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RANDOM ATTRACTOR FOR STOCHASTIC PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH FINITE DELAY

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Abstract. This paper deals with a class of stochastic partial functional differential equations with finite delay. We give some sufficient conditions to guarantee the existence of a unique random attractor which attracts any tempered random set in the phase space.

1. INTRODUCTION

As is known to all, random attractor is a helpful tool to understand the dynamics of stochastic systems, which was introduced in [5] as an extension to stochastic systems of the theory of attractors for deterministic system [6]. Many works have been devoted to the existence of random attractors for some stochastic PDEs, such as reaction-diffusion equations and Navier-Stokes equations, see for example [3, 4, 10].

In this paper, we investigate the asymptotic behavior of solutions for the following stochastic partial functional differential equations on a separable Banach space $(E, \|\cdot\|)$

$$dx(t) = Ax(t)dt + f(x_t)dt + \sigma dW(t), \tag{1.1}$$

where A is a linear operator on E and f is a nonlinear operator satisfying the global Lipschitz condition; $\sigma \in D(A)$ and $W(t)$ is a real-valued two-sided Winer process. Note that, in the previous works concerned with random attractors, the authors mainly focus their attentions on the case that the linear part is a densely defined operator. As far as the linear part is not densely defined concerned, to our best knowledge, no literature in this area can be found.

Motivated by the previous works on the random attractor of the explicit partial differential equations, in the present paper, we consider the existence of random attractors for a more general form of equation as Eq. (1.1), where the linear operator $A : D(A) \subset E \to E$ is unnecessarily densely defined but satisfies the following Hille-Yosida condition

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(H1) there exist two constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\|(\lambda I - A)^{-n}\|_{\mathcal{L}} \leq \frac{M}{(\lambda - \omega)^n}, \quad \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $\|\cdot\|_{\mathcal{L}}$ denotes the operator norm.

In fact, operators with non-dense domain occurs in many situations due to restrictions on the space where the equations are considered. For example, periodic continuous functions and Hölder continuous functions are not dense in the space of continuous functions, we refer to [9] for more examples. Besides, the boundary conditions may also give rise to operators with non-dense domains, e.g., the age-structured problem given in the last section of the present paper.

Now we turn to Eq. (1.1). Since the linear part A is not densely defined, we could not consider its random attractor via the theory of C_0-semigroup directly. Fortunately, it is known that the non-densely defined Hille-Yosida operator generates integrated semigroup, which is introduced in [1] and more properties about which are established later, for example [7, 11]. For the convenience, we study Eq. (1.1) on the space $C := C([-r, 0], E)$, the space of continuous functions from $[-r, 0]$ to E with the supreme norm.

The rest of the paper is organized as follows. In section 2, we present some basic concepts and properties for integrated semigroup theory and random dynamical systems. In section 3, we convert Eq. (1.1) to a deterministic equation with a random parameter. In section 4, by proving the existence of random absorbing set and the asymptotic compactness for the solution operator, we establish the existence of random attractor. In the last section, as an application of our theory, we use the age-structured problem with white noise to demonstrate our result.

2. Preliminary Results

In this section, we recall some basic definitions and theories about integrated semigroup and general random dynamical systems, see [1, 2, 3, 7, 11].

Consider an abstract evolution equation on a general Banach space E

$$\frac{dx(t)}{dt} = Ax(t) + f(x_t), \quad t > 0 \tag{2.1}$$

with initial function $x_0 = \xi$, where A is a Hille-Yosida operator, that is, A satisfies (H1).

Definition 2.1. [1]. Let $T > 0$. A continuous function $x : [-r, T] \to E$ is called an integral solution of equation (2.1) if

(i) $\int_0^t x(s)ds \in D(A)$ for $t \in [0, T]$;

(ii) $x(t) = \xi(0) + A(\int_0^t x(s)ds) + \int_0^t f(x_s)ds$;

(iii) $x_0 = \xi$.

Remark 2.1. From (i) we know that if x is an integral solution of (2.1), then $x_t(0) = x(t) \in \overline{D(A)}$ for $t \in [0, T]$. In particular, $\xi(0) \in \overline{D(A)}$, which is a necessary condition for the existence of an integral solution.

Definition 2.2. [1] An integrated semigroup is a family $S(t)$, $t \geq 0$, of bounded linear operators on E with the following properties:

(i) $S(0) = 0$;

(ii) $t \mapsto S(t)$ is strongly continuous;

(iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))\, dr$, for all $t, s \geq 0$.

Lemma 2.1. [7]. *The following assertions are equivalent:*

(i) *A is the generator of a locally Lipschitz continuous integrated semigroup;*

(ii) *A is a Hille-Yosida operator.*

Now we introduce the part A_0 of A in $\overline{D(A)}$:

$$A_0 = A \quad \text{on} \quad D(A_0) = \{x \in D(A);\ Ax \in \overline{D(A)}\ \}.$$

Proposition 1. [11]. *The part A_0 of A in $\overline{D(A)}$ generates a strongly continuous semigroup on $\overline{D(A)}$.*

Now we turn to the random dynamical systems. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra on Ω generated by the compact open topology, and $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$. $(X, \|\cdot\|_X)$ is a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$.

Definition 2.3. $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical systems, if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$ measurable, $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$, for all t, $s \in \mathbb{R}$, and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.4. A continuous random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

such that the following properties hold:

(1) $\phi(0, \omega, x) = x$ for all $\omega \in \Omega$ and $x \in X$;

(2) $\phi(t+s,\omega,\cdot) = \phi(t,\theta_s\omega,\phi(s,\omega,\cdot))$ for all $t,s \geq 0$ and $\omega \in \Omega$;

(3) ϕ is continuous in t and x.

Definition 2.5.

