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THE ABSOLUTE LENGTH OF ALGEBRAIC INTEGERS WITH POSITIVE REAL PARTS

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Abstract. Let α be a nonzero algebraic integer of degree d, all of whose conjugates α_i are confined to a sector $|\arg(\alpha_i)| \leq \theta$ with $0 < \theta < \pi/2$. Let $P = X^d + b_1 X^{d-1} + \cdots + b_d$ be the minimal polynomial of α. We give in this paper the greatest lower bounds $\rho_{\mathcal{L}}(\theta)$ of the absolute length $\mathcal{L}(P) = (1 + \sum_{i=1}^{d} |b_i|)^{1/d}$ of all but finitely many such α, for ten different values of θ.

1. INTRODUCTION

Let α be a nonzero algebraic integer of degree d, and let $\alpha_1 = \alpha, \alpha_2, \cdots, \alpha_d$ be its conjugates, with $P = X^d + b_1 X^{d-1} + \cdots + b_{d-1} X + b_d \in \mathbb{Z}[X]$ its minimal polynomial. The *length* of α is given by

$$L(P) = 1 + |b_1| + \cdots + |b_d|,$$

and $L(P) \geq 2$(as $P \neq x$). The *absolute length* of α is given by

$$\mathcal{L}(P) = L(P)^{\frac{1}{d}}.$$

The length $L(P)$ is an important measure of a nonzero algebraic integer. We have the inequality[3] $M(P) \leq L(P) \leq 2^d M(P)$, where $M(P)$ is Mahler measure of P which is given by $M(P) = \prod_{i=1}^{d} \max(1, |\alpha_i|)$. From Kronecker's theorem and Lehmer's conjecture, we know that $M(P)$ is either 1 (if P is cyclotomic) or thought to be bounded away from 1 by an absolute constant (if P is not cyclotomic)[1][2]. From a result of Langevin[5], we know that there is a constant $C_\Omega(V) > 1$ such that the absolute Mahler measure $\Omega(P) := M(P)^{1/d}$ is either 1 or else satisfies $\Omega(P) \geq$

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$C_\Omega(V)$, when zeros of P are restricted to the closed set V which does not contain the whole unit circle. In the case where V is the sector $\{z : |\arg(z)| \le \theta\}$ where $0 \le \theta \le \pi$, G. Rhin and C. Smyth[7] succeeded in finding $c(\theta)$ exactly for θ in nine intervals, where $c(\theta)$ denote the largest value of $C_\Omega(V)$. In 2005, G. Rhin and the first author[8] improved the result to thirteen subintervals of $[0, \pi]$ and extended some known subintervals.

The absolute length $\mathcal{L}(P)$ is thought to be greater than an absolute constant $C_{\mathcal{L}}(V)$, when all the zeros of P are restricted to a set V. In fact, from $M(P) \le L(P) \le 2^d M(P)$, on taking the dth root, that $\Omega(P) \le \mathcal{L}(P) \le 2\Omega(P)$. Hence, from Langevin's result, we can deduce the existence of $C_{\mathcal{L}}(V) > 1$ for the same V for which Langevin's result is valid.

If P is the minimal polynomial of totally positive algebraic integer α (different from $x-1$), then $L(P) = \prod_{i=1}^{d}(1+\alpha_i)$. In 1995, Flammang[3] succeeded in finding a good value for the constant $\rho_{\mathcal{L}}$. She proved that the absolute length of totally positive algebraic integer α satisfies $\mathcal{L}(P) \ge \rho_{\mathcal{L}} = 2.36110147\cdots$ with for five exceptions in the spectrum given by 7 algebraic integers, whose minimal polynomials are $x^2-3x+1, x^3-5x^2+6x-1, x^3-6x^2+5x-1, x^4-7x^3+13x^2-7x+1, x^4-7x^3+14x^2-8x+1, x^4-8x^3+14x^2-7x+1, x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1$. Recently, Mu and the first author[6] improved these results to $\rho_{\mathcal{L}} = 2.364950\cdots$, with the same exceptions.

