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MULTI-PARAMETER TRIEBEL-LIZORKIN AND BESOV SPACES ASSOCIATED WITH ZYGMUND DILATION

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Abstract. In this paper, the authors use the discrete Littlewood-Paley-Stein theory to develop a theory of multi-parameter Triebel-Lizorkin and Besov spaces associated with Zygmund dilation. They also obtain the boundedness of Ricci-Stein singular integral operators on multi-parameter Triebel-Lizorkin and Besov spaces associated with Zygmund dilation.

1. Introduction and Statement of Main Results

The multi-parameter theory has been developed by many authors over the past decades. We refer the reader to the work of Carleson [1], Fefferman and Stein [3], Gundy and Stein [5], etc. The Ricci-Stein singular integral operators were first considered by Ricci and Stein in [9], they obtained that the operators are bounded on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$. Fefferman and Pipher in [4] further showed that the Ricci-Stein singular integral operators are bounded in weighted L^p_w spaces for all $1 < p < \infty$ when the weight w satisfies an analogous condition of Muckenhoupt associated with Zygmund dilation.

Recently, Han and Lu in [6, 7] developed multi-parameter Hardy spaces $H^p_Z(\mathbb{R}^3)$ associated with Zygmund dilation via the discrete Littlewood-Paley-Stein theory and discrete Calderón's identity. For $0 < p \leq 1$, they proved the boundedness of Ricci-Stein singular integral operators from $H^p_Z(\mathbb{R}^3)$ to $H^p_Z(\mathbb{R}^3)$ and from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$. The dual space of $H^p_Z(\mathbb{R}^3)$ was also obtained.

The main purpose of this paper is to develop the theory of multi-parameter Triebel-Lizorkin spaces $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$ and Besov spaces $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$ associated with the Zygmund dilation. The boundedness of Ricci-Stein singular integral operators on $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$ and $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$ are also established.

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Let $\mathcal{S}(\mathbb{R}^n)$ denote Schwartz functions on $\mathbb{R}^n$. Let a test function ψ be defined on $\mathbb{R}^3$ by

$$\psi(x,y,z) = \psi^{(1)}(x)\psi^{(2)}(y,z),$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R})$, $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$, and satisfy

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi_1)|^2 = 1$$

for all $\xi_1 \in \mathbb{R}\backslash\{0\}$, and

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}\xi_2, 2^{-k}\xi_3)|^2 = 1$$

for all $(\xi_2, \xi_3) \in \mathbb{R}^2\backslash\{(0,0)\}$, and the cancellation conditions

$$\int_{\mathbb{R}} x^\alpha \psi^{(1)}(x)\, dx = \int_{\mathbb{R}^2} y^\beta z^\gamma \psi^{(2)}(y,z)\, dydz = 0$$

for all nonnegative integers α, β and γ.

Let $f \in L^p$ for $1 < p < \infty$, then Littlewood-Paley-Stein square function of f associated with Zygmund dilation is defined by

$$g_Z(f)(x,y,z) = \left\{ \sum_{j,k} |\psi_{j,k} * f(x,y,z)|^2 \right\}^{1/2},$$

where

$$\psi_{j,k}(x,y,z) = 2^{2(j+k)}\psi^{(1)}(2^j x)\psi^{(2)}(2^k y, 2^{j+k} z). \tag{1}$$

From the Fourier transform, it is easy to see the following continuous Calderón's identity holds on $L^2(\mathbb{R}^3)$,

$$f(x,y,z) = \sum_{j,k} \psi_{j,k} * \psi_{j,k} * f(x,y,z).$$

We now introduce the product test function space on $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$.

Definition 1.1. A Schwartz test function $f(x,y,z)$ defined on $\mathbb{R}^3$ is said to be a product test function on $\mathbb{R} \times \mathbb{R}^2$, if $f \in \mathcal{S}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}} x^\alpha f(x,y,z)\, dx = \int_{\mathbb{R}^2} y^\beta z^\gamma f(x,y,z)\, dydz = 0,$$

for all nonnegative integers α, β and γ.

If f is a product test function on $\mathbb{R} \times \mathbb{R}^2$, we denote $f \in \mathcal{S}_Z(\mathbb{R}^3)$ and the norm of f is defined by the norm of Schwartz test function.

We denote by $(\mathcal{S}_Z(\mathbb{R}^3))'$ the dual space of $\mathcal{S}_Z(\mathbb{R}^3)$.

We also denote $\mathcal{S}_{Z,M}(\mathbb{R}^3)$ by the collection of Schwartz test function $f(x,y,z)$ defined on $\mathbb{R}^3$ with

$$\| f \|_{\mathcal{S}_{Z,M}} = \sup_{x\in\mathbb{R},y\in\mathbb{R},z\in\mathbb{R}} (1+|x|+|y|+|z|)^M \sum_{|\alpha|\le M,|\beta|\le M,|\gamma|\le M} \left| \frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial y^\beta}\frac{\partial^\gamma}{\partial z^\gamma} f(x,y,z)\right| < \infty,$$

and

$$\int_{\mathbb{R}} x^\alpha f(x,y,z)\, dx = \int_{\mathbb{R}^2} y^\beta z^\gamma f(x,y,z)\, dydz = 0$$

for all nonnegative integers $\alpha, \beta, \gamma \le M$.

Similarly, we denote $(\mathcal{S}_{Z,M}(\mathbb{R}^3))'$ the dual space of $\mathcal{S}_{Z,M}(\mathbb{R}^3)$.

Since the functions $\psi_{j,k}$ constructed above belong to $\mathcal{S}_Z(\mathbb{R}^3)$, so the Littlewood-Paley-Stein square function g_Z can be defined for all distributions in $(\mathcal{S}_Z(\mathbb{R}^3))'$. Thus the authors in [7] defined the multi-parameter Hardy space associated with Zygmund dilation as follows.

Definition 1.2. Let $0 < p < \infty$. The multi-parameter Hardy space associated with the Zygmund dilation is defined as $H^p_Z(\mathbb{R}^3) = \{f \in (\mathcal{S}_Z(\mathbb{R}^3))' : g_Z(f) \in L^p(\mathbb{R}^3)\}$. If $f \in H^p_Z(\mathbb{R}^3)$, the norm of f is defined by $\|f\|_{H^p_Z} = \|g_Z(f)\|_p$.

Clearly, it follows that $H^p_Z(\mathbb{R}^3) = L^p(\mathbb{R}^3)$ for $1 < p < \infty$.

It was proved in [7] that the definition is independent of the choice of functions $\psi_{j,k}$ and the following boundedness results of convolution type Ricci-Stein singular integral operators on $\mathbb{R}^3$ were established.