(1) A set-valued mapping $\omega \mapsto D(\omega) : \Omega \to 2^X$ is said to be a random set if the mapping $\omega \mapsto d(x, D(\omega))$ is measurable for any $x \in X$. If $D(\omega)$ is also closed (compact) for each $\omega \in \Omega$, the mapping $\omega \mapsto D(\omega)$ is called a random closed (compact) set. A random set $\omega \mapsto D(\omega)$ is said to be bounded if there exist $x_0 \in X$ and a random variable $R(\omega) > 0$ such that

$$D(\omega) \subset \{x \in X : \|x - x_0\|_X \leq R(\omega)\} \quad \text{for all } \omega \in \Omega.$$

(2) A random set $\omega \mapsto D(\omega)$ is called tempered provided for $\mathbb{P}$-a.s. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} \sup\{\|b\|_X : b \in D(\theta_t\omega)\} = 0 \quad \text{for all } \beta > 0.$$

(3) A random set $\omega \mapsto B(\omega)$ is said to be a random absorbing set if for any tempered random set $\omega \mapsto D(\omega)$, there exists $t_0(\omega)$ such that

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega) \quad \text{for all } t \geq t_0,\ \omega \in \Omega.$$

(4) A random set $\omega \mapsto B_1(\omega)$ is said to be a random attracting set if for any tempered random set $\omega \mapsto D(\omega)$, we have

$$\lim_{t\to\infty} d(\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), B_1(\omega)) = 0, \quad \text{for all } \omega \in \Omega.$$

(5) A random compact set $\omega \mapsto A(\omega)$ is said to be a random attractor if it is an random attracting set and $\phi(t,\omega,A(\omega)) = A(\theta_t\omega)$ for all $\omega \in \Omega$ and $t \geq 0$,

where d is the Hausdorff semi-metric given by $d(Y,Z) = \sup_{y\in Y} \inf_{z\in Z} \|y - z\|_X$ for any $Y \subset X$, $Z \subset X$.

Definition 2.6. ϕ is called pullback asymptotically compact on X if for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\omega \mapsto B(\omega)$ is tempered.

In what follows, we recall the definition of the Kuratowski's measure of non-compactness for a bounded set B of a Banach space E, which is defined as

$$\kappa(B) = \inf\{d > 0 : B \text{ has a finite cover of diameter} < d\}, \tag{2.2}$$

and plays an important role in proving the pullback asymptotically compact in section 4.

Theorem 2.1. [3]. *Let ϕ be a continuous random dynamical system over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. Suppose that $\omega \mapsto K(\omega)$ is a closed random absorbing set, and ϕ is pullback asymptotically compact on X. Then ϕ has a unique random attractor $\omega \mapsto A(\omega)$, where*

$$A(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}, \quad \omega \in \Omega.$$

3. Problem Transformation

In this section, we focus our attention on associating a continuous random dynamical system with Eq. (1.1). To do this, we need to convert the stochastic equation into a deterministic equation with a random parameter. In the sequel, we take

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$$

and identify $\omega(t) = W(t)$. Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega,\ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system. Now, we first consider the one-dimensional Ornstein-Uhlenbeck equation

$$d\tilde{z} + \tilde{z}dt = dW(t).$$

It is obvious that its unique stationary solution can be described by

$$\tilde{z}(\theta_t\omega) = -\int_{-\infty}^{0} e^s\omega(t+s)ds + \omega(t), \quad t \in R. \tag{3.1}$$

Note that the random variable $|\tilde{z}(\omega)|$ is tempered and $t \mapsto \log|\tilde{z}(\theta_t\omega)|$ is $\mathbb{P}$-a.e. continuous, it follows from [2, Proposition 4.3.3] that for any $\epsilon > 0$, there is a tempered random variable $\tilde{r}(\omega) > 0$ such that

$$\frac{1}{\tilde{r}(\omega)} \leq |\tilde{z}(\omega)| \leq \tilde{r}(\omega),$$

where $\tilde{r}(\omega)$ satisfies for $\mathbb{P}$-a.s $\omega \in \Omega$,

$$e^{-\epsilon|t|}\tilde{r}(\omega) \leq \tilde{r}(\theta_t\omega) \leq e^{\epsilon|t|}\tilde{r}(\omega). \tag{3.2}$$

Putting $z(\theta_t\omega) = \sigma\tilde{z}(\theta_t\omega)$. Then it solves

$$dz + zdt = \sigma dW(t).$$

Moreover, the above analysis guarantees the following lemmas.

Lemma 3.1. *For any* $\epsilon > 0$*, there is a tempered random variable* $r(\omega) > 0$ *such that*

$$\|z(\theta_t\omega)\| \leq e^{\epsilon|t|} r(\omega),$$

where $r(\omega) = \|\sigma\|\tilde{r}(\omega)$ *satisfies for* $\mathbb{P}$*-a.s* $\omega \in \Omega$*,*

$$e^{-\epsilon|t|} r(\omega) \leq r(\theta_t\omega) \leq e^{\epsilon|t|} r(\omega). \tag{3.3}$$

Lemma 3.2. *For any* $\epsilon > 0$*, there is a tempered random variable* $r'(\omega) > 0$ *such that*

$$\|Az(\theta_t\omega)\| \leq e^{\epsilon|t|} r'(\omega),$$

where $r'(\omega) = \|A\sigma\|\tilde{r}(\omega)$ *satisfies for* $\mathbb{P}$*-a.s* $\omega \in \Omega$*,*

$$e^{-\epsilon|t|} r'(\omega) \leq r'(\theta_t\omega) \leq e^{\epsilon|t|} r'(\omega). \tag{3.4}$$

Let $y(t) = x(t) - z(\theta_t\omega)$. Then $y(t)$ satisfies the following evolution equation with random variable.

$$\frac{dy(t)}{dt} = Ay(t) + F(\theta_t\omega, y_t), \tag{3.5}$$

with initial function

$$y_0(s) = x_0(s) - z(\theta_s\omega), \quad -r \leq s \leq 0,$$

where $F(\theta_t\omega, y_t) := f(y_t + z(\theta_{t+\cdot}\omega)) + Az(\theta_t\omega) + z(\theta_t\omega)$. Therefore, in order to study the asymptotic behavior of x in C, it suffices to investigate Eq. (3.5) with each initial function $y_0 \in C$.

