TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 1, pp. 305-312, February 2014
DOI: 10.11650/tjm.18.2014.2594
This paper is available online at http://journal.taiwanmathsoc.org.tw

NEW BOUNDS FOR EIGENVALUES OF THE HADAMARD PRODUCT AND THE FAN PRODUCT OF MATRICES

Guang-Hui Cheng

Abstract. In this paper, we proposed some lower bounds for the minimum eigenvalue of the Fan product of M-matrices, and a upper bound for the spectral radius of the Hadamard product of nonnegative matrices. These improve two existing results. To illustrate our results, two simple examples are considered.

1. INTRODUCTION

For convenience, the set $\{1, 2, \ldots, n\}$ is denoted by N, where n is any positive integer. For any two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, the Hadamard product of A and B is defined by $A \circ B = (a_{ij}b_{ij})$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a nonnegative matrix if $a_{ij} \geq 0$. A matrix $A \in \mathbb{R}^{n\times n}$ is called a nonsingular M-matrix [4] if there exists $P \geq 0$ and $\alpha > 0$ such that $A = \alpha I - P \quad \text{and} \quad \alpha > \rho(P)$, where $\rho(P)$ is the spectral radius of the nonnegative matrix P, I is the $n \times n$ identity matrix. Denote by $\mathcal{M}_n$ the set of all $n \times n$ nonsingular M-matrices. Denote $\tau(A) = \min\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}$, and $\sigma(A)$ denotes the spectrum of A. If $A \in \mathcal{M}_n$, then [3]

$$\tau(A) = \frac{1}{\rho(A^{-1})}$$

is a positive real eigenvalue, and the corresponding eigenvector is nonnegative.

A matrix A is irreducible if there does not exist a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix},$$

Received November 3, 2012, accepted July 31, 2013.
Communicated by Wen-Wei Lin.
2010 *Mathematics Subject Classification*: 15A18, 15A42.
Key words and phrases: Hadamard product, Fan product, Spectral radius, Nonnegative matrix, Minimum eigenvalue, M-matrix.
This research was supported by the National Natural Science Foundation of China (Grant No. 11301056).

where $A_{1,1}$ and $A_{2,2}$ are square matrices.

It is known [3, p.358] that if $A, B \in \mathbb{R}^{n\times n}$ are nonnegative matrices, then

$$\rho(A \circ B) \leq \rho(A)\rho(B).$$

Evidently, this equality can be very weak in some cases. For example, if $A = I$ and $B = J$, where J is the $n \times n$ matrix of all ones, then

$$\rho(A \circ B) = \rho(A) = 1 \ll \rho(A)\rho(B) = n$$

when n is very large. See [2] for some generalizations.

Recently, Zheng and Cui obtained the following result in [8].

Theorem 1. *If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ nonnegative matrices, then*

$$\rho(A \circ B) \leq \min\{\max_{1\leq i\leq n}\{(a_{ii} - \alpha_i)b_{ii} + \alpha_i\rho(B)\}, \max_{1\leq i\leq n}\{(b_{ii} - \beta_i)a_{ii} + \beta_i\rho(A)\}\}, \tag{1}$$

and

$$\rho(A\circ B) \leq \max \frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj})+4\beta_i\beta_j(\rho(A)-a_{ii})(\rho(A)-a_{jj})]^{\frac{1}{2}}\}, \tag{2}$$

where $\alpha_i = \max_{k\neq i}\{a_{ik}\}$ and $\beta_i = \max_{k\neq i}\{b_{ik}\},\ \forall i \in N$.

In fact, since the Hadamard product is commutative, if A and B are switched, we can easily obtain the following results from the inequality (2).

Theorem 2. *If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ nonnegative matrices, then*

$$\rho(A\circ B) \leq \max \frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj})+4\alpha_i\alpha_j(\rho(B)-b_{ii})(\rho(B)-b_{jj})]^{\frac{1}{2}}\right\}, \tag{3}$$

where $\alpha_i = \max_{k\neq i}\{a_{ik}\}$ and $\beta_i = \max_{k\neq i}\{b_{ik}\},\ \forall i \in N$.