Theorem 1.1. [7]. *Let $T_{\mathcal{Z}} f = K * f$ be the Ricci-Stein singular integral operator on $\mathbb{R}^3$, kernel K is defined by*

$$K(x,y,z) = \sum_{j,k\in\mathbb{Z}} 2^{-2(k+j)} \varphi_{j,k}\Big(\frac{x}{2^j}, \frac{y}{2^k}, \frac{z}{2^{k+j}}\Big),$$

where the functions $\varphi_{j,k}$ are test functions in $\mathcal{S}_Z(\mathbb{R}^3)$. Then $T_{\mathcal{Z}}$ is bounded on $H^p_Z(\mathbb{R}^3)$ and bounded from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ for all $0 < p \le 1$.

Moreover, the similar results for the nonconvolution type Ricci-Stein singular integral operators were also established in [7]. For a fixed large positive integer N, we defined $\mathcal{S}_N(\mathbb{R}^3 \times \mathbb{R}^3)$ to be a collection of functions $\psi(x,y,z,u,v,w) \in C^N(\mathbb{R}^3 \times \mathbb{R}^3)$ with finite norm $\|\psi\|_{\mathcal{S}_N}$ defined by

$$\sup_{(x,y,z)\in\mathbb{R}^3,(u,v,w)\in\mathbb{R}^3} (1+|(x-u,y-v,z-w)|)^N \sum_{|\alpha|,|\beta|\le N} |\partial^\alpha_{x,y,z}\partial^\beta_{u,v,w}\psi(x,y,z,u,v,w)|,$$

where $\partial^{\alpha}_{x,y,z} = \partial^{\alpha_1}_x \partial^{\alpha_2}_y \partial^{\alpha_3}_z$, $\partial^{\beta}_{u,v,w} = \partial^{\beta_1}_u \partial^{\beta_2}_v \partial^{\beta_3}_w$. We further assume that the following cancellation conditions on ψ:

$$\int_{\mathbb{R}} \psi(x,y,z,u,v,w)x^{\alpha_1}\,dx = \int_{\mathbb{R}} \psi(x,y,z,u,v,w)u^{\alpha_2}\,du = 0$$

and

$$\int_{\mathbb{R}^2} \psi(x,y,z,u,v,w)y^{\beta_1}z^{\gamma_1}\,dydz = \int_{\mathbb{R}^2} \psi(x,y,z,u,v,w)v^{\beta_2}z^{\gamma_2}\,dydz = 0$$

for all $0 \le \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \le N$. We also use the notation $\mathcal{S}_\infty(\mathbb{R}^3 \times \mathbb{R}^3) = \bigcap_{N>1} \mathcal{S}_N(\mathbb{R}^3 \times \mathbb{R}^3)$.

Thus, the authors in [7] obtained the following results.

Theorem 1.2. [7]. *Let $T_{\mathcal{NC}}$ be the nonconvolution type Ricci-Stein singular integral operator, namely*

$$T_{\mathcal{NC}}f(x,y,z) = \int_{\mathbb{R}^3} K(x,y,z,u,v,w)f(u,v,w)\,dudvdw,$$

where K is defined by

$$K(x,y,z,u,v,w) = \sum_{j,k\in\mathbb{Z}} 2^{-2(k+j)}\varphi_{j,k}\Big(\frac{x}{2^k}, \frac{y}{2^j}, \frac{z}{2^{k+j}}, \frac{u}{2^k}, \frac{v}{2^j}, \frac{w}{2^{k+j}}\Big),$$

$\varphi_{j,k} \in \mathcal{S}_\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. *Then $T_{\mathcal{NC}}$ is bounded on $H^p_Z(\mathbb{R}^3)$ and bounded from $H^p_Z(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ for all $0 < p \le 1$.*

Having obtained the boundedness of Ricci-Stein singular integral operator on multi-parameter Hardy spaces associated with Zygmund dilation, we will develop a theory of the multi-parameter Triebel-Lizorkin and Besov spaces associated with Zygmund dilation. First we give the definitions.

Definition 1.3. Let $0 < p, q < \infty$, $s = (s_1, s_2) \in \mathbb{R}^2$. The Triebel-Lizorkin space $\dot F^{s,q}_{p,Z}(\mathbb{R}^3)$ associated with the Zygmund dilation is defined by

$$\dot F^{s,q}_{p,Z}(\mathbb{R}^3) = \{f \in (\mathcal{S}_Z(\mathbb{R}^3))' : \|f\|_{\dot F^{s,q}_{p,Z}} < \infty\},$$

where

$$\|f\|_{\dot F^{s,q}_{p,Z}} = \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{-js_1q}2^{-ks_2q}|\psi_{j,k} * f|^q \right)^{1/q} \right\|_p.$$

The Besov space $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$ associated with the Zygmund dilation is defined by

$$\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3) = \{f \in (\mathcal{S}_Z(\mathbb{R}^3))' : \|f\|_{\dot{B}^{s,q}_{p,Z}} < \infty\},$$

where

$$\|f\|_{\dot{B}^{s,q}_{p,Z}} = \left(\sum_{j,k\in\mathbb{Z}} 2^{-js_1q}2^{-ks_2q}\|\psi_{j,k} * f\|_p^q\right)^{1/q}.$$

We will show that the definitions of $\dot{F}^{s,q}_{p,Z}$ and $\dot{B}^{s,q}_{p,Z}$ are independent of the choice function $\psi_{j,k}$ by Min-Max comparison principle. The main tool to prove the Min-Max comparison principle is the following Calderón's identity in [7].

Theorem 1.3. [7]. *Suppose that $\psi_{j,k}$ are the same as in (1). Then*

$$f(x,y,z) = \sum_{j,k}\sum_{I,J,R} |I||J||R|\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R)(\psi_{j,k} * f)(x_I,y_J,z_R),$$

where $\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R) \in \mathcal{S}_{Z,M}(\mathbb{R}^3)$, $I \subset \mathbb{R}, J \subset \mathbb{R}, R \subset \mathbb{R}$ are dyadic intervals with interval-length $l(I) = 2^{-j-N}$, $l(J) = 2^{-k-N}$ and $l(R) = 2^{-j-k-2N}$ for a fixed large integer N, x_I, y_J, z_R are any fixed points in I, J, R, respectively. The above series converges in the norm of $\mathcal{S}_{Z,M}(\mathbb{R}^3)$ and in the dual space $(\mathcal{S}_{Z,M}(\mathbb{R}^3))'$.

The Min-Max comparison principle in Triebel-Lizorkin spaces are given as follows. We use the notation $A \approx B$ to denote that two quantities A and B are comparable independent of other substantial quantities involved in the paper.