Let P be the minimal polynomial of algebraic integer α of degree d whose conjugates have positive real parts, i.e. $\Re(\alpha_i) > 0$ for $1 \le i \le d$. As $P(-x)$ is a product of terms $x+\alpha$ for α real and terms $(x+\alpha)(x+\overline{\alpha}) = x^2 + 2\Re(\alpha)x + \alpha\overline{\alpha}$ otherwise and so has positive coefficients, then the length of α can be written as

$$L(P) = |P(-1)| = |(-1-\alpha_1)(-1-\alpha_2)\cdots(-1-\alpha_d)| = \prod_{i=1}^{d} |1+\alpha_i|.$$

Then

$$\mathcal{L}(P) = \left(\prod_{i=1}^{d} |1+\alpha_i|\right)^{\frac{1}{d}}.$$

The aim of this paper is to find not only the value for the constant $C_{\mathcal{L}}(V(\theta))$ but also a good value for a constant $\rho_{\mathcal{L}}(\theta) > C_{\mathcal{L}}(V(\theta))$ such that $\mathcal{L}(P) \ge \rho_{\mathcal{L}}(\theta)$ for all but an explicit finite list of P when all the zeros of P are restricted to a set $V(\theta)$, where $V(\theta)$ is the sector $\{z : |\arg(z)| \le \theta\}$ for a fixed θ with $0 < \theta < \pi/2$. It is clear that $\rho_{\mathcal{L}}(\theta)$ is a non-increasing function of θ. We succeed in finding $\rho_{\mathcal{L}}(\theta)$ exactly for θ with ten different values. We have

Theorem 1. *Let P be the minimal polynomial of algebraic integer α of degree d whose conjugates have positive real parts. Let $V(\theta)$, $\mathcal{L}(P)$, $\rho_{\mathcal{L}}(\theta)$ and $C_{\mathcal{L}}(V(\theta))$ be* *defined as above. If all the zeros of P are restricted to the set $V(\theta_k)$ for each θ_k in Table 1, then the absolute length of P satisfies $\mathcal{L}(P) \geqslant \rho_{\mathcal{L}}(\theta_k)$ respectively, except for those algebraic integers whose minimal polynomials are denoted Q_j^* in Table 1. In particular, the value $C_{\mathcal{L}}(V(\theta))$ of $\mathcal{L}(P)$ for such P is attained by $\mathcal{L}(P)$ as given in the 4th column of Table 1.*

Remark 1. In Table 1 $Q_{16}^* = (x^3 - 5x^2 + 6x - 1)(x^3 - 6x^2 + 5x - 1)$, $Q_{30}^* = (x^4 - 7x^3 + 14x^2 - 8x + 1)(x^4 - 8x^3 + 14x^2 - 7x + 1)$.

In Section 2, we prove Theorem 1 by using explicit auxiliary functions. We briefly describe the research method in Section 3.

2. The Explicit Auxiliary Function for the Absolute Length of P

2.1. The explicit auxiliary function for the absolute length of P

For a fixed θ_k, we consider an explicit auxiliary function of the type

$$f_k(z) = \frac{1}{2}\log(1+z)(1+\overline{z}) - \sum_{j=1}^{J} e_{kj} \log |Q_{kj}(z)|, \tag{2.1}$$

where z is a complex number, the numbers e_{kj} are positive real numbers and the polynomials Q_{kj} are nonzero elements of $\mathbb{Z}[X]$. The numbers e_{kj} and the polynomials Q_{kj} are always chosen to maximize the minimum of $f_k(z)$ on $V(\theta_k)$. We denote by m_k the minimum of $f_k(z)$ for $z \in V(\theta_k)$. Since the function f_k is harmonic in this sector outside the union of arbitrarily small disks around the roots of the polynomials Q_{kj}, this minimum is taken on the boundary of $V(\theta_k)$.