Theorem 3. *If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ nonnegative matrices, then*

$$\begin{aligned} &\rho(A \circ B) \\ &\leq \min\Big\{\max \frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj})+4\beta_i\beta_j(\rho(A)-a_{ii})(\rho(A)-a_{jj})]^{\frac{1}{2}}\}, \\ &\max \frac{1}{2}\{a_{ii}b_{ii} + a_{jj}b_{jj} + [(a_{ii}b_{ii} - a_{jj}b_{jj}) + 4\alpha_i\alpha_j(\rho(B) - b_{ii})(\rho(B) - b_{jj})]^{\frac{1}{2}}\}\Big\}, \end{aligned} \tag{4}$$

where $\alpha_i = \max_{k\neq i}\{a_{ik}\}$ and $\beta_i = \max_{k\neq i}\{b_{ik}\},\ \forall i \in N$.

Remark 1. Without loss of generality, for $i \neq j$, assume that

$$(a_{ii} - \beta_i)b_{ii} + \beta_i\rho(B) \geq (a_{jj} - \beta_j)b_{jj} + \beta_j\rho(B),$$

i.e.,

$$a_{ii}b_{ii} - a_{jj}b_{jj} + \beta_i(\rho(B) - b_{ii}) \geq \beta_j(\rho(B) - b_{jj}) \geq 0, \tag{5}$$

From (1) and (5), we have

$$\begin{aligned} &\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}+[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\beta_i\beta_j(\rho(B)-b_{ii})(\rho(B)-b_{jj})]^{\frac{1}{2}}\} \\ &\leq \frac{1}{2}\{2a_{ii}b_{ii} + 2\beta_i(\rho(B) - b_{ii})\} \\ &= a_{ii}b_{ii} + \beta_i(\rho(B) - b_{ii}). \end{aligned} \tag{6}$$

Hence, from (1), (4) and (6), it is easy to know that the result of Theorem 3 is sharper than one of Theorem 1.

Let $A, B \in \mathbb{C}^{n\times n}$, the Fan product of A and B is denoted by $A \star B \equiv C = (c_{ij}) \in \mathbb{C}^{n\times n}$ and is defined by

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & i \neq j, \\ a_{ii}b_{ii}, & i = j. \end{cases}$$

If $A, B \in \mathcal{M}_n$, then $A \star B$ is M-matrix. There are some inequalities for the minimum eigenvalue of the Fan product of M-matrices as follows.

Theorem 4. [6]. *If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ nonsingular M-matrices, then*

$$\begin{aligned} \tau(A \star B) \geq \min_{i\neq j} \frac{1}{2}\Bigg\{ &a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg[(a_{ii}b_{ii} - a_{jj}b_{jj}) \\ &+4(a_{ii} - \tau(A))(b_{ii} - \tau(B))(a_{jj} - \tau(A))(b_{jj} - \tau(B))\Bigg]^{\frac{1}{2}}\Bigg\}. \end{aligned} \tag{7}$$

Theorem 5. [9]. *If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ nonsingular M-matrices, then*

$$\tau(A\star B) \geq \max\left\{ \min_{1\leq i\leq n}\{(a_{ii}-\alpha_i)b_{ii}+\alpha_i\tau(B)\},\ \min_{1\leq i\leq n}\{(b_{ii}-\beta_i)a_{ii}+\beta_i\tau(A)\}\right\}, \tag{8}$$

where $\alpha_i = \max_{k\neq i}\{|a_{ik}|\}$ and $\beta_i = \max_{k\neq i}\{|b_{ik}|\}$, $\forall i \in N$.

Remark 2. From (7), we must know $\tau(A)$ and $\tau(B)$ before the bound of $\tau(A \star B)$ can be computed. But from (8), if we know one of $\tau(A)$ and $\tau(B)$, then the bound of $\tau(A \star B)$ will be also obtained.

Based on Remark 2, in section 2, we will give some sharp results for the Fan product of two nonsingular M-matrices which can be calculated by one of $\tau(A)$ and $\tau(B)$.