According to the first part in section 2, if A satisfies (H1), then it generates an integrated semigroup $S(t)$, $t \geq 0$, and its part A_0 generates a C_0-semigroup $T_0(t)$, $t \geq 0$. Moreover, the author in [11] gives the relationship between $S(t)$ and $T_0(t)$:

$$S(t)x = \lim_{\lambda \to +\infty} \int_0^t T_0(s)\lambda(\lambda I - A)^{-1} x ds, \quad \text{for} \quad x \in E, t \geq 0. \tag{3.6}$$

On the other hand, if we denote $F^\omega(t, \xi) := F(\theta_t\omega, y_t)$, it is easy to see that $F^\omega : \mathbb{R}^+ \times C \to C$ is continuous in t and globally Lipschitz continuous in ξ for each $\omega \in \Omega$. By the classical theory concerning the existence and uniqueness of the solutions, we obtain that

Proposition 2. *For* $\mathbb{P}$*-a.e* $\omega \in \Omega$ *and each* $y_0 \in C$*, if* $y_0(0) \in \overline{D(A)}$*, Eq.* (3.5) *possesses a unique global integral solution* $y(\cdot, \omega, y_0) \in C([-r, +\infty), E)$ *with* $y(0, \omega, \xi) = y_0$*, which can be expressed as*

$$y(t, \omega, y_0) = \begin{cases} T_0(t)y_0(0) + \lim_{\lambda \to +\infty} \int_0^t T_0(t-\tau)\lambda(\lambda I - A)^{-1} F(\theta_\tau\omega, y_\tau(\cdot, \omega, y_0))d\tau, & t > 0, \\ y_0(t), & -r \leq t \leq 0. \end{cases} \tag{3.7}$$

Here $y_0(0) \in \overline{D(A)}$ is a necessary condition for the existence of integral solutions, see Remark 2.1.

Denote

$$X = \{\xi \in C : \xi(0) \in \overline{D(A)}\},$$

which is also a separable Banach space. Then Eq. (3.5) generates a random dynamical system ϕ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$, where

$$\phi(t, \omega, y_0) = y_t(\cdot, \omega, y_0), \quad \forall (t, \omega, y_0) \in \mathbb{R}^+ \times \Omega \times X. \tag{3.8}$$

Define $\varphi : \mathbb{R} \times \Omega \times X \to X$ by

$$\varphi(t, \omega, x_0) = x_t(\cdot, \omega, x_0) = y_t(\cdot, \omega, y_0) + z(\theta_{t+\cdot}\omega), \quad \forall (t, \omega, \xi) \in \mathbb{R}^+ \times \Omega \times X. \tag{3.9}$$

Then φ is a continuous random dynamical systems associated with Eq. (1.1) on X.

Note that the two random dynamical systems are equivalent. It is easy to check that φ has a random attractor provided ϕ possesses a random attractor. Then, we only need to consider the random dynamical system ϕ.

4. Existence of Random Attractors

In this section, we establish the existence of random attractor by proving the existence of random absorbing set and the asymptotic compactness for ϕ. To this end, we suppose the nonlinear function $f : C \to E$ satisfies

(H2) there exists a constant $L > 0$ such that

$$\|f(\phi_1) - f(\phi_2)\| \le L\|\phi_1 - \phi_2\|_C, \quad \text{for any} \quad \phi_1, \phi_2 \in C;$$

and we need the following assumption on the C_0-semigroup $T_0(t)$ (generated by the part A_0 of A), $t \ge 0$.

(H3) $\|T_0\|_{\mathcal{L}} \le e^{-\alpha t}$, for some $\alpha > 0$.

Lemma 4.1. *For $0 \le \tau \le t$, we have*

$$\|F(\theta_{\tau-t}\omega, 0)\| \le (L+1)e^{\epsilon(t-\tau)}r(\omega) + \|f(0)\|.$$

Proof. Let $\epsilon < \gamma$, where ϵ is the one in (3.4). By the definition of F, we obtain the following estimation

$$\begin{aligned}
\|F(\theta_{\tau-t}\omega, 0)\| &= \|f(z(\theta_{\tau-t+\cdot}\omega)) + Az(\theta_{\tau-t}\omega) + z(\theta_{\tau-t}\omega)\| \\
&\le \|f(z(\theta_{\tau-t+\cdot}\omega)) - f(0)\| + \|f(0)\| + \|Az(\theta_{\tau-t}\omega)\| + \|z(\theta_{\tau-t}\omega)\| \\
&\le L\|z(\theta_{\tau-t+\cdot}\omega)\|_C + \|f(0)\| + \|Az(\theta_{\tau-t}\omega)\| + \|z(\theta_{\tau-t}\omega)\| \\
&= L \sup_{-r\le s\le 0} \|z(\theta_{\tau-t+s}\omega)\| + \|f(0)\| + e^{\epsilon|\tau-t|}r'(\omega) + e^{\epsilon|\tau-t|}r(\omega) \\
&\le (Le^{\epsilon r} + 1)e^{\epsilon(t-\tau)}r(\omega) + e^{\epsilon(t-\tau)}r'(\omega) + \|f(0)\|.
\end{aligned}$$