We have

$$\sum_{1 \le i \le d} f_k(\alpha_i) \ge dm_k$$

and

$$\log L(P) \ge dm_k + \sum_{1 \le j \le J} e_{kj} \log \left| \prod_{1 \le i \le d} Q_{kj}(\alpha_i) \right|.$$

$\prod_{1 \le i \le d} Q_{kj}(\alpha_i)$ is equal to the resultant of P and Q_{kj}. If we assume now that polynomial P does not divide any polynomial Q_{kj}, then this resultant is a nonzero integer. Therefore

$$\log L(P) \ge dm_k,$$

so that

$$\mathcal{L}(P) \ge e^{m_k}. \tag{2.2}$$

2.2. The proof of the Theorem 1

For each θ_k in Table 1, we take Q_{kj} in the auxiliary function f_k as Q_j(which is given in Table 3) in the kth row of Table 1 and e_{kj} respectively in the kth row of Table 2. With (2.2), by numerical computation, we then obtain Theorem 1.

3. The Method

In order to get the largest lower bound for $\mathcal{L}(P)$, we only need to find the greatest m_k. If, in the auxiliary function of (2.1), we replace the real numbers e_{kj} by rational numbers we may write

$$f_k(z) = \frac{1}{2}\log(1+z)(1+\overline{z}) - \frac{t}{h_k}\log|H_k(z)|, \tag{3.1}$$

where H_k is in $\mathbb{Z}[X]$ of degree h_k and t is a positive real number. We want to obtain a function f_k whose minimum m_k in $V(\theta_k)$ is as large as possible. That is to say, we seek a polynomial $H_k \in \mathbb{Z}[X]$ such that

$$\sup_{z\in V(\theta_k)} |H_k(z)|^{t/h_k}\,((1+z)(1+\overline{z}))^{-1/2} \le e^{-m_k}.$$

Now, if we suppose that t is fixed, say $t=1$, it is clear that we need to get an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}(V(\theta_k)) = \liminf_{\substack{h_k\ge 1\\ h_k\to\infty}} \inf_{\substack{H_k\in\mathbb{Z}[X]\\ \deg H_k = h_k}} \sup_{z\in V(\theta_k)} |H_k(z)|^{t/h_k}\varphi(z)$$

in which we use the weight $\varphi(z) = ((1+z)(1+\overline{z}))^{-1/2}$. To get an upper bound for $t_{\mathbb{Z},\varphi}(V(\theta_k))$, it is sufficient to get an explicit polynomial $H_k \in \mathbb{Z}[X]$ and then to use the sequence of the successive powers of H_k.

The function $t_{\mathbb{Z},\varphi}(V(\theta_k))$ is a generalization of the integer transfinite diameter. For any $h \ge 1$ we say that a polynomial H (not always unique) is an *Integer Chebyshev Polynomial* if the quantity $\sup_{z\in V(\theta)} |H(z)|^{t/h}\varphi(z)$ is minimum. With the first author's algorithm[10], we compute the polynomials H of degree less than 30 and take their irreducible factors as the polynomials Q_j. We start with the polynomial $x-1$, get the best e_1 and take $t = e_1$. When we have computed J polynomials, we optimize the numbers e_j with a refinement of the semi-infinite linear programming method that has been introduced into number theory by Smyth[9]. This gives us a new number t. We continue by induction to get $J+1$ polynomials. More details can be found in [4].

We use also the LLL algorithm to find candidates for Q_j. The optimal function f is obtained by semi-infinite linear programming[10]. Moreover, technical improvements allow us to find the polynomials Q_j with higher degrees than before. Table 1 shows the

10 θ_k's, the greatest value for the constant $\rho_{\mathcal{L}}(\theta_k)$ and the absolute constant $C_{\mathcal{L}}(V(\theta_k))$ when all the zeros of P are restricted to the set $V(\theta_k)$, for each k. The last column in Table 1 gives the polynomials Q_{kj} which are used in the auxiliary functions $f_k(z)$. The corresponding polynomials are those in Table 3. All the coefficients e_{kj} in the auxiliary functions $f_k(z)$ can be found in Table 2.