Theorem 1.4. *Suppose $\phi^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R})$, $\phi^{(2)}, \psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$ and $\psi_{j,k}$ is defined using $\psi^{(1)}$ and $\psi^{(2)}$ as in (1), $\phi_{j,k}$ is defined similarly. Then for $f \in (\mathcal{S}_{Z,M}(\mathbb{R}^3))'$, where M depends on p and q with $0 < p, q < \infty$, and $s = (s_1, s_2) \in \mathbb{R}^2$,*

$$\left\|\left(\sum_{j,k\in\mathbb{Z}} 2^{-js_1q}2^{-ks_2q}\sum_{I,J,R}\sup_{u\in I, v\in J, w\in R}|\psi_{j,k} * f(u,v,w)|^q\chi_I\chi_J\chi_R\right)^{1/q}\right\|_p$$
$$\approx \left\|\left(\sum_{j,k\in\mathbb{Z}} 2^{-js_1q}2^{-ks_2q}\sum_{I,J,R}\inf_{u\in I, v\in J, w\in R}|\phi_{j,k} * f(u,v,w)|^q\chi_I\chi_J\chi_R\right)^{1/q}\right\|_p$$

where $I \subset \mathbb{R}, J \subset \mathbb{R}, R \subset \mathbb{R}$ are dyadic intervals with interval-length $l(I) = 2^{-j-N}$, $l(J) = 2^{-k-N}$ and $l(R) = 2^{-j-k-2N}$ for a fixed large integer N, x_I, y_J, z_R are any fixed points in I, J, R, respectively.

Similarly, we have the Min-Max comparison principle in Besov spaces.

Theorem 1.5. *Suppose* $\phi^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R})$, $\phi^{(2)}, \psi^{(2)} \in \mathcal{S}(\mathbb{R}^2)$, *and* $\psi_{j,k}$ *is defined using* $\psi^{(1)}$ *and* $\psi^{(2)}$ *as in (1),* $\phi_{j,k}$ *is defined similarly. Then for* $f \in (\mathcal{S}_{Z,M}(\mathbb{R}^3))'$, *where M depends on p and q with* $0 < p, q < \infty$, $s = (s_1, s_2) \in \mathbb{R}^2$,

$$\left(\sum_{j,k\in\mathbb{Z}} 2^{-js_1q}2^{-ks_2q}\left\|\sum_{I,J,R}\sup_{u\in I,v\in J,w\in R}|\psi_{j,k}*f(u,v,w)|\chi_I\chi_J\chi_R\right\|_p^q\right)^{1/q}$$
$$\approx\left(\sum_{j,k\in\mathbb{Z}} 2^{-js_1q}2^{-ks_2q}\left\|\sum_{I,J,R}\inf_{u\in I,v\in J,w\in R}|\phi_{j,k}*f(u,v,w)|\chi_I\chi_J\chi_R\right\|_p^q\right)^{1/q}$$

where $I \subset \mathbb{R}, J \subset \mathbb{R}, R \subset \mathbb{R}$ *are dyadic intervals with interval-length* $l(I) = 2^{-j-N}$, $l(J) = 2^{-k-N}$ *and* $l(R) = 2^{-j-k-2N}$ *for a fixed large integer* N, x_I, y_J, z_R *are any fixed points in* I, J, R, *respectively.*

Using discrete Calderón's identity and Min-Max comparison principle, we can prove the following theorems:

Theorem 1.6. *Let* $T_{\mathcal{Z}}f = K * f$ *be the Ricci-Stein singular integral operator on* $\mathbb{R}^3$, *kernel* K *is defined by*

$$K(x,y,z) = \sum_{j,k\in\mathbb{Z}} 2^{-2(k+j)}\varphi_{j,k}\left(\frac{x}{2^j},\frac{y}{2^k},\frac{z}{2^{k+j}}\right)$$

where $\varphi_{j,k}$ *are test functions in* $\mathcal{S}_Z(\mathbb{R}^3)$. *Then* $T_{\mathcal{Z}}$ *is bounded on* $\dot{F}^{s,q}_{p,Z}$. *Namely, for all* $0 < p, q < \infty$ *and* $s = (s_1, s_2) \in \mathbb{R}^2$, *there exists a constant* C_p *such that*

$$\|T_{\mathcal{Z}}(f)\|_{\dot{F}^{s,q}_{p,Z}} \le C_p\|f\|_{\dot{F}^{s,q}_{p,Z}}.$$

Theorem 1.7. *Let* $T_{\mathcal{Z}}f = K * f$ *be the Ricci-Stein singular integral operator on* $\mathbb{R}^3$*with the kernel* $K(x,y,z)$ *satisfying the same conditions as in Theorem 1.6. Then* $T_{\mathcal{Z}}$ *is bounded on* $\dot{B}^{s,q}_{p,Z}$. *Namely, for all* $0 < p, q < \infty$ *and* $s = (s_1, s_2) \in \mathbb{R}^2$, *there exists a constant* C_p *such that*

$$\|T_{\mathcal{Z}}(f)\|_{\dot{B}^{s,q}_{p,Z}} \le C_p\|f\|_{\dot{B}^{s,q}_{p,Z}}.$$

Theorem 1.8. *Let* $T_{\mathcal{NC}}$ *be the nonconvolution type Ricci-Stein singular integral operator, namely*

$$T_{\mathcal{NC}}f(x,y,z) = \int_{\mathbb{R}^3} K(x,y,z,u,v,w)f(u,v,w)\,dudvdw,$$

where K is defined by

$$K(x,y,z,u,v,w)=\sum_{j,k\in\mathbb{Z}}2^{-2(k+j)}\varphi_{j,k}(\frac{x}{2^k},\frac{y}{2^j},\frac{z}{2^{k+j}},\frac{u}{2^k},\frac{v}{2^j},\frac{w}{2^{k+j}}),$$

with $\varphi_{j,k}\in\mathcal{S}_\infty(\mathbb{R}^3\times\mathbb{R}^3)$. Then $T_{\mathcal{NC}}$ is bounded on $\dot{F}^{s,q}_{p,Z}$. Namely, for all $0<p,q<\infty$ and $s=(s_1,s_2)\in\mathbb{R}^2$, there exists a constant C_p such that

$$\|T_{\mathcal{NC}}(f)\|_{\dot{F}^{s,q}_{p,Z}}\le C_p\|f\|_{\dot{F}^{s,q}_{p,Z}}.$$

Theorem 1.9. *Let $T_{\mathcal{NC}}$ be the nonconvolution type Ricci-Stein singular integral operator with the kernel $K(x,y,z,u,v,w)$ satisfying the same conditions as in Theorem 1.8. Then $T_{\mathcal{NC}}$ is bounded on $\dot{B}^{s,q}_{p,Z}$. Namely, for all $0<p,q<\infty$ and $s=(s_1,s_2)\in\mathbb{R}^2$, there exists a constant C_p such that*

$$\|T_{\mathcal{NC}}(f)\|_{\dot{B}^{s,q}_{p,Z}}\le C_p\|f\|_{\dot{B}^{s,q}_{p,Z}}.$$

2. The Min-max Comparison Principle in Triebel-lizorkin and Besov Spaces

In this section, we establish the Min-Max comparison principle in Triebel-Lizorkin and Besov Spaces associated with Zygmund dilation. We first give the almost orthogonal estimates.