■

Lemma 4.2. *Assume (H1)-(H3) holds. For $\mathbb{P}$-a.s $\omega \in \Omega$, we have*

$$
\begin{aligned}
&\|y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C\\
&\le \Big(e^{\alpha r}\|y_0\|_C+\frac{L(Le^{\epsilon r}+1)e^{2\alpha r}}{(\alpha-Le^{\alpha r})(\epsilon-Le^{\alpha r})}r(\omega)\\
&\quad+\frac{Le^{2\alpha r}}{(\alpha-Le^{\alpha r})(\epsilon-Le^{\alpha r})}r'(\omega)-\frac{e^{\alpha r}}{\alpha-Le^{\alpha r}}\|f(0)\|\Big)e^{(Le^{\alpha r}-\alpha)t}\\
&\quad+\Big(\frac{\epsilon(Le^{\epsilon r}+1)e^{\alpha r}}{(\alpha-\epsilon)(Le^{\alpha r}-\epsilon)}r(\omega)+\frac{\epsilon e^{\alpha r}}{(\alpha-\epsilon)(Le^{\alpha r}-\epsilon)}r'(\omega)\Big)e^{(\epsilon-\alpha)t}\\
&\quad+\Big(\frac{\alpha(Le^{\epsilon r}+1)e^{\alpha r}}{(\alpha-\epsilon)(\alpha-Le^{\alpha r})}r(\omega)+\frac{\alpha e^{\alpha r}}{(\alpha-\epsilon)(\alpha-Le^{\alpha r})}r'(\omega)+\frac{e^{\alpha r}}{\alpha-Le^{\alpha r}}\|f(0)\|\Big).
\end{aligned}
$$

where ϵ is the one in (3.4), $Le^{\alpha r}\neq\epsilon$, $\alpha\neq\epsilon$, $\alpha\neq Le^{\alpha r}$.

Proof. For $0\le t\le r$, from (3.7) we deduce that

$$
\begin{aligned}
&\|e^{\alpha\cdot}y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C=\sup_{-r\le s\le 0}\|e^{\alpha s}y(t+s,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|\\
&\le\max\Big\{\sup_{-t<s\le 0}e^{\alpha s}\Big(\|T(t+s)y_0(0)\|\\
&\quad+\lim_{\lambda\to+\infty}\int_0^{t+s}\|T(t+s-\tau)\lambda(\lambda I-A)^{-1}F(\theta_{\tau-t}\omega,y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|d\tau\Big),\\
&\quad\sup_{-r\le s\le -t}e^{\alpha s}\|y_0(t+s)\|\Big\}.
\end{aligned}
$$

For simplicity, we take $M=1$ in (H1), i.e.,

$$\|(\lambda I-A)^{-1}\|_{\mathcal{L}}\le\frac{1}{\lambda-\omega}\quad\text{for any }\lambda>\omega.$$

In fact, this can be done if we employ the renorming lemma in [8, Page 17] to introduce an equivalent norm in E. Therefore,

$$
\begin{aligned}
&\|e^{\alpha\cdot}y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C\\
&\le\max\Big\{\sup\nolimits_{-t<s\le 0}e^{\alpha s}e^{-\alpha(t+s)}\|y_0(0)\|+\sup\nolimits_{-t<s\le 0}e^{\alpha s}e^{-\alpha(t+s)}\\
&\quad\textstyle\int_0^{t+s}e^{\alpha\tau}(L\|y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C+\|F(\theta_{\tau-t}\omega,0)\|)d\tau,e^{-\alpha t}\|y_0\|_C\Big\}\\
&\le\max\Big\{e^{-\alpha t}\|y_0(0)\|+Le^{-\alpha t}\textstyle\int_0^t e^{\alpha\tau}\|y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C d\tau\\
&\quad+e^{-\alpha t}\textstyle\int_0^t e^{\alpha\tau}\Big(Le^{\epsilon r}+1)e^{\epsilon(t-\tau)}r(\omega)+e^{\epsilon(t-\tau)}r'(\omega)+\|f(0)\|\Big)d\tau,e^{-\alpha t}\|y_0\|_C\Big\}\\
&\le e^{-\alpha t}\|y_0\|_C+Le^{-\alpha t}\textstyle\int_0^t e^{\alpha\tau}\|y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C d\tau\\
&\quad+\frac{Le^{\epsilon r}+1}{\alpha-\epsilon}e^{-\alpha t}(e^{\alpha t}-e^{\epsilon t})r(\omega)+\frac{1}{\alpha-\epsilon}e^{-\alpha t}(e^{\alpha t}-e^{\epsilon t})r'(\omega)\\
&\quad+\frac{1}{\alpha}e^{-\alpha t}(e^{\alpha t}-1)\|f(0)\|.
\end{aligned}
$$

For $t \geq r$, we have

$$\begin{aligned}
&\|e^{\alpha\cdot}y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C = \sup_{-r\leq s\leq 0}\|e^{\alpha s}y(t+s,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\| \\
&\leq \sup_{-r\leq s\leq 0} e^{\alpha s}\Big(\|T(t+s)y_0(0)\| \\
&\quad + \lim_{\lambda\to+\infty}\int_0^{t+s}\|T(t+s-\tau)\lambda(\lambda I-A)^{-1}F(\theta_{\tau-t}\omega,y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|d\tau\Big) \\
&\leq e^{-\alpha t}\|y_0(0)\| + e^{-\alpha t}\int_0^t e^{\alpha\tau}(L\|y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C + \|F(\theta_{\tau-t}\omega,0)\|)d\tau \\
&\leq e^{-\alpha t}\|y_0\|_C + Le^{-\alpha t}\int_0^t e^{\alpha\tau}\|y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C d\tau \\
&+e^{-\alpha t}\int_0^t e^{\alpha\tau}\Big(Le^{\epsilon r}+1)e^{\epsilon(t-\tau)}r(\omega) + e^{\epsilon(t-\tau)}r'(\omega) + \|f(0)\|\Big)d\tau \\
&\leq e^{-\alpha t}\|y_0\|_C + Le^{-\alpha t}\int_0^t e^{\alpha\tau}\|y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C d\tau \\
&\quad +\frac{Le^{\epsilon r}+1}{\alpha-\epsilon}e^{-\alpha t}(e^{\alpha t}-e^{\epsilon t})r(\omega) + \frac{1}{\alpha-\epsilon}e^{-\alpha t}(e^{\alpha t}-e^{\epsilon t})r'(\omega) \\
&\quad +\frac{1}{\alpha}e^{-\alpha t}(e^{\alpha t}-1)\|f(0)\|.
\end{aligned}$$

