $n, n' \geq 3$ then $\chi^*_=(K_{1,n}\Box K_{1,n'}) = 4$ except for $\chi^*_=(K_{1,n}\Box K_{1,n'}) = 3$, when $(n-2)(n'-2) \leq 5$.*

Beside Theorem 6, Lin and Chang in [4] also determined $\chi^*_=(G\Box H)$ for some other classes of bipartite graphs. For instance, for bipartite graph H, $\chi_=(P_{2n+1}\Box H) = \chi^*_=(P_{2n+1}\Box H) = 3$ except that $\chi_=(P_{2n+1}\Box H) = \chi^*_=(P_{2n+1}\Box H) = 2$, when $\chi_=(H) \leq 2$; $\chi_=(C_{2\ell+2}\Box K_{m,n}) = \chi^*_=(C_{2\ell+2}\Box K_{m,n}) = \chi_=(P_{2\ell}\Box K_{m,n}) = \chi^*_=(P_{2\ell}\Box K_{m,n}) = 2$ except for $\chi^*_=(C_4\Box K_{m,n}) = \chi^*_=(P_2\Box K_{m,n}) = 4$, when $m + n + 2 < 3\min\{m,n\}$. Based on these exact values, they raised the following conjecture.

Conjecture 7. ([4, Conjecture 2]). $\chi^*_=(G\Box H) \leq 4$ for bipartite graphs G and H.

It is easy to see that Conjecture 7 is true if it holds for complete bipartite graphs. Hence, we can restate the conjecture as the following theorem.

Theorem 8. $\chi^*_=(K_{m,n}\Box K_{m',n'}) \leq 4$.

In this paper, we prove Theorem 8.

2. Proof of Theorem 8

In what follows, we always assume $m \leq n$ and $m' \leq n'$. Noting that Theorem 8 is true when one factor is $K_{1,1} = P_2$, or $K_{1,2} = P_3$, or $K_{2,2} = C_4$, so it is sufficient to consider for $n \geq 3$ and $n' \geq 3$.

We say that $K_{m,n}$ is *almost balanced* if $|m - n| \leq 1$. First, when one factor is almost balanced, we apply the following theorem given by Lin and Chang.

Theorem 9. ([4, Theorem 9]). *If m, n, m' and n' are positive integers such that $m \leq n$, $m' \leq n'$, $m + n \geq 4$ and $m' + n' \geq 4$, then $K_{m,n}\Box K_{m',n'}$ is equitably k-colorable for $k \geq \lceil \frac{(m+n)(m'+n')}{\max\{m(n'-1),m'(n-1)\}+1} \rceil$.*

Lemma 10. *For positive integers n, m' and n' with $n \geq 3$, $n' \geq 3$ and $n' \geq m'$, $\chi^*_=(K_{n-1,n}\Box K_{m',n'}) \leq 5$ and $\chi^*_=(K_{n,n}\Box K_{m',n'}) \leq 5$.*

Proof. We consider four cases as follows.

Case 1. $K_{n-1,n}\Box K_{m',n'}$ with $m' < n'$. In this case, we have $(n-1)(n'-1) - (n-1)m' = (n-1)(n'-m'-1) \geq 0$ and $5((n-1)(n'-1)+1) - (2n-1)(m'+n') = (2n-1)(n'-m'-1) + (n-3)(n'-3) \geq 0$. By Theorem 9, $\chi^*_=(K_{n-1,n}\Box K_{m',n'}) \leq \lceil \frac{(2n-1)(m'+n')}{\max\{(n-1)(n'-1),(n-1)m'\}+1} \rceil = \lceil \frac{(2n-1)(m'+n')}{(n-1)(n'-1)+1} \rceil \leq 5$.

Case 2. $K_{n,n}\Box K_{m',n'}$ with $m' < n'$. In this case, we have $n(n'-1)-(n-1)m' = n(n'-m'-1)+m' \geq 0$ and $5(n(n'-1)+1) - 2n(m'+n') = n(n'-3) + 2n(n'-m'-1) + 5 \geq 0$. By Theorem 9, $\chi^*_=(K_{n,n}\Box K_{m',n'}) \leq \lceil \frac{2n(m'+n')}{\max\{n(n'-1),(n-1)m'\}+1} \rceil = \lceil \frac{2n(m'+n')}{n(n'-1)+1} \rceil \leq 5$. This case includes the case of $K_{n-1,n}\Box K_{n',n'}$.