Lemma 2.1. [7]. *If we allow N_1,N_2,M_1,M_2 to be any positive integers, $\psi,\phi\in\mathcal{S}_Z(\mathbb{R}^3)$ with cancellation conditions of any order, and $\psi_{ts}(x,y,z,u,v,w)=t^{-2}s^{-2}\psi(\frac{x}{t},\frac{y}{s},\frac{z}{ts},\frac{u}{t},\frac{v}{s},\frac{w}{ts})$ and $\phi_{t's'}$ is defined similarly. Then for any positive integers L,M_1,M_2, there exists $C=C(L,M_1,M_2)$ such that*

$$\begin{aligned}&\int_{\mathbb{R}^3}\psi_{ts}(x,y,z,u,v,w)\phi_{t's'}(u,v,w,x_0,y_0,z_0)\,dudvdw\\ \le& C(\frac{t'}{t}\wedge\frac{t}{t'})^L(\frac{s'}{s}\wedge\frac{s}{s'})^L\frac{(t\vee t')^{M_1}}{(t\vee t'+|x-x_0|)^{1+M_1}}\frac{(s\vee s')^{M_2}}{t^*(s\vee s'+|y-y_0|+\frac{|z-z_0|}{t^*})^{2+M_2}}\end{aligned}$$

where $t^*=t$ if $s>s'$ and $t^*=t'$ if $s\le s'$.

Lemma 2.2. [7]. *If $f,g\in\mathcal{S}_Z(\mathbb{R}^3)$ and $f_{ts}(x,y,z)=t^{-2}s^{-2}f(\frac{x}{t},\frac{y}{s},\frac{z}{ts})$ and $g_{t's'}$ is defined similarly. Then for any positive integers L , M_1 and M_2, there exists a constant $C=C(L,M_1,M_2)$ such that*

$$|f_{ts}*g_{t's'}(x,y,z)|\le C(\frac{t'}{t}\wedge\frac{t}{t'})^L(\frac{s'}{s}\wedge\frac{s}{s'})^L\frac{(t\vee t')^{M_1}}{(t\vee t'+|x|)^{1+M_1}}\frac{(s\vee s')^{M_2}}{t^*(s\vee s'+|y|+\frac{|z|}{t^*})^{2+M_2}}$$

where $t^*=t$ if $s>s'$ and $t^*=t'$ if $s\le s'$.

Next, we give the following lemma which is crucial in dealing with the Triebel-Lizorkin and Besov Spaces theory.

Lemma 2.3. *Given a large positive integer* N *and integers* $j, k, j^{'}, k^{'}$. *Let* $I, J, R, I^{'}, J^{'}, R^{'}$ *be dyadic intervals in* $\mathbb{R}$ *such that their side-lengths are* $l(I) = 2^{-j-N}$, $l(J) = 2^{-k-N}$, $l(R) = 2^{-j-k-2N}$, $l(I^{'}) = 2^{-j^{'}-N}$, $l(J^{'}) = 2^{-k^{'}-N}$, $l(R^{'}) = 2^{-j^{'}-k^{'}-2N}$. *Let* $x_I^{'} \in I^{'}$, $y_J^{'} \in J^{'}$, *and* $z_R^{'} \in R^{'}$. *Then for any* $u, u^* \in I$, $v, v^* \in J$, $w, w^* \in R$, *we have*

$$\sum_{I^{'},J^{'},R^{'}} \frac{2^{-(j\wedge j^{'})M_1}2^{-(k\wedge k^{'})M_2}|I^{'}||J^{'}||R^{'}|}{(2^{-(j\wedge j^{'})}+|u-x_{I^{'}}|)^{1+M_1}2^{-j^*}(2^{-(k\wedge k^{'})}+|v-y_{J^{'}}|+\frac{|w-z_{R^{'}}|}{2^{-j^*}})^{2+M_2}}$$
$$\times |\psi_{j^{'},k^{'}} * f(x_{I^{'}}, y_{J^{'}}, z_{R^{'}})|$$
$$\leq C2^{4N(\frac{1}{r}-1)}2^{\tau}\left\{ M_{\mathcal{Z}}\left(\sum_{I^{'},J^{'},R^{'}} |\psi_{j^{'},k^{'}} * f(x_{I^{'}}, y_{J^{'}}, z_{R^{'}})|^r \chi_{I^{'}}\chi_{J^{'}}\chi_{R^{'}} \right)\right\}^{1/r} (u^*, v^*, w^*),$$

where $j^* = j$ *if* $k < k'$ *and* $j^* = j^{'}$ *if* $k \geq k^{'}$, *and* M_Z *is the Zygmund maximal function on* $\mathbb{R}^3$ *and max*$\{\frac{2}{1+M_1}, \frac{2}{2+M_2}\} < r \leq 1$, *and the summation is taken for all* $I^{'}, J^{'}, R^{'}$ *with the fixed side-length, and* τ *is defined as follows:* $\tau = (\frac{2}{r}-2)(j^{'}+k^{'}-j-k)$ *if* $j < j^{'}$ *and* $k < k^{'}$; $\tau = (\frac{2}{r}-1)(j^{'}-j)$ *if* $j < j^{'}$ *and* $k \geq k^{'}$; $\tau = (\frac{2}{r}-2)(+k^{'}-k)+j-j^{'}$ *if* $j \geq j^{'}$ *and* $k < k^{'}$; $\tau = 0$ *if* $j \geq j^{'}$ *and* $k \geq k^{'}$.

The proof of Lemma 2.3 is similar to that of Lemma 3.7 in [8] with only minor modification. We omit the details here.

Now we prove Theorem 1.4 and Theorem 1.5.