Since

$$\begin{aligned}
\sup_{-r\leq s\leq 0}\|e^{\alpha s}y(t+s,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\| &= \sup_{-r\leq s\leq 0} e^{\alpha s}\|y(t+s,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\| \\
&\geq e^{-\alpha r}\|y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C,
\end{aligned}$$

then for any $t \geq 0$,

$$\begin{aligned}
&e^{\alpha t}\|y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C \\
&\leq e^{\alpha r}\|y_0\|_C + \frac{(Le^{\epsilon r}+1)e^{\alpha r}}{\alpha-\epsilon}(e^{\alpha t}-e^{\epsilon t})r(\omega) + \frac{e^{\alpha r}}{\alpha-\epsilon}(e^{\alpha t}-e^{\epsilon t})r'(\omega) \\
&\quad + \frac{e^{\alpha r}}{\alpha}(e^{\alpha t}-1)\|f(0)\| \\
&\quad + Le^{\alpha r}\int_0^t e^{\alpha\tau}\|y_\tau(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C d\tau.
\end{aligned}$$

By the generalized Gronwall inequality, we obtain that

$$\begin{aligned}
&e^{\alpha t}\|y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C \\
&\leq e^{\alpha r}\|y_0\|_C + \frac{(Le^{\epsilon r}+1)e^{\alpha r}}{\alpha-\epsilon}(e^{\alpha t}-e^{\epsilon t})r(\omega) + \frac{e^{\alpha r}}{\alpha-\epsilon}(e^{\alpha t}-e^{\epsilon t})r'(\omega) \\
&\quad + \frac{e^{\alpha r}}{\alpha}(e^{\alpha t}-1)\|f(0)\| \\
&\quad + Le^{\alpha r}\int_0^t\Big(e^{\alpha r}\|y_0\|_C + \frac{(Le^{\epsilon r}+1)e^{\alpha r}}{\alpha-\epsilon}(e^{\alpha s}-e^{\epsilon s})r(\omega)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{e^{\alpha r}}{\alpha-\epsilon}(e^{\alpha s}-e^{\epsilon s})r'(\omega)+\frac{e^{\alpha r}}{\alpha}(e^{\alpha s}-1)\|f(0)\|\Big)e^{Le^{\alpha r}(t-s)}ds\\
\leq&\Big(e^{\alpha r}\|y_0\|_C+\frac{L(Le^{\epsilon r}+1)e^{2\alpha r}}{(\alpha-Le^{\alpha r})(\epsilon-Le^{\alpha r})}r(\omega)\\
&+\frac{Le^{2\alpha r}}{(\alpha-Le^{\alpha r})(\epsilon-Le^{\alpha r})}r'(\omega)-\frac{e^{\alpha r}}{\alpha-Le^{\alpha r}}\|f(0)\|\Big)e^{Le^{\alpha r}t}\\
&+\Big(\frac{\epsilon(Le^{\epsilon r}+1)e^{\alpha r}}{(\alpha-\epsilon)(Le^{\alpha r}-\epsilon)}r(\omega)+\frac{\epsilon e^{\alpha r}}{(\alpha-\epsilon)(Le^{\alpha r}-\epsilon)}r'(\omega)\Big)e^{\epsilon t}\\
&+\Big(\frac{\alpha(Le^{\epsilon r}+1)e^{\alpha r}}{(\alpha-\epsilon)(\alpha-Le^{\alpha r})}r(\omega)+\frac{\alpha e^{\alpha r}}{(\alpha-\epsilon)(\alpha-Le^{\alpha r})}r'(\omega)+\frac{e^{\alpha r}}{\alpha-Le^{\alpha r}}\|f(0)\|\Big)e^{\alpha t},
\end{aligned}
$$

which implies the conclusion. ■

Lemma 4.3. *Let (H1)-(H3) holds and $\alpha > Le^{\alpha r}$. Then there exists a tempered random set $\omega \mapsto K(\omega)$ attracting any tempered random set $\omega \mapsto B(\omega)$, that is, for $\mathbb{P}$-a.e $\omega \in \Omega$, there is $T_B(\omega) > 0$ such that*

$$\phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega) \subset K(\omega), \quad \forall t \geq T_B(\omega).$$

Proof. For $y_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, by Lemma 4.2 we have

$$
\begin{aligned}
&\|\phi(t,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C=\|y_t(\cdot,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C\\
\leq&\Big(e^{\alpha r}\|y_0\|_C+\frac{L(Le^{\epsilon r}+1)e^{2\alpha r}}{(\alpha-Le^{\alpha r})(\epsilon-Le^{\alpha r})}r(\omega)\\
&+\frac{Le^{2\alpha r}}{(\alpha-Le^{\alpha r})(\epsilon-Le^{\alpha r})}r'(\omega)-\frac{e^{\alpha r}}{\alpha-Le^{\alpha r}}\|f(0)\|\Big)e^{(Le^{\alpha r}-\alpha)t}\\
&+\Big(\frac{\epsilon(Le^{\epsilon r}+1)e^{\alpha r}}{(\alpha-\epsilon)(Le^{\alpha r}-\epsilon)}r(\omega)+\frac{\epsilon e^{\alpha r}}{(\alpha-\epsilon)(Le^{\alpha r}-\epsilon)}r'(\omega)\Big)e^{(\epsilon-\alpha)t}\\
&+\Big(\frac{\alpha(Le^{\epsilon r}+1)e^{\alpha r}}{(\alpha-\epsilon)(\alpha-Le^{\alpha r})}r(\omega)+\frac{\alpha e^{\alpha r}}{(\alpha-\epsilon)(\alpha-Le^{\alpha r})}r'(\omega)+\frac{e^{\alpha r}}{\alpha-Le^{\alpha r}}\|f(0)\|\Big).
\end{aligned}
$$