Table 1 $\rho_{\mathcal{L}}(\theta_k)$, $C_{\mathcal{L}}(V(\theta_k))$ for θ_k and Q_{kj} used in the auxiliary functions $f_k(z)$

k	θ_k	$\rho_{\mathcal{L}}(\theta_k)$	$C_{\mathcal{L}}(V(\theta_k))$	Q_{kj}
1	$0.01875 \approx 0.00597\pi$	$2.35961291\cdots$	$\mathcal{L}(Q_2)$	$Q_1, Q_2^*, Q_4^*, Q_5, Q_{11}^*, Q_{12}, Q_{16}^*, Q_{29}^*, Q_{30}^*, Q_{34}, Q_{48}, Q_{49}, Q_{57}$
2	$0.03757 \approx 0.01196\pi$	$2.35341723\cdots$	$\mathcal{L}(Q_2)$	$Q_1, Q_2^*, Q_4^*, Q_5, Q_{11}^*, Q_{12}, Q_{13}, Q_{16}^*, Q_{29}^*, Q_{30}, Q_{34}$
3	$0.04341 \approx 0.01382\pi$	$2.35133701\cdots$	$\mathcal{L}(Q_2)$	$Q_1, Q_2^*, Q_4^*, Q_5, Q_{11}^*, Q_{12}, Q_{16}^*, Q_{29}, Q_{30}, Q_{40}, Q_{56}$
4	$0.12529 \approx 0.03988\pi$	$2.32059849\cdots$	$\mathcal{L}(Q_2)$	$Q_1, Q_2^*, Q_4^*, Q_5, Q_{11}^*, Q_{33}, Q_{55}, Q_{63}$
5	$0.31743 \approx 0.10104\pi$	$2.23607259\cdots$	$\mathcal{L}(Q_2)$	$Q_1, Q_2^*, Q_4^*, Q_{15}, Q_{27}, Q_{28}, Q_{39}, Q_{47}, Q_{52}, Q_{53}, Q_{54}, Q_{62}$
6	$0.74808 \approx 0.23812\pi$	$2.00000207\cdots$	$\mathcal{L}(Q_2)$	$Q_1, Q_2^*, Q_8, Q_{10}, Q_{26}, Q_{32}, Q_{38}, Q_{46}, Q_{59}, Q_{60}, Q_{61}, Q_{65}, Q_{69}$
7	$0.95637 \approx 0.30442\pi$	$1.89883252\cdots$	$\mathcal{L}(Q_8)$	$Q_1, Q_3, Q_8^*, Q_9, Q_{23}, Q_{24}, Q_{25}, Q_{37}, Q_{45}, Q_{58}, Q_{66}, Q_{67}, Q_{68}$
8	$1.16605 \approx 0.37117\pi$	$1.77828481\cdots$	$\mathcal{L}(Q_3)$	$Q_1, Q_3^*, Q_7^*, Q_{20}, Q_{22}, Q_{44}, Q_{51}$
9	$1.24066 \approx 0.39491\pi$	$1.73205380\cdots$	$\mathcal{L}(Q_3)$	$Q_1, Q_3^*, Q_6, Q_{14}, Q_{19}, Q_{21}, Q_{35}, Q_{36}, Q_{43}$
10	$1.39314 \approx 0.44345\pi$	$1.62657883\cdots$	$\mathcal{L}(Q_6)$	$Q_1, Q_6^*, Q_{14}, Q_{17}, Q_{18}, Q_{31}, Q_{41}, Q_{42}, Q_{50}, Q_{64}$