Case 3. $K_{n,n}\Box K_{n',n'}$ except $n = n' = 3$. In this case, we may assume $n' \geq n$. Then we have $n(n'-1) - (n-1)n' = n' - n \geq 0$ and $5(n(n'-1)+1) - 4nn' = n(n'-5) + 5 \geq 0$. By Theorem 9, $\chi^*_=(K_{n,n}\Box K_{n',n'}) \leq \lceil \frac{4nn'}{\max\{n(n'-1),(n-1)n'\}+1} \rceil = \lceil \frac{4nn'}{n(n'-1)+1} \rceil \leq 5$.

Case 4. $K_{3,3}\Box K_{3,3}$. In this case, we show that $K_{3,3}\Box K_{3,3}$ is equitably 5-colorable by giving an equitable 5-coloring illustrated in Fig. 1. ∎

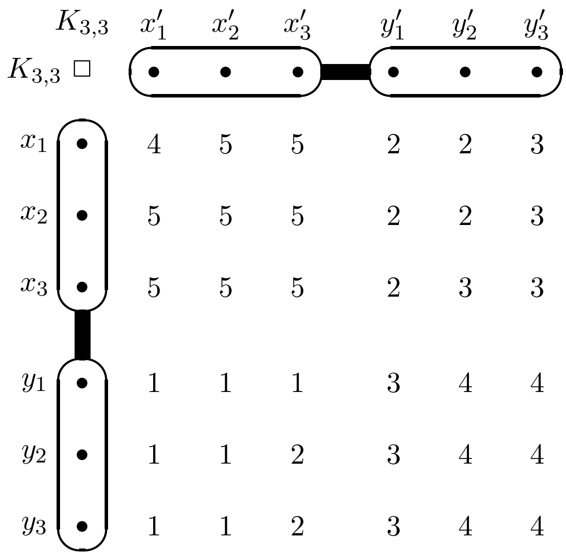

Figure 1: An equitable 5-coloring of $K_{3,3}\square K_{3,3}$.

By commutativity of Cartesian product, we can assume $m' \geq m$. Moreover, if $m' = 1$ then $m = 1$ and hence Theorem 8 holds by Theorem 6. Therefore, we can assume $m' \geq 2$. Now we give the following lemma to deal with the remaining cases.

Lemma 11. *For positive integers m, n, m' and n' with $n \geq m+2$, $n' \geq m'+2$, $m' \geq m$ and $m' \geq 2$, $\chi^*_=(K_{m,n}\square K_{m',n'}) \leq 5$.*

Proof. We shall give a particular ordering of $V(K_{m,n}\square K_{m',n'})$ and show that any set consisting of consecutive vertices in this ordering of size no more than $\lceil \frac{1}{5}(m+n)(m'+n')\rceil$ is an independent set. Then, for each $k \geq 5$, we can obtain an equitable k-coloring of $K_{m,n}\square K_{m',n'}$ by partitioning its vertex set consecutively in the ordering into k sets of size $\lfloor \frac{1}{k}(m+n)(m'+n')\rfloor$ or $\lceil \frac{1}{k}(m+n)(m'+n')\rceil$, each of which is clearly independent.

Namely, we say the bipartition of $K_{m,n}$ consists of $\{x_1,\ldots,x_m\}$ and $\{y_1,\ldots,y_n\}$ and the bipartition of $K_{m',n'}$ consists of $\{x'_1,\ldots,x'_{m'}\}$ and $\{y'_1,\ldots,y'_{n'}\}$. We order the vertices of $K_{m,n}\square K_{m',n'}$ as follows, where $X_1,\ldots,X_9$ are shown in Fig. 2, and label the vertices in this ordering as $v_1, v_2, \ldots, v_{(m+n)(m'+n')}$.