Take $\epsilon < \alpha$, then there exists $T_B(\omega) > 0$ such that for all $t \geq T_B(\omega)$,

$$\|\phi(t,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_C \leq c_1 r(\omega)+c_2 r'(\omega)+c_3,$$

where

$$c_1=\frac{\alpha(Le^{\epsilon r}+1)e^{\alpha r}}{(\alpha-\epsilon)(\alpha-Le^{\alpha r})}+1, \quad c_2=\frac{\alpha e^{\alpha r}}{(\alpha-\epsilon)(\alpha-Le^{\alpha r})}+1,$$

and

$$c_3=\frac{e^{\alpha r}}{\alpha-Le^{\alpha r}}\|f(0)\|+1.$$

Given $\omega \in \Omega$, we denote by

$$K(\omega) = \{\xi \in C_\gamma : \|\xi\|_C \le c_1 r(\omega) + c_2 r'(\omega) + c_3\}.$$

Then $\omega \mapsto K(\omega)$ is a tempered random set because $r(\omega)$ and $r'(\omega)$ are tempered. Moreover, it is absorbing. ■

Lemma 4.4. *For any* $y_{01}, y_{02} \in B(\omega)$, *where* $\omega \mapsto B(\omega)$ *is a tempered random set, we have*

$$\|\phi(t,\omega,y_{01}) - \phi(t,\omega,y_{02})\|_C \le e^{\alpha r} e^{(Le^{\alpha r}-\alpha)t}\|y_{01}-y_{02}\|_C, \quad \forall t \ge 0, \omega \in \Omega. \tag{4.1}$$

Proof. By (3.8) and (3.7), for $0 \le t \le r$, we have

$$\begin{aligned}
&\|e^{\alpha\cdot}(y_t(\cdot,\omega,y_{01}) - y_t(\cdot,\omega,y_{02}))\|_C \\
&\le \max\Big\{\sup_{-t<s\le 0} e^{\alpha s}\|T_0(t+s)(y_{01}(0)-y_{02}(0))\| \\
&\quad + \sup_{-t<s\le 0} e^{\alpha s} \lim_{\lambda\to+\infty}\int_0^{t+s} \|T_0(t+s-\tau)\lambda(\lambda I - A)^{-1}(F(\theta_\tau\omega, y_\tau(\cdot,\omega,y_{01})) \\
&\quad - F(\theta_\tau\omega, y_\tau(\cdot,\omega,y_{02})))\|d\tau, \sup_{-r\le s\le -t} e^{\alpha s}\|y_{01}(t+s) - y_{02}(t+s)\|\Big\} \\
&\le e^{-\alpha t}\|y_{01}-y_{02}\|_C + Le^{-\alpha t}\int_0^t e^{\alpha\tau}\|y_\tau(\cdot,\omega,y_{01}) - y_\tau(\cdot,\omega,y_{02})\|_C d\tau.
\end{aligned}$$

For $t > r$, one easily deduces that

$$\begin{aligned}
&\|\phi(t,\omega,y_{01}) - \phi(t,\omega,y_{01})\|_C = \|y_t(\cdot,\omega,y_{01}) - y_t(\cdot,\omega,y_{02})\|_C \\
&\le \sup_{-r\le s\le 0} \|T_0(t+s)(y_{01}(0)-y_{02}(0))\| \\
&\quad + \sup_{-r\le s\le 0} \lim_{\lambda\to+\infty}\int_0^{t+s} \|T_0(t+s-\tau)\lambda(\lambda I - A)^{-1}(F(\theta_\tau\omega, y_\tau(\cdot,\omega,y_{01})) \\
&\quad - F(\theta_\tau\omega, y_\tau(\cdot,\omega,y_{02})))\|d\tau \\
&\le e^{-\alpha t}e^{\alpha r}\|y_{01}-y_{02}\|_C + Le^{-\alpha t}e^{\alpha r}\int_0^t e^{\alpha\tau}\|y_\tau(\cdot,\omega,y_{01}) - y_\tau(\cdot,\omega,y_{02})\|_C d\tau.
\end{aligned}$$

Then

$$\begin{aligned}
&e^{\alpha t}\|y_t(\cdot,\omega,y_{01}) - y_t(\cdot,\omega,y_{02})\|_C \\
&\le e^{\alpha r}\|y_{01}-y_{02}\|_C + Le^{\alpha r}\int_0^t e^{\alpha\tau}\|y_\tau(\cdot,\omega,y_{01}) - y_\tau(\cdot,\omega,y_{02})\|_C d\tau.
\end{aligned}$$

By the classical Gronwall inequality, we arrive at

$$e^{\alpha t}\|y_t(\cdot,\omega,y_{01}) - y_t(\cdot,\omega,y_{02})\|_C \le e^{\alpha r}e^{Le^{\alpha r}t}\|y_{01}-y_{02}\|_C,$$

which implies the conclusion. ■

Lemma 4.5. *Let (H1)-(H3) hold and $\alpha > Le^{\alpha r}$. Then ϕ is pullback asymptotically compact.*