Table 2 e_{kj} used in the auxiliary functions $f_k(z)$

k	e_{kj}
1	0.31640461, 0.11635268, 0.03905736, 0.00207354, 0.01327430, 0.00057312, 0.00485278, 0.00495405, 0.00255289, 0.00021242, 0.00040214, 0.00038123, 0.00068269
2	0.31729055, 0.11920741, 0.04221770, 0.00302601, 0.01470128, 0.00125251, 0.00054243, 0.00490601, 0.00524017, 0.00169688, 0.00048067
3	0.31812098, 0.11937264, 0.04203181, 0.00213928, 0.01604043, 0.00037157, 0.00545934, 0.00422952, 0.00108730, 0.00001212, 0.00093495
4	0.32527235, 0.13499569, 0.04828147, 0.00454245, 0.01898362, 0.00097494, 0.00109277, 0.00031353
5	0.33437678, 0.15973207, 0.05749201, 0.00009375, 0.00272716, 0.00049440, 0.00116907, 0.00005410, 0.00048092, 0.00036906, 0.00002697, 0.00007655
6	0.34233462, 0.20460902, 0.00670147, 0.00315100, 0.00112223, 0.00002894, 0.00117557, 0.00035031, 0.00014383, 0.00044087, 0.00060545, 0.00055713, 0.00043618
7	0.35373279, 0.04932791, 0.02403626, 0.00233610, 0.00121050, 0.00676016, 0.00029411, 0.00061858, 0.00004358, 0.00005472, 0.00009809, 0.00029626, 0.00019071
8	0.34289602, 0.05332589, 0.02473867, 0.00041985, 0.00478406, 0.00179133, 0.00261823
9	0.35637893, 0.05279506, 0.02040707, 0.00425959, 0.00413851, 0.00037100, 0.00153723, 0.00074740, 0.00078389
10	0.29377193, 0.01878965, 0.00229036, 0.01590297, 0.00567345, 0.00379205, 0.00149646, 0.00083798, 0.00134531, 0.00294081

Table 3 Polynomials Q_j used in the auxiliary functions.