Proof of Theorem 1.4. For $(u^{'}, v^{'}, w^{'}) \in I^{'} \times J^{'} \times R^{'}$, by discrete Calderón's identity, $f \in \mathcal{S}_Z$ can be represented by

$$f(x,y,z) = \sum_{j,k}\sum_{I,J,R} |I||J||R|\widetilde{\phi}_{j,k}(x,y,z,x_I,y_J,z_R)(\phi_{j,k} * f)(x_I,y_J,z_R),$$

by Lemma 2.2 and Lemma 2.3, for any $x_I, u^* \in I$, $y_J, v^* \in J$, $z_R, w^* \in R$, we have

$$(\psi_{j^{'},k^{'}} * f)(u^{'}, v^{'}, w^{'})$$
$$= \left(\psi_{j^{'},k^{'}} * \sum_{j,k}\sum_{I,J,R} |I||J||R|\widetilde{\phi}_{j,k}(\cdot,\cdot,\cdot,x_I,y_J,z_R)\right)(u^{'}, v^{'}, w^{'})(\phi_{j,k} * f)(x_I,y_J,z_R)$$
$$= \sum_{j,k}\sum_{I,J,R} |I||J||R|(\psi_{j^{'},k^{'}} * \widetilde{\phi}_{j,k})(u^{'}, v^{'}, w^{'}, x_I, y_J, z_R)(\phi_{j,k} * f)(x_I,y_J,z_R)$$

$$\leq \sum_{j,k} 2^{-(|j-j'|+|k-k'|)L} \sum_{I,J,R} |I||J||R| \frac{2^{-(j\wedge j')M_1}}{(2^{-(j\wedge j')} + |u' - x_I|)^{M_1+1}}$$

$$\times \frac{2^{-(k\wedge k')M_2}}{2^{-j^*}(2^{-(k\wedge k')} + |v' - y_I| + 2^{j^*}|w' - z_R|)^{M_2+2}} (\phi_{j,k} * f)(x_I, y_J, z_R)$$

$$\leq C \sum_{j,k} 2^{-(|j-j'|+|k-k'|)L} 2^{4N(\frac{1}{r}-1)} 2^{\tau}$$

$$\times \left\{ M_{\mathcal{Z}} \left(\sum_{I,J,R} |(\phi_{j,k} * f)(x_I, y_J, z_R)| \chi_I \chi_J \chi_R \right)^r \right\}^{1/r} (u^*, v^*, w^*)$$

where $M_{\mathcal{Z}}$ is the Zygmung maximal function, $\max\{\frac{2}{1+M_1}, \frac{2}{2+M_2}\} < r < \min\{p, q\}$. Applying Hölder's inequality and summing over j', k', I', J', R', we obtain

$$\left\{ \sum_{j',k'} \sum_{I',J',R'} 2^{-j's_1q} 2^{-k's_2q} \sup_{(u',v',w')\in I'\times J'\times R'} |(\psi_{j',k'} * f)(u', v', w')|^q \chi_{I'} \chi_{J'} \chi_{R'} \right\}^{1/q}$$

$$\leq C \left\{ \sum_{j,k} 2^{-js_1q} 2^{-ks_2q} \left\{ M_{\mathcal{Z}} \left(\sum_{I,J,R} |(\phi_{j,k} * f)(x_I, y_J, z_R)| \chi_I \chi_J \chi_R \right)^r \right\}^{q/r} \right\}^{1/q}.$$

Since (x_I, y_J, z_R) is an arbitrary point in $I \times J \times R$, we have

$$\left\{ \sum_{j',k'} \sum_{I',J',R'} 2^{-j's_1q} 2^{-k's_2q} \sup_{(u',v',w')\in I'\times J'\times R'} |(\psi_{j',k'} *) f(u', v', w')|^q \chi_{I'} \chi_{J'} \chi_{R'} \right\}^{1/q}$$

$$\leq C \left\{ \sum_{j,k} 2^{-js_1q} 2^{-ks_2q} \left\{ M_{\mathcal{Z}} \left(\sum_{I,J,R} \inf_{(u,v,w)\in I\times J\times R} |(\phi_{j,k} *) f(u, v, w)| \chi_I \chi_J \chi_R \right)^r \right\}^{q/r} \right\}^{1/q}.$$

Hence, by the Fefferman-Stein vector-valued maximal function inequality with $r < \min\{p, q\}$, we get

$$\left\| \left\{ \sum_{j',k'} \sum_{I',J',R'} 2^{-j's_1q} 2^{-k's_2q} \sup_{(u',v',w')\in I'\times J'\times R'} |(\psi_{j',k'} * f)(u', v', w')|^q \chi_{I'} \chi_{J'} \chi_{R'} \right\}^{1/q} \right\|_p$$

$$\leq C \left\| \left\{ \sum_{j,k} \sum_{I,J,R} 2^{-js_1q} 2^{-ks_2q} \inf_{(u,v,w)\in I\times J\times R} |(\phi_{j,k} * f)(u, v, w)|^q \chi_I \chi_J \chi_R \right\}^{1/q} \right\|_p,$$

thus, we obtain desired result in Theorem 1.4.

Proof of Theorem 1.5. As in the proof of Theorem 1.4, $f \in \mathcal{S}_Z$ can be represented by

$$f(x,y,z) = \sum_{j,k} \sum_{I,J,R} |I||J||R| \widetilde{\phi}_{j,k}(x,y,z,x_I,y_J,z_R)(\phi_{j,k} * f)(x_I,y_J,z_R),$$

Arguing as in the proof Theorem 1.4, we have

$$\begin{aligned}(\psi_{j',k'} * f)(u^{'},v^{'},w^{'}) \leq & C\sum_{j,k} 2^{-(|j-j^{'}|+|k-k^{'}|)L} 2^{4N(\frac{1}{r}-1)} 2^{\tau} \\ & \times\left\{M_{\mathcal{Z}}\left(\sum_{I,J,R} |(\phi_{j,k} * f)(x_I,y_J,z_R)|\chi_I\chi_J\chi_R\right)^r\right\}^{1/r}(u^*,v^*,w^*).\end{aligned}$$

Therefore

$$\begin{aligned}&\sup_{(u',v',w')\in I'\times J'\times R'} (\psi_{j',k'} * f)(u^{'},v^{'},w^{'})\chi_{I'}(u^*)\chi_{J'}(v^*)\chi_{R'}(w^*) \\ \leq & C\sum_{j,k} 2^{-(|j-j^{'}|+|k-k^{'}|)L} 2^{4N(\frac{1}{r}-1)} 2^{\tau} \\ & \times \left\{M_{\mathcal{Z}}\left(\sum_{I,J,R} |(\phi_{j,k} * f)(x_I,y_J,z_R)|\chi_I\chi_J\chi_R\right)^r\right\}^{1/r}(u^*,v^*,w^*).\end{aligned}$$

When $1 \leq p < \infty$, by the Fefferman-Stein vector-valued maximal function inequality with $r < \min(p,q)$, we get

$$\begin{aligned}&\left\|\sum_{I',J',R'} \sup_{(u',v',w')\in I'\times J'\times R'} |(\psi_{j',k'} * f)(u^{'},v^{'},w^{'})|\chi_{I'}(u^*)\chi_{J'}(v^*)\chi_{R'}(w^*)\right\|_p \\ \leq & C\sum_{j,k} 2^{-(|j-j^{'}|+|k-k^{'}|)L} 2^{4N(\frac{1}{r}-1)} 2^{\tau}\left\|\sum_{I,J,R} |(\phi_{j,k} * f)(x_I,y_J,z_R)|\chi_I\chi_J\chi_R\right\|_p.\end{aligned}$$