$X_1:$ $(x_1,y'_1),(x_2,y'_1),\ldots,(x_m,y'_1)$, $(x_1,y'_2),(x_2,y'_2),\ldots,(x_m,y'_2)$, $\ldots,\ldots$, $(x_1,y'_{\lfloor\frac{n'}{2}\rfloor}),(x_2,y'_{\lfloor\frac{n'}{2}\rfloor}),\ldots,(x_m,y'_{\lfloor\frac{n'}{2}\rfloor})$;

$X_2:$ $(y_1,x'_1),(y_2,x'_1),\ldots,(y_{\lfloor\frac{n}{2}\rfloor},x'_1)$, $(y_1,x'_2),(y_2,x'_2),\ldots,(y_{\lfloor\frac{n}{2}\rfloor},x'_2)$, $\ldots,\ldots$, $(y_1,x'_{m'}),(y_2,x'_{m'}),\ldots,(y_{\lfloor\frac{n}{2}\rfloor},x'_{m'})$;

$\ldots, \ldots, \ldots, \ldots;$

$X_9: (x_1, y'_{\lfloor \frac{n'}{2} \rfloor+1}), (x_2, y'_{\lfloor \frac{n'}{2} \rfloor+1}), \ldots, (x_m, y'_{\lfloor \frac{n'}{2} \rfloor+1}), (x_1, y'_{\lfloor \frac{n'}{2} \rfloor+2}),$
$(x_2, y'_{\lfloor \frac{n'}{2} \rfloor+2}), \ldots, (x_m, y'_{\lfloor \frac{n'}{2} \rfloor+2}), \ldots, \ldots, (x_1, y'_{n'}), (x_2, y'_{n'}), \ldots, (x_m, y'_{n'}).$

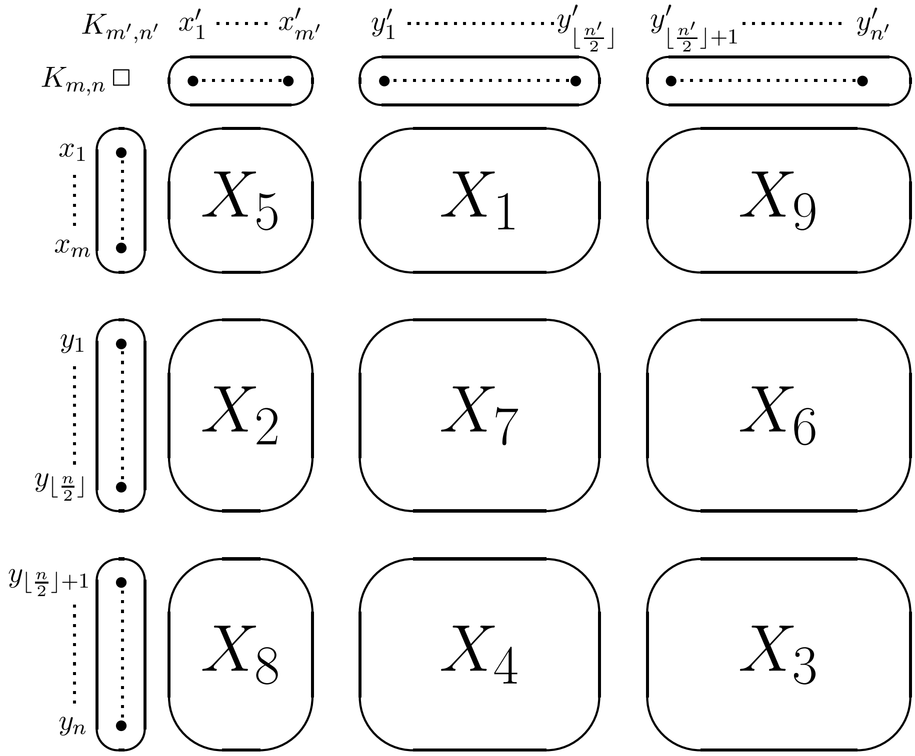

Figure 2: Vertex ordering for $K_{m,n} \square K_{m',n'}$.