j	d	$\mathcal{L}(Q)$	$\arg(Q_j)$	First half coefficients of Q_j except $d=1$								
1	1	1.000000	0.00000	1	0							
2	1	2.000000	0.00000	1	-1							
3	2	1.732050	1.04719	1	-1							
4	2	2.236067	0.00000	1	-3							
5	2	2.449489	0.00000	1	-4							
6	4	1.626576	1.34033	1	-1	3						
7	4	1.778279	1.11851	1	-2	4						
8	4	1.898828	0.86138	1	-3	5						
9	4	1.934336	0.94978	1	-3	6						
10	4	2.030543	0.67488	1	-4	7						
11	4	2.320595	0.00000	1	-7	13						
12	4	2.396781	0.00000	1	-8	15						
13	4	2.414736	0.00000	1	-8	16						
14	6	1.686376	1.35402	1	-2	6	-5					
15	6	2.158010	1.62009	1	-9	29	-43					
16	6	2.351334	0.00000	1	-11	41	-63					
17	8	1.650233	1.37283	1	-2	8	-9	15				
18	8	1.685055	1.37767	1	-2	10	-10	19				
19	8	1.718310	1.27411	1	-3	10	-14	20				
20	8	1.726646	1.31167	1	-3	10	-15	21				
21	8	1.747591	1.26279	1	-3	11	-16	25				
22	8	1.791278	1.17990	1	-4	13	-21	28				
23	8	1.853006	1.03603	1	-5	16	-29	37				
24	8	1.911183	0.93113	1	-6	20	-38	48				
25	8	1.923004	0.95711	1	-6	21	-40	51				
26	8	1.959103	0.84836	1	-7	24	-47	59				
27	8	2.234274	0.32922	1	-12	58	-143	193				
28	8	2.286084	0.29597	1	-13	67	-173	238				
29	8	2.353416	0.00000	1	-15	83	-220	303				
30	8	2.359611	0.00000	1	-15	84	-225	311				
31	10	1.644889	1.43314	1	-2	11	-14	32	-25			
32	10	1.978479	0.89591	1	-9	40	-107	189	-227			
33	10	2.334173	0.09449	1	-18	130	-492	1069	-1381			
34	10	2.339943	0.13388	1	-18	131	-501	1098	-1423			
35	12	1.750704	1.23109	1	-5	21	-51	104	-146	173		
36	12	1.752454	1.25381	1	-5	21	-51	105	-148	177		
37	12	1.915501	0.96609	1	-9	44	-136	296	-464	540		
38	12	1.992226	0.80058	1	-11	60	-203	468	-763	897		
39	12	2.234102	0.36158	1	-18	141	-628	1756	-3219	3935		
40	12	2.344418	1.61290	1	-22	102	-1014	3076	-5906	7327		
41	14	1.637776	1.40600	1	-3	16	-31	82	-108	178	-161	
42	14	1.664113	1.41432	1	-3	19	-36	104	-132	230	-199	
43	14	1.703361	1.66874	1	-5	22	-57	128	-208	290	-309	
44	14	1.755450	1.30510	1	-6	28	-80	187	-318	453	-493	
45	14	1.900761	0.99620	1	-10	55	-197	509	-980	1445	-1641	
46	14	2.001518	0.79988	1	-13	84	-343	974	-2009	3081	-3549	
47	14	2.268769	0.32292	1	-22	215	-1225	4503	-11190	19214	-22994	
48	14	2.372419	0.00000	1	-27	309	-1979	7895	-20676	36527	-44101	
49	14	2.375410	0.00000	1	-27	310	-1995	7997	-21021	37220	-44971	
50	16	1.664957	1.40739	1	-4	22	-55	149	-253	434	-519	613
51	16	1.803649	1.18284	1	-8	43	-153	422	-892	1523	-2074	2312
52	16	2.234418	0.37705	1	-24	260	-1678	7183	-21516	46426	-73292	85281
53	16	2.238845	0.36225	1	-24	261	-1695	7309	-22050	47859	-75846	88371
54	16	2.260152	0.32803	1	-25	281	-1874	8253	-25306	55575	-88711	103614
55	16	2.317191	0.15656	1	-28	345	-2473	11499	-36646	82525	-133568	156691
56	16	2.336486	0.13848	1	-29	368	-2702	12803	-41378	94078	-153111	179941
57	16	2.366799	0.00000	1	-31	413	-3141	15261	-50187	115410	-189036	222621

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j	d	$\mathcal{L}(Q)$	$\arg(Q_j)$	First half coefficients of Q_j except $d=1$							
58	18	1.907615	0.97760	1	-13	91	-425	1460	-3857	8057	-13511
				18368	-20335						
59	18	1.929765	1.34801	1	-15	112	-540	1868	-4900	10085	-16650
				22371	-24663						
60	18	1.998772	0.84159	1	-17	142	-759	2883	-8212	18102	-31478
				43688	-48699						
61	18	2.006439	0.77638	1	-17	143	-773	2976	-8596	19194	-33726
				47126	-52655						
62	18	2.257346	0.32627	1	-28	357	-2742	14155	-51926	139672	-280505
				424856	-487682						
63	18	2.328193	0.17545	1	-32	457	-3858	21505	-83778	235655	-487972
				752292	-868541						
64	20	1.645451	1.45107	1	-4	27	-73	249	-488	1088	-1596
				2546	-2831	3363					
65	20	2.004631	0.76251	1	-19	178	-1075	4653	-15248	39078	-79933
				132212	-178280	196867					
66	22	1.908812	1.00181	1	-16	136	-777	3297	-10921	29120	-63713
				115886	-176714	227171	-246924				
67	22	1.910192	0.98156	1	-16	136	-778	3308	-10987	29382	-64474
				117571	-179644	231228	-251444				
68	24	1.932179	0.97636	1	-18	172	-1109	5332	-20103	61285	-153994
				323092	-570849	854251	-1086553	1177035			
69	24	2.007814	0.80754	1	-23	261	-1923	10259	-41975	136372	-359858
				783207	-1420894	2164363	-2781154	3022729			

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