If $q \geq 1$, applying Hölder's inequality and if $0 < q < 1$ by using usual inequality, summing over $j^{'}, k^{'}$, we get

$$\begin{aligned}&\left(\sum_{j',k'} 2^{-j^{'}s_1q} 2^{-k^{'}s_2q}\left\|\sum_{I',J',R'} \sup_{(u',v',w')\in I'\times J'\times R'} |(\psi_{j',k'} * f)(u^{'},v^{'},w^{'})|\chi_{I'}\chi_{J'}\chi_{R'}\right\|_p^q\right)^{1/q} \\ \leq & C\left(\sum_{j,k} 2^{-js_1q}2^{-ks_2q}\left\|\sum_{I,J,R} |(\phi_{j,k} * f)(x_I,y_J,z_R)|\chi_I\chi_J\chi_R)\right\|_p^q\right)^{1/q}.\end{aligned}$$

Since (x_I, y_J, z_R) is an arbitrary point in $I \times J \times R$, thus, we can get the desired result, namely

$$\left(\sum_{j',k'} 2^{-j's_1q}2^{-k's_2q}\left\|\sum_{I',J',R'}\sup_{(u',v',w')\in I'\times J'\times R'}|(\psi_{j',k'}*f)(u',v',w')|\chi_{I'}\chi_{J'}\chi_{R'}\right\|_p^q\right)^{1/q}$$

$$\leq C\left(\sum_{j,k}2^{-js_1q}2^{-ks_2q}\left\|\sum_{I,J,R}\inf_{(u,v,w)\in I\times J\times R}|(\phi_{j,k}*f)(u,v,w)|\chi_I\chi_J\chi_R)\right\|_p^q\right)^{1/q}.$$

When $0 < p < 1$, by the Fefferman-Stein vector-valued maximal function inequality with $r < p$, we get

$$\int_{\mathbb{R}^3}\left(\sum_{I',J',R'}\sup_{(u',v',w')\in I'\times J'\times R'}|(\psi_{j',k'}*f)(u',v',w')|\chi_{I'}\chi_{J'}\chi_{R'}\right)^p du^*dv^*dw^*$$

$$\leq C\sum_{j,k}2^{-(|j-j'|+|k-k'|)Lp}2^{4Np(\frac{1}{r}-1)}2^{p\tau}$$

$$\times\int_{\mathbb{R}^3}\left\{M_Z\left(\sum_{I,J,R}|(\phi_{j,k}*f)(x_I,y_J,z_R)|\chi_I\chi_J\chi_R\right)^r(u^*,v^*,w^*)\right\}^{p/r}du^*dv^*dw^*$$

$$\leq C\sum_{j,k}2^{-(|j-j'|+|k-k'|)Lp}2^{4Np(\frac{1}{r}-1)}2^{p\tau}$$

$$\times\int_{\mathbb{R}^3}\left\{\sum_{I,J,R}|(\phi_{j,k}*f)(x_I,y_J,z_R)|\chi_I(u^*)\chi_J(v^*)\chi_R(w^*)\right\}^p du^*dv^*dw^*,$$

so if $q/p \geq 1$, applying Hölder's inequality and if $0 < q/p < 1$ by using usual inequality, we get

$$\left(\sum_{j',k'} 2^{-j's_1q}2^{-k's_2q}\left\|\sum_{I',J',R'}\sup_{(u',v',w')\in I'\times J'\times R'}|(\psi_{j',k'}*f)(u',v',w')|\chi_{I'}\chi_{J'}\chi_{R'}\right\|_p^q\right)^{1/q}$$

$$\leq C\left(\sum_{j,k}2^{-js_1q}2^{-ks_2q}\left\|\sum_{I,J,R}|(\phi_{j,k}*f)(x_I,y_J,z_R)|\chi_I\chi_J\chi_R\right\|_p^q\right)^{1/q}.$$

Since (x_I, y_J, z_R) is an arbitrary point in $I \times J \times R$, thus, we can get the desired result, namely

$$\left(\sum_{j',k'} 2^{-j's_1q}2^{-k's_2q}\left\|\sum_{I',J',R'}\sup_{(u',v',w')\in I'\times J'\times R'}|(\psi_{j',k'}*f)(u',v',w')|\chi_{I'}\chi_{J'}\chi_{R'}\right\|_p^q\right)^{1/q}$$
$$\le C\left(\sum_{j,k} 2^{-js_1q}2^{-ks_2q}\left\|\sum_{I,J,R}\inf_{(u,v,w)\in I\times J\times R}|(\phi_{j,k}*f)(u,v,w)|\chi_I\chi_J\chi_R\right\|_p^q\right)^{1/q}.$$

As a consequence of Theorem 1.4 and Theorem 1.5, it is easy to see that Triebel-Lizorkin Spaces $\dot{F}^{s,q}_{p,Z}$ and Besov Spaces $\dot{B}^{s,q}_{p,Z}$ are independent of the choice of functions $\psi_{j,k}$. We also obtain the following results.

Proposition 2.1. *Let $0 < p, q < \infty$ and $s = (s_1, s_2) \in \mathbb{R}^2$. Then we have*

$$\|f\|_{\dot{B}^{s,q}_{p,Z}} \approx \left(\sum_{j,k\in\mathbb{Z}} 2^{-js_1q}2^{-ks_2q}\left\|\sum_{I,J,R}|(\psi_{j,k}*f)(x_I,y_J,z_R)|\chi_I(x)\chi_J(y)\chi_R(z)\right\|_p^q\right)^{1/q}$$

and

$$\|f\|_{\dot{F}^{s,q}_{p,Z}} \approx \left\|\left(\sum_{j,k\in\mathbb{Z}}\sum_{I,J,R} 2^{-js_1q}2^{-ks_2q}|(\psi_{j,k}*f)(x_I,y_J,z_R)|^q\chi_I(x)\chi_J(y)\chi_R(z)\right)^{1/q}\right\|_p,$$

where $j, k, x_I, y_J, z_R, \chi_I, \chi_J, \chi_R, \psi_{j,k}$ are the same as in Theorem 1.4.

3. Boundedness of Ricci-stein Singular Integral Operators

The main purpose of this section is to obtain the boundedness of Ricci-Stein singular integral operators on multi-parameter Triebel-Lizorkin and Besov Spaces associated with Zygmund dilation. We first give some propositions.