2. Some LoweR Bounds for the Minimum Eigenvalue of the Fan Product of M-Matrices

Firstly, we will give some lemmas in this section. Secondly, we will propose some lower bounds for the minimum eigenvalue of the Fan product of M-matrices.

Lemma 1. [1]. *If $A \geq 0$ is an irreducible $n \times n$ matrix, then there exists a positive eigenvector x such that $Ax = \rho(A)x$.*

Lemma 2. [7]. *Let $A,\ B$ be two nonsingular M-matrices and if D and E are two positive diagonal matrices, then*

$$D(A \star B)E = (DAE) \star B = (DA) \star (BE) = (AE) \star (DB) = A \star (DBE).$$

Lemma 3. *Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, $n \geq 2$. Then all the eigenvalues of A lie inside the union of $\frac{n(n-1)}{2}$ ovals of Cassini, i.e.,*

$$\sigma(A) \subseteq \cup \left\{ z \in \mathbb{C} :\ |z - a_{ii}| \cdot |z - a_{jj}| \leq (\sum_{k=1, k\neq i}^{n} |a_{ik}|)(\sum_{k=1, k\neq j}^{n} |a_{jk}|),\ i \neq j \right\}.$$

Lemma 4. *Let $A = (a_{ij}) \in \mathcal{M}_n$, $n \geq 2$. Then*

$$\tau(A) \geq \min_{i\neq j} \frac{1}{2} \left\{ a_{ii} + a_{jj} - [(a_{ii} - a_{jj})^2 + 4(\sum_{k=1, k\neq i}^{n} |a_{ik}|)(\sum_{k=1, k\neq j}^{n} |a_{jk}|)]^{\frac{1}{2}} \right\}.$$

Proof. Since $A - \tau(A)I$ is a singular M-matrix, Theorem 6.4.16 of [4] yields that

$$a_{ii} - \tau(A) > 0, \quad \forall i \in N. \tag{9}$$

By Lemma 3, there exist $i_0,\ j_0 (i_0 \neq j_0)$ such that

$$|\tau(A) - a_{ii}||\tau(A) - a_{jj}| \leq (\sum_{k=1, k\neq i_0}^{n} |a_{i_0 k}|)(\sum_{k=1, k\neq j_0}^{n} |a_{j_0 k}|). \tag{10}$$

By (9) and (10), we have

$$(\tau(A) - a_{ii})(\tau(A) - a_{jj}) \leq (\sum_{k=1, k\neq i_0}^{n} |a_{i_0 k}|)(\sum_{k=1, k\neq j_0}^{n} |a_{j_0 k}|). \tag{11}$$

Solving inequality (11), we get

$$\tau(A) \geq \frac{1}{2} \left\{ a_{i_0 i_0} + a_{j_0 j_0} - [(a_{i_0 i_0} - a_{j_0 j_0})^2 + 4(\sum_{k=1, k\neq i_0}^{n} |a_{i_0 k}|)(\sum_{k=1, k\neq j_0}^{n} |a_{j_0 k}|)]^{\frac{1}{2}} \right\}.$$

Hence,

$$\tau(A) \geq \min_{i\neq j} \frac{1}{2} \left\{ a_{ii} + a_{jj} - [(a_{ii} - a_{jj})^2 + 4(\sum_{k=1, k\neq i}^{n} |a_{ik}|)(\sum_{k=1, k\neq j}^{n} |a_{jk}|)]^{\frac{1}{2}} \right\}.$$

Theorem 6. *If* $A = (a_{ij}),\ B = (b_{ij}) \in \mathcal{M}_n$, $n \geq 2$, *then*

$$\tau(A \star B) \geq \min_{i \neq j} \frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\alpha_i\alpha_j(b_{ii}-\tau(B))(b_{jj}-\tau(B))]^{\frac{1}{2}}\right\}. \tag{12}$$

where $\alpha_i = \max_{k\neq i}\{|a_{ik}|\},\ \forall i \in N.$

Proof. In this proof, two cases will be discussed in the following.