More precisely, vertices in X_1 appear first followed by those in $X_2, X_3, \ldots, X_9$ continuously, and we always have $s < t$ for any two distinct vertices $v_s = (x_i, x'_j)$ (resp. (x_i, y'_j), (y_i, x'_j) or (y_i, y'_j)) and $v_t = (x_p, x'_q)$ (resp. (x_p, y'_q), (y_p, x'_q) or (y_p, y'_q)) in the same X_ℓ for $\ell = 1, \ldots, 9$ if either $j < q$, or $j = q$ and $i < p$.

Let $\gamma = \min\{t-s-1\colon 1 \le s < t \le (m+n)(m'+n'), v_s v_t \in E(K_{m,n} \square K_{m',n'})\}$. Clearly γ is well-defined since $K_{m,n} \square K_{m',n'}$ is non-empty, and any set consisting of consecutive vertices in the ordering of size no more than $\gamma + 1$ is independent.

By the definition of Cartesian product, one easily check that $X_3 \cup X_4 \cup \cdots \cup X_7$ and $X_i \cup X_{i+1} \cup X_{i+2}$ for $i \in \{1, 2, 5, 6, 7\}$ are independent. Hence, γ attends only when $v_s \in X_i$ and $v_t \in X_{i+3}$ for some $i \in \{1, 2, 5, 6\}$. When the minimality of $t - s - 1$ occurs with $v_s \in X_1$ and $v_t \in X_4$, if $m \le \lceil \frac{n}{2} \rceil$, then $v_s = (x_m, y'_1)$ and $v_t = (y_{\lfloor \frac{n}{2} \rfloor+1}, y'_1)$ which gives $\gamma = |X_1|+|X_2|+|X_3|-m$; otherwise, if $m > \lceil \frac{n}{2} \rceil$, then

$v_s = (x_m, y'_{\lfloor \frac{n'}{2} \rfloor})$ and $v_t = (y_{\lfloor \frac{n}{2} \rfloor+1}, y'_{\lfloor \frac{n'}{2} \rfloor})$ which gives $\gamma = |X_2| + |X_3| + |X_4| - \lceil \frac{n}{2} \rceil$. Similar consideration for other three possible values of i leads that $\gamma = \min\{|X_1| + |X_2| + |X_3| - m, |X_2| + |X_3| + |X_4| - \lceil \frac{n}{2} \rceil, |X_2| + |X_3| + |X_4| - \lfloor \frac{n}{2} \rfloor, |X_3| + |X_4| + |X_5| - m, |X_5| + |X_6| + |X_7| - m, |X_6| + |X_7| + |X_8| - \lceil \frac{n}{2} \rceil, |X_6| + |X_7| + |X_8| - \lfloor \frac{n}{2} \rfloor, |X_7| + |X_8| + |X_9| - m\}$. For simplicity, in what follows let $a_i = |X_i|$ for $i = 1, \ldots, 9$, see Table 1 for exact values of a_i's.

Table 1: List of exact values of a_i's

i	1	2	3	4	5	6	7	8	9
a_i	$m\lfloor \frac{n'}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor m'$	$\lceil \frac{n}{2} \rceil \lceil \frac{n'}{2} \rceil$	$\lceil \frac{n}{2} \rceil \lfloor \frac{n'}{2} \rfloor$	mm'	$\lfloor \frac{n}{2} \rfloor \lceil \frac{n'}{2} \rceil$	$\lfloor \frac{n}{2} \rfloor \lfloor \frac{n'}{2} \rfloor$	$\lceil \frac{n}{2} \rceil m'$	$m\lceil \frac{n'}{2} \rceil$

Since $\lceil \frac{n}{2} \rceil \geq \lfloor \frac{n}{2} \rfloor$, $(a_2 + a_3 + a_4 - \lceil \frac{n}{2} \rceil) - (a_6 + a_7 + a_8 - \lceil \frac{n}{2} \rceil) = (\lfloor \frac{n}{2} \rfloor m' + \lceil \frac{n}{2} \rceil n' - \lceil \frac{n}{2} \rceil) - (\lfloor \frac{n}{2} \rfloor n' + \lceil \frac{n}{2} \rceil m' - \lceil \frac{n}{2} \rceil) = (\lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor)(n' - m') \geq 0$, and $(a_3 + a_4 + a_5 - m) - (a_5 + a_6 + a_7 - m) = (\lceil \frac{n}{2} \rceil n' + mm' - m) - (mm' + \lfloor \frac{n}{2} \rfloor n' - m) = (\lceil \frac{n}{2} \rceil - \lfloor \frac{n}{2} \rfloor) n' \geq 0$, we have $\gamma = \min\{a_1 + a_2 + a_3 - m, a_5 + a_6 + a_7 - m, a_6 + a_7 + a_8 - \lceil \frac{n}{2} \rceil, a_7 + a_8 + a_9 - m\}$.