Proposition 3.1. *Let $0 < p, q < \infty$ and $s = (s_1, s_2) \in \mathbb{R}^2$, then $\mathcal{S}_Z(\mathbb{R}^3)$ is dense in $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$ and $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$.*

Proof. Suppose $f \in \dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$, we get

$$f(x,y,z) = \sum_{j,k}\sum_{I,J,R}|I||J||R|\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R)(\psi_{j,k}*f)(x_I,y_J,z_R),$$

where the series converges in $(\mathcal{S}_Z(\mathbb{R}^3))'$. It suffices to show that

$$F = F_{M_1,M_2,s}(x,y,z,x_I,y_J,z_R)$$
$$= \sum_{|j|\le M_1,|k|\le M_2}\sum_{I\times J\times R\subseteq B(0,s)}|I||J||R|\widetilde{\psi}_{j,k}(x,y,z,x_I,y_J,z_R)(\psi_{j,k}*f)(x_I,y_J,z_R)$$

converges to f in $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$, as M_1, M_2 and s tend to infinity, where $B(0,s) = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < s^2\}$. To do this, let W the set $\{(j,k,I,J,R) : I \times J \times R \subseteq B(0,s), |j| \le M_1, |k| \le M_2\}$, where the I, J, R are dyadic intervals in $\mathbb{R}$ with side-length $2^{-j-N}, 2^{-k-N}, 2^{-j-k-2N}$, respectively, and let W^c be the complement of W. Let also $V = \{(j,k) : |j| \le M_1, |k| \le M_2\}$ and V^c denotes its complement.

Then for $(x_{I'}, y_{J'}, z_{R'}) \in I' \times J' \times R'$, we have

$$\begin{aligned}
&\left|\psi_{j',k'} * \sum_{(j,k,I,J,R)\in W^c} |I||J||R|\widetilde{\psi}_{j,k}(\cdot,\cdot,\cdot,x_I,y_J,z_R)(x_{I'},y_{J'},z_{R'})(\psi_{j,k}*f)(x_I,y_J,z_R)\right| \\
\le& C \sum_{(j,k,I,J,R)\in W^c} 2^{-|j-j'|L}2^{-|k-k'|L}|I||J||R|\frac{2^{-(j\wedge j')M_1}}{(2^{-(j\wedge j')}+|x_{I'}-x_I|)^{1+M_1}} \\
&\times \frac{2^{-(k\wedge k')M_2}}{2^{-j*}(2^{-(k\wedge k')}+|y_{J'}-y_J|+2^{j*}|z_{R'}-z_R|)^{2+M_2}}|(\psi_{j,k}*f)(x_I,y_J,z_R)| \\
\le& C \sum_{(j,k)\in V^c} 2^{-|j-j'|L}2^{-|k-k'|L}4^{N(\frac{1}{r}-1)}2^{\tau} \\
&\times \left\{ M_{\mathcal{Z}}\left(\sum_{I,J,R:(j,k,I,J,R)\in W^c} |(\psi_{j,k}*f)(x_I,y_J,z_R)|\chi_I\chi_J\chi_R\right)^r\right\}^{1/r},
\end{aligned}$$

where $j^* = j$ if $k < k'$ and $j^* = j'$ if $k \ge k'$, τ as in lemma 2.3, $\max\left\{\frac{2}{1+M_1} \frac{2}{1+M_2}\right\} < r <\min(p,q)$. Repeating the proof of Min-Max comparison principle of $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$, we get

$$\begin{aligned}
&\left\|\left\{\sum_{j',k'}\sum_{I',J',R'} 2^{-j's_1q}2^{-k's_2q}|(\psi_{j',k'}*F)|^q\chi_{I'}\chi_{J'}\chi_{R'}\right\}^{1/q}\right\|_p \\
&\quad\le \left\|\left\{\sum_{(j,k,I,J,R)\in W^c} 2^{-js_1q}2^{-ks_2q}|(\psi_{j,k}*f)|^q\chi_I\chi_J\chi_R\right\}^{1/q}\right\|_p,
\end{aligned}$$

where the last term tends to zero as M_1, M_2 and r tend to infinity whenever $f \in \dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$.

When $f \in \dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$, we can similarly get desired result.

Since $\mathcal{S}_Z(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$, as a consequence of proposition 3.1, it is immediate that

Proposition 3.2. *$L^2(\mathbb{R}^3)$ is dense in $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$ and $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$ for $0 < p,q < \infty$ and $s = (s_1,s_2) \in \mathbb{R}^2$.*

We now prove the $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$ and $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$ boundedness of convolution type Ricci-Stein singular integral operators.

Proof of Theorem 1.6. Applying discrete Calderón's identity, for $f \in L^2(\mathbb{R}^3) \cap \dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$, we have for any $(u^{''}, v^{''}, w^{''}) \in I^{''} \times J^{''} \times R^{''}$,

$$\begin{aligned}&(\psi_{j'',k''} * T_{\mathcal{Z}} f)(u^{''}, v^{''}, w^{''})\\ =&\sum_{j',k'} \sum_{j,k} \sum_{I,J,R} |I||J||R|(\psi_{j'',k''} * \phi_{j',k'} * \widetilde{\psi}_{j,k})(u^{''}, v^{''}, w^{''}, x_I, y_J, z_R)(\psi_{j,k} * f)(x_I, y_J, z_R).\end{aligned}$$

We note that $\widetilde{\psi}_{j,k}(\cdot,\cdot,\cdot,x_I,y_J,z_R)$ is a function in $\mathcal{S}_Z(\mathbb{R}^3)$ and thus for any given positive integers L, M_1 and M_2, we have

$$\begin{aligned}|(\phi_{j',k'} * \widetilde{\psi}_{j,k})(x,y,z,x_I,y_J,z_R)| \leq & C2^{-|j-j'|L}2^{-|k-k'|L} \frac{|I||J||R|2^{-(j\wedge j')M_1}}{(2^{-(j\wedge j')} + |x_{I'} - x_I|)^{1+M_1}}\\ &\times \frac{2^{-(k\wedge k')M_2}}{2^{-j*}(2^{-(k\wedge k')} + |y_{J'} - y_J| + 2^{j*}|z_{R'} - z_R|)^{2+M_2}}\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} (\phi_{j',k'} * \widetilde{\psi}_{j,k})(x,y,z,x_I,y_J,z_R) x^{\alpha}\, dx\\ =&\int_{\mathbb{R}^2} (\phi_{j',k'} * \widetilde{\psi}_{j,k})(x,y,z,x_I,y_J,z_R) y^{\beta} z^{\gamma}\, dydz = 0\end{aligned}$$

for all nonnegative integers α, β, and γ. This implies $(\phi_{j',k'} * \widetilde{\psi}_{j,k})(\cdot,\cdot,\cdot,x_I,y_J,z_R) \in \mathcal{S}_Z(\mathbb{R}^3)$, and

$$\|(\phi_{j',k'} * \widetilde{\psi}_{j,k})(\cdot,\cdot,\cdot,x_I,y_J,z_R)\|_{\mathcal{S}_Z(\mathbb{R}^3)} \leq C2^{-|j-j'|L}2^{-|k-k'|L}.$$