Case 1. If A and B are irreducible, then $A \star B$ is irreducible, and B^{-1} is nonnegative and irreducible. By Lemma 1, there exists a positive vector $v = (v_1, \ldots, v_n)^T$ such that

$$Bv = \tau(B)v.$$

Hence, we have

$$b_{ii}v_i - \sum_{j\neq i} |b_{ij}|v_j = \tau(B)v_i, \quad \forall i \in N,$$

i.e.,

$$\frac{\sum\limits_{j\neq i} |b_{ij}|v_j}{v_i} = b_{ii} - \tau(B), \quad \forall i \in N. \tag{13}$$

Define a positive diagonal matrix $V = (v_1, \ldots, v_n)^T$. Let $\widetilde{B} = (\tilde{b}_{ij}) = V^{-1}BV$, then we get

$$\widetilde{B} = (\tilde{b}_{ij}) = V^{-1}BV = \begin{bmatrix} a_{11} & \frac{a_{12}v_2}{v_1} & \ldots & \frac{a_{1n}v_n}{v_1} \\ \frac{a_{21}v_1}{v_2} & a_{22} & \ldots & \frac{a_{2n}v_n}{v_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}v_1}{v_n} & \frac{a_{n2}v_2}{v_n} & \ldots & a_{nn} \end{bmatrix}.$$

According to Lemma 2, we have

$$V^{-1}(A \star B)V = A \star (V^{-1}BV) = A \star \widetilde{B}.$$

Hence, $\tau(A \star B) = \tau(A \star \widetilde{B})$.

Let us denote $\alpha_i = \max_{k\neq i}\{|a_{ik}|\},\ \forall i \in N$. Since A is an irreducible nonnegative matrix, $\alpha_i > 0,\ \forall i \in N$. By Lemma 4 and the equality (13), we can obtain

$$\begin{aligned}\tau(A \star B) \geq\ & \min_{i\neq j} \frac{1}{2}\Bigg\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ & +4(\sum_{k=1,k\neq i}^{n} \frac{|a_{ik}b_{ik}v_k|}{v_i})(\sum_{k=1,k\neq j}^{n} \frac{|a_{jk}b_{jk}v_k|}{v_j})]^{\frac{1}{2}}\Bigg\} \\ \geq\ & \min_{i\neq j} \frac{1}{2}\Bigg\{a_{ii}b_{ii} + a_{jj}b_{jj} - [(a_{ii}b_{ii} - a_{jj}b_{jj})^2\end{aligned}$$

$$
\begin{aligned}
&+4\alpha_i\alpha_j(\sum_{k=1,k\neq i}^{n}\frac{|b_{ik}|v_k}{v_i})(\sum_{k=1,k\neq j}^{n}\frac{|b_{jk}|v_k}{v_j})]^{\frac{1}{2}}\Bigg\} \\
= &\min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\
&+4\alpha_i\alpha_j(b_{ii}-\tau(B))(b_{jj}-\tau(B))]^{\frac{1}{2}}\Bigg\}.
\end{aligned}
$$

Case 2. If either A or B is reducible, then $A\star B$ must be reducible. Let $T=(t_{ij})$ be the permutation matrix such that $t_{12}=t_{23}=\cdots=t_{n-1,n}=t_{n,1}=1$ and the remaining $t_{ij}=0$. Then there exists a positive real number ϵ such that $A-\epsilon T$ and $B-\epsilon T$ are two irreducible M-matrices. Apply the first case on them and then use continuity argument and to complete the proof. ■

Remark 3. If $a_{ii}\geq\tau(A)+\alpha_i$ for all $i=1,\ldots,n$, then $(a_{ii}-\tau(A))(a_{jj}-\tau(A))\geq\alpha_i\alpha_j$ for all $1\leq i\neq j\leq n$, *i.e.*, the bound of (12) is the better than the one of (7).

Since the Fan product is commutative, the following result can be immediately obtained.