Proof. We need to prove that for every sequence $t_n \to +\infty$ and $\mathbb{P}$-a.e $\omega \in \Omega$, the sequence $\{\phi(t_n, \theta_{-t_n}\omega, y_0(\theta_{-t_n}\omega))\}_{n=1}^{+\infty}$ has a convergent subsequence as $t_n \to +\infty$, where $y_0(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$ with $\omega \mapsto B(\omega)$ tempered. To this end, we show that the Kuratowski's measure of non-compactness satisfies the following

$$\kappa\Big(\phi(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega))\Big) \to 0, \quad t_n \to +\infty.$$

Replacing t by t_n and ω by $\theta_{-t_n}\omega$ in (4.1), it follows that, for any $y_{01}(\theta_{-t_n}\omega)$, $y_{02}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$ and

$$\begin{aligned}&\|\phi(t_n, \theta_{-t_n}\omega, y_{01}(\theta_{-t_n}\omega)) - \phi(t_n, \theta_{-t_n}\omega, y_{02}(\theta_{-t_n}\omega))\|_C \\ &\leq e^{\alpha r} e^{(Le^{\alpha r}-\alpha)t}\|y_{01}(\theta_{-t_n}\omega)) - y_{02}(\theta_{-t_n}\omega))\|_C.\end{aligned}$$

Since $\omega \mapsto B(\omega)$ is tempered, for any $\epsilon > 0$ and each $\omega \in \Omega$, there exist tempered random sets $B_i(\theta_{-t_n}\omega)$, $i = 1, 2, \cdots, m$, such that $B(\theta_{-t_n}\omega) \subset \bigcup_{i=1}^m B_i(\theta_{-t_n}\omega)$ and

$$\text{diam}(B_i(\theta_{-t_n}\omega)) \leq \kappa(B(\theta_{-t_n}\omega)) + \epsilon, \quad i = 1, 2, \cdots, m.$$

For any u, $v \in \phi(t_n, \theta_{-t_n}\omega, B_i(\theta_{-t_n}\omega))$, there exist u_0, $v_0 \in B_i(\theta_{-t_n}\omega)$ such that $u = \phi(t_n, \theta_{-t_n}\omega, u_0)$ and $v = \phi(t_n, \theta_{-t_n}\omega, v_0)$. Thus,

$$\begin{aligned}\|u - v\|_C &= \|\phi(t_n, \theta_{-t_n}\omega, u_0) - \phi(t_n, \theta_{-t_n}\omega, v_0)\|_C \\ &\leq e^{\alpha r} e^{(Le^{\alpha r}-\alpha)t_n}\|u_0 - v_0\|_C \\ &\leq e^{\alpha r} e^{(Le^{\alpha r}-\alpha)t_n}\text{diam}(B_i(\theta_{-t_n}\omega)) \\ &\leq e^{\alpha r} e^{(Le^{\alpha r}-\alpha)t_n}\kappa(B(\theta_{-t_n}\omega)) + \epsilon,\end{aligned}$$

which implies that

$$\text{diam}(\phi(t_n, \theta_{-t_n}\omega, B_i(\theta_{-t_n}\omega)) \leq e^{\alpha r} e^{(Le^{\alpha r}-\alpha)t_n}\kappa(B(\theta_{-t_n}\omega)) + \epsilon.$$

Therefore,

$$\kappa\Big(\phi(t_n, \theta_{-t_n}\omega, B_i(\theta_{-t_n}\omega)\Big) \leq e^{\alpha r} e^{(Le^{\alpha r}-\alpha)t_n}\kappa(B(\theta_{-t_n}\omega)) + \epsilon,$$

and hence

$$\kappa\Big(\phi(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega)\Big) \leq e^{\alpha r} e^{(Le^{\alpha r}-\alpha)t_n}\kappa(B(\theta_{-t_n}\omega)) + \epsilon.$$

By the arbitrary of ϵ, we obtain that

$$\kappa\Big(\phi(t_n, \theta_{-t_n}\omega, B(\theta_{-t_n}\omega)\Big) \leq e^{\alpha r} e^{(Le^{\alpha r}-\alpha)t_n}\kappa(B(\theta_{-t_n}\omega)) \to 0, \quad t_n \to +\infty. \qquad \blacksquare$$

As a consequence of Theorem 2.1, Lemmas 4.3 and 4.5, we have already proved the main result of this paper.

Theorem 4.1. *Suppose that (H1)-(H3) hold. If* $\alpha > Le^{\alpha r}$*, the continuous random dynamical system* ϕ *defined in (3.8) possesses a unique random attractor* $\omega \mapsto A(\omega) \subset X$*, where*

$$A(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}, \quad \omega \in \Omega, \tag{4.2}$$

with $K(\omega)$ *given in Lemma 4.3.*

Corrollary 4.1. *Suppose that* (*H1*)-(*H3*) *hold. If* $\alpha > Le^{\alpha r}$*, the continuous random dynamical system* ψ *associated with* (*1.1*) *possesses a unique random attractor* $\omega \mapsto A(\omega) + z(\theta.\omega) \subset X$*, where* $A(\omega)$ *is given in (4.2),* $z(\theta_s\omega) = \sigma\tilde{z}(\theta_s\omega)$*,* $s \leq 0$*, with* $\tilde{z}$ *is given in (3.1)*

5. Example

As an application of Theorem 4.1, we consider the following age-structured model with white noise

$$\begin{cases} \partial_t u + \partial_a u = -\mu(a)u(t,a) \\ \qquad + \displaystyle\int_0^{+\infty} f(a,b,u(t-r,b))dbds + \delta(a)dW(t), & t > 0,\ a > 0, \\ u(t,0) = \beta \displaystyle\int_0^{+\infty} u(t,a)da, & t > 0, \\ u(s,a) = u_0(s,a), & a > 0 \end{cases} \tag{5.1}$$

with $u(t,\cdot) \in L^1(0,+\infty)$, the space of Lebesgue integrable functions with values in $\mathbb{R}$; $\mu \in L^1(0,+\infty)$ with nonnegative values; $\delta \in H^1(0,+\infty)$, $\delta(0) = 0$; $\beta \geq 0$ and $W(t)$ being the white noise. For the information about Eq. (5.1) without the white noise, we refer the reader to the book [12].