Lastly, we shall prove a stronger result that $\gamma \geq \frac{1}{5}(m+n)(m'+n')$ by checking all the four expressions $a_1 + a_2 + a_3 - m$, $a_5 + a_6 + a_7 - m$, $a_6 + a_7 + a_8 - \lceil \frac{n}{2} \rceil$, $a_7 + a_8 + a_9 - m$ are not less than $\frac{1}{5}(m+n)(m'+n')$. Note that $\lceil \frac{n}{2} \rceil \geq \frac{n}{2}$, $\lfloor \frac{n}{2} \rfloor \geq \frac{n-1}{2}$ and $\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor = n$. Thus, by the assumption that $n \geq m+2$, $n' \geq m'+2$, $m' \geq m$ and $m' \geq 2$, we have the following four inequalities.

$$
\begin{aligned}
&20(a_1 + a_2 + a_3 - m) - 4(m+n)(m'+n') \\
&= 20\left(m \left\lfloor \frac{n'}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor m' + \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n'}{2} \right\rceil - m\right) - 4(m+n)(m'+n') \\
&\geq 10m(n'-1) + 10(n-1)m' + 5nn' - 20m - 4(m+n)(m'+n') \\
&= (m+n)(n'+m'-5) + 5(n-m-2)(m'+1) + 5m(n'-4) + 10 \\
&\geq 0.
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
&20(a_5 + a_6 + a_7 - m) - 4(m+n)(m'+n') \\
&= 20\left(mm' + \left\lfloor \frac{n}{2} \right\rfloor n' - m\right) - 4(m+n)(m'+n') \\
&\geq 20mm' + 10(n-1)n' - 20m - 4(m+n)(m'+n') \\
&= (m+n)(m'+n'-5) + 5(n-m-2)(n'-m'+1) + 10(m-1)(m'-1) \\
&\geq 0.
\end{aligned} \tag{2}
$$

$$\begin{aligned} & 20(a_6+a_7+a_8-\left\lceil\frac{n}{2}\right\rceil)-4(m+n)(m'+n') \\ & =20(\left\lfloor\frac{n}{2}\right\rfloor n'+\left\lceil\frac{n}{2}\right\rceil(m'-1))-4(m+n)(m'+n') \\ & \geq 10(n-1)n'+10n(m'-1)-4(m+n)(m'+n') \\ & =(m+n)(m'+n'-5)+5(n-m-2)(n'+m'-1)+10(m'-1) \\ & \geq 0. \end{aligned} \tag{3}$$

$$\begin{aligned} & 20(a_7+a_8+a_9-m)-4(m+n)(m'+n') \\ & =20(\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n'}{2}\right\rfloor+\left\lceil\frac{n}{2}\right\rceil m'+m\left\lceil\frac{n'}{2}\right\rceil-m)-4(m+n)(m'+n') \\ & \geq 5(n-1)(n'-1)+10nm'+10mn'-20m-4(m+n)(m'+n') \\ & =(m+n)(m'+n'-5)+5(n-m-2)m'+5(m-1)(n'-3)+10(m'-1) \\ & \geq 0. \end{aligned} \tag{4}$$

By inequalities (1) to (4) and the definition of γ, we have $\gamma \geq \frac{1}{5}(m+n)(m'+n')$ and hence $\chi^*_=(K_{m,n}\Box K_{m',n'}) \leq \lceil\frac{(m+n)(m'+n')}{\gamma+1}\rceil \leq 5$. ■

According to Theorem 5, Lemmas 10 and 11, we have $\chi^*_=(K_{m,n}\Box K_{m',n'}) \leq 4$ for $n \geq 3$, $n' \geq 3$, $m' \geq m$ and $m' \geq 2$, and this completes the proof of Theorem 8.

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