Theorem 7. *If* $A=(a_{ij}),\ B=(b_{ij})\in\mathcal{M}_n$, $n\geq 2$, *then*

$$
\begin{aligned}
\tau(A\star B)\geq &\min_{i\neq j}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2 \\
&+4\beta_i\beta_j(a_{ii}-\tau(A))(a_{jj}-\tau(A))]^{\frac{1}{2}}\Bigg\}.
\end{aligned}
\tag{14}
$$

where $\beta_i=\max_{k\neq i}\{|b_{ik}|\},\ \forall i\in N$.

According to Theorem 6 and Theorem 7, the following theorem can be immediately obtained.

Theorem 8. *If* $A=(a_{ij}),\ B=(b_{ij})\in\mathcal{M}_n$, $n\geq 2$, *then*

$$
\begin{aligned}
&\tau(A\star B) \\
\geq &\max\Bigg\{\min_{i\neq j}\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\alpha_i\alpha_j(b_{ii}-\tau(B))(b_{jj}-\tau(B))]^{\frac{1}{2}}\}, \\
&\min_{i\neq j}\frac{1}{2}\{a_{ii}b_{ii}+a_{jj}b_{jj}-[(a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\beta_i\beta_j(a_{ii}-\tau(A))(a_{jj}-\tau(A))]^{\frac{1}{2}}\}\Bigg\}.
\end{aligned}
$$

where $\alpha_i=\max_{k\neq i}\{|a_{ik}|\},\ \beta_i=\max_{k\neq i}\{|b_{ik}|\},\ \forall i\in N$.

Remark 4. Similar to Remark 1, we can prove that the result of Theorem 8 is sharper than one of Theorem 4.

3. Examples

In this section, we will consider two examples for validating our results.

Example 1. Consider two 3×3 nonnegative matrices as follows.

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

By direct calculation, $\rho(A \circ B) = 10.7568$. According to inequalities (1) and (4), we have

$$\rho(A \circ B) \le 12.3852,$$

and

$$\rho(A \circ B) \le 11.3278,$$

respectively.

Example 2. Consider two 3×3 M-matrices as follows.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -0.5 \\ -0.5 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -0.25 & -0.25 \\ -0.5 & 1 & -0.25 \\ -0.25 & -0.5 & 1 \end{bmatrix}.$$

By direct calculation, $\tau(A \star B) = 0.9377$. According to Theorem 4 and Theorem 8, we have

$$\tau(A \star B) \ge 0.7701,$$

and

$$\tau(A \star B) \ge 0.8536,$$

respectively.

From the two examples above, we can conclude our results are better than the relevant ones. The bounds for eigenvalues have definitely improved.

Acknowledgment

The author would like to thank the editor and the anonymous reviewer for their constructive suggestions and helpful comments, which led to significant improvement of the original manuscript.

References

1. R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

2. L. Elsner, C. R. Johnson and J. A. Dias da Silva, The Perron root of a weighted geometric mean of nonnegative matrices, *Linear and Multilinear Algebra*, **24** (1988), 1-13.

3. R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

4. A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics in Applied Mathematics, Vol. 9, SIAM, Philadelphia, 1994.

5. M. Z. Fang, Bounds on eigenvalues of the Hadamard product and the Fan product of matrices, *Linear Alegebra. Appl.*, **425** (2007), 7-15.

6. Q. B. Liu and G. L. Chen, On two inequalities for the Hadamard product and the Fan product of matrices, *Linear Algebra. Appl.*, **431** (2009), 974-984.

7. Q. B. Liu, G. L. Chen and L. L. Zhao, Some new bounds on the spectral radius of matrices, *Linear Algebra. Appl.*, **432** (2010), 936-948.

8. Y. M. Zheng and R. Q. Cui, Upper bound of the spectral radius for Hadamard product of nonnegative matrices, *Journal of Henan Polytechnic University* (*Natrual Science*), **29(4)** (2010), 543-546.

9. G. H. Cheng and J. Li, New bounds for the minimum eigenvalue of the Fan product of two M-matrices, appear to *Czechoslovak Mathematical Journal*.

Guang-Hui Cheng
School of Mathematical Sciences
University of Electronic Science and Technology
Chengdu, Sichuan, 611731
P. R. China
E-mail: ghcheng@126.com