Let

$$E = \mathbb{R} \times L^1(0,+\infty)$$

with the usual product norm of $\mathbb{R} \times L^1(0,+\infty)$. Define $A : D(A) \subset E \to E$ as following

$$A\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} -\phi(0) \\ -\phi' - \mu\phi \end{pmatrix}, \ \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in D(A), \tag{5.2}$$

where

$$D(A) = \{0\}_{\mathbb{R}} \times \{\phi \in L^1(0,+\infty) : \phi' \in L^1(0,+\infty), \phi(0) = 0\}.$$

Clearly $E_0 = \overline{D(A)} = \{0\}_{\mathbb{R}} \times L^1(0,+\infty) \neq E$. Denote by $C = \mathbb{R} \times C([-r,0], L^1(0,+\infty))$ with the norm

$$\left\| \begin{pmatrix} a \\ \phi \end{pmatrix} \right\|_C = |a| + \sup_{-r \leq s \leq 0} \|\phi(s)\|_{L^1},$$

and define the nonlinear term $F : C \to E$ as following

$$F\left(\begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) = \begin{pmatrix} \beta \int_0^{+\infty} \phi(0)(a)da \\ \int_0^{+\infty} f(a,b,\phi(-r)(b))db \end{pmatrix}. \tag{5.3}$$

Set $v(t) = \begin{pmatrix} 0 \\ u(t,\cdot) \end{pmatrix} \in E_0$, $v_t = \begin{pmatrix} 0 \\ u_t \end{pmatrix} \in C$ and $\begin{pmatrix} 0 \\ \xi \end{pmatrix} = v_0 \in C$ where $\xi(s)(a) = u_0(s,a)$, and $\sigma = \begin{pmatrix} 0 \\ \delta \end{pmatrix} \in D(A)$, then Eq. (5.1) can be written as

$$\begin{cases} dv(t) = Av(t)dt + F(v_t)dt + \sigma dW(t), & t > 0, \\ v(0) = v_0 \in C. \end{cases} \tag{5.4}$$

Proposition 3. *Suppose that there exists a constant $\underline{\mu} > 0$, such that*

$$\mu(a) > \underline{\mu}, \quad \forall a \geq 0. \tag{5.5}$$

Then we have

(i) *The operator A defined in (5.2) is a Hille-Yosida operator with $(-\underline{\mu}, +\infty) \subset \rho(A)$ and*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}} \leq \frac{1}{\lambda + \underline{\mu}}, \quad \forall \lambda > -\underline{\mu};$$

(ii) *the C_0-semigroup $T_0(t)$, generated by A_0 on X_0, satisfies that*

$$\|T_0(t)\|_{\mathcal{L}} \leq e^{-\underline{\mu}t}, \quad \forall t \geq 0.$$

Proof. (i) From (5.2), we know that

$$(\lambda I - A)\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \phi(0) \\ \phi' + (\lambda + \mu)\phi \end{pmatrix}.$$

Set $y = \phi(0)$ and $\psi = \phi' + (\lambda + \mu)\phi$. Then

$$\phi(a) = e^{-\lambda a - \int_0^a \mu(s)ds} y + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau - s)d\tau} \psi(s)ds. \tag{5.6}$$

By (5.5), $\phi \in L^1(0,+\infty)$ provided that $\lambda > -\underline{\mu}$. Therefore, for any $\lambda > -\underline{\mu}$,

$$(\lambda I - A)^{-1}\begin{pmatrix} y \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi \end{pmatrix}$$

if and only if (5.6) holds. A simple calculation shows that

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}} \le \frac{1}{\lambda + \underline{\mu}}, \quad \forall \lambda > -\underline{\mu}.$$

(ii) The C_0-semigroup $T_0(t)$, generated by A_0 on E_0, possesses the following form

$$T_0(t)\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{T}_0(t)\phi \end{pmatrix}, \tag{5.7}$$

where

$$\widetilde{T}_0(t)\phi = \begin{cases} 0, & a < t, \\ \phi(a-t)e^{-\int_{a-t}^{a}\mu(\tau)d\tau}, & a \ge t. \end{cases} \tag{5.8}$$

Then

$$\begin{aligned} \left\| T_0(t)\begin{pmatrix} 0 \\ \phi \end{pmatrix} \right\| &= \|\widetilde{T}_0(t)\phi\|_{L^1} \\ &= \int_t^{+\infty} |\phi(a-t)e^{-\int_{a-t}^{a}\mu(\tau)d\tau}| da \\ &= \int_0^{+\infty} |\phi(a)| e^{-\int_a^{a+t}\mu(\tau)d\tau} da \\ &\le e^{-\underline{\mu}t}\|\phi\|_{L^1}, \end{aligned}$$

which implies that $\|T_0(t)\|_{\mathcal{L}} \le e^{-\underline{\mu}t}$, $\forall t \ge 0$. ■

In order to obtain the existence of random attractor of Eq. (5.1), we need the following assumptions on f.

(H_f) There exists a nonnegative function $L(\cdot) \in L^1(0,\infty)$ such that

$$\int_0^{+\infty} |f(a,b,\phi_1(s)(b)) - f(a,b,\phi_2(s)(b))| db \le L(a)\|\phi_1(s) - \phi_2(s)\|_{L^1}, \quad \forall a \ge 0.$$

Then F is globally Lipschitz continuous with Lipschitzian constant $\beta + \|L\|_{L^1}$.

Theorem 5.1. *Suppose that (H_f) and (5.5) hold true with*

$$\underline{\mu} > (\beta + \|L\|_{L^1})e^{\underline{\mu}r},$$

then Eq. (5.1) has a random attractor.

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