Therefore

$$\sum_{j',k'} (\phi_{j',k'} * \widetilde{\psi}_{j,k})(\cdot,\cdot,\cdot,x_I,y_J,z_R) \in \mathcal{S}_Z(\mathbb{R}^3)$$

and

$$\|\sum_{j',k'} (\phi_{j',k'} * \widetilde{\psi}_{j,k})(\cdot,\cdot,\cdot,x_I,y_J,z_R)\|_{\mathcal{S}_Z(\mathbb{R}^3)} \leq C.$$

Set $\widetilde{\widetilde{\psi}}_{j,k} = \sum_{j',k'} (\phi_{j',k'} * \widetilde{\psi}_{j,k})(\cdot,\cdot,\cdot,x_I,y_J,z_R)$, then $\widetilde{\widetilde{\psi}}_{j,k} \in \mathcal{S}_Z(\mathbb{R}^3)$. Applying lemma 2.3 again, we have for any $(u'', v'', w'') \in I'' \times J'' \times R''$,

$$
\begin{aligned}
&(\psi_{j'',k''} * T_{\mathcal{Z}} f)(u'', v'', w'') \\
&\leq \sum_{j,k} \sum_{I,J,R} C 2^{-|j-j''|L} 2^{-|k-k''|L} |I||J||R| \frac{2^{-(j \wedge j'')M_1}}{(2^{-(j\wedge j'')} + |u'' - x_I|)^{1+M_1}} \\
&\quad \times \frac{2^{-(k\wedge k'')M_2}}{2^{-j*}(2^{-(k\wedge k'')} + |v'' - y_J| + 2^{j*}|w'' - z_R|)^{2+M_2}} (\psi_{j,k} * f)(x_I, y_J, z_R) \\
&\leq \sum_{j,k} C 2^{-|j-j''|L} 2^{-|k-k''|L} 2^{4N(1/r-1)} 2^{\tau} \\
&\quad \times \left\{ M_{\mathcal{Z}} \left(\sum_{I,J,R} |(\psi_{j,k} * f)| \chi_I(\cdot)\chi_J(\cdot)\chi_R(\cdot) \right)^r \right\}^{1/r} (u^*, v^*, w^*),
\end{aligned}
$$

where in the above we use the same notation $j^* = j$ if $k < k''$ and $j^* = j''$ if $k'' < k$, $\max\{\frac{2}{1+M_1}, \frac{2}{2+M_2}\} < r < \min\{p, q\}$. Applying Hölder's inequality and summing over $j', k', j'', k'', I'', J''$ and R'', we get

$$
\begin{aligned}
&\left\{ \sum_{j'',k''} \sum_{I'',J'',R''} 2^{-j''s_1 q} 2^{-k''s_2 q} |(\psi_{j'',k''} * T_{\mathcal{Z}} f)(x_{I''}, y_{J''}, z_{R''})|^q \chi_{I''}(\cdot)\chi_{J''}(\cdot)\chi_{R''}(\cdot) \right\}^{1/q} \\
&\leq C \left\{ \sum_{j,k} 2^{-js_1 q} 2^{-ks_2 q} \left\{ M_{\mathcal{Z}} \left(\sum_{I,J,R} |(\psi_{j,k} * f)(x_I, y_J, z_R)|^r \chi_I(\cdot)\chi_J(\cdot)\chi_R(\cdot) \right)^r \right\}^{q/r} \right\}^{1/q}
\end{aligned}
$$

Hence, by the Fefferman-Stein vector-valued maximal function inequality with $r<\min(p,q)$, we have

$$
\begin{aligned}
&\left\| \left\{ \sum_{j'',k''} \sum_{I'',J'',R''} 2^{-j''s_1 q} 2^{-k''s_2 q} |(\psi_{j'',k''} * T_{\mathcal{Z}} f)(x_{I''}, y_{J''}, z_{R''})|^q \chi_{I''}\chi_{J''}\chi_{R''} \right\}^{1/q} \right\|_p \\
&\leq C \left\| \left\{ \sum_{j,k} \sum_{I,J,R} 2^{-js_1 q} 2^{-ks_2 q} |(\psi_{j,k} * f)(x_I, y_J, z_R)|^q \chi_I \chi_J \chi_R \right\}^{1/q} \right\|_p ,
\end{aligned}
$$

namely,

$$
\|T_{\mathcal{Z}}(f)\|_{\dot{F}^{s,q}_{p,Z}} \leq \|f\|_{\dot{F}^{s,q}_{p,Z}}.
$$

Since $L^2(\mathbb{R}^3)$ is dense in $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$, then $T_{\mathcal{Z}}$ can be extended to be a boundedness operator on $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$. This finishes the proof of Theorem 1.6.

Proof of Theorem 1.7. As in the proof of Theorem 1.6, we have

$$(\psi_{j'',k''} * T_{\mathcal{Z}} f)(u'', v'', w'') \leq \sum_{j,k} C 2^{-|j-j''|L} 2^{-|k-k''|L} 2^{4N(1/r-1)} 2^{\tau} \times C \left\{ M_{\mathcal{Z}} \left(\sum_{I,J,R} |(\psi_{j,k} * f)| \chi_I(\cdot)\chi_J(\cdot)\chi_R(\cdot) \right)^r \right\}^{1/r} (u^*, v^*, w^*)$$

When $1 \leq p < \infty$, by the Fefferman-Stein vector-valued maximal function inequality with $r < p$, we have

$$\begin{aligned} &\left\| \sum_{I'',J'',R''} |(\psi_{j'',k''} * T_{\mathcal{Z}} f)|(u'', v'', w'')\chi_{I''}\chi_{J''}\chi_{R''} \right\|_p \\ &\leq C \sum_{j,k} 2^{-|j-j''|L} 2^{-|k-k''|L} 2^{4N(1/r-1)} 2^{\tau} \\ &\quad \times \left\| \left\{ M_{\mathcal{Z}} \left(\sum_{I,J,R} |(\psi_{j,k} * f)| \chi_I\chi_J\chi_R \right)^r \right\}^{1/r} \right\|_p \\ &\leq C \sum_{j,k} 2^{-|j-j''|L} 2^{-|k-k''|L} 2^{4N(1/r-1)} 2^{\tau} \left\| \sum_{I,J,R} |(\psi_{j,k} * f)| \chi_I\chi_J\chi_R \right\|_p . \end{aligned}$$

If $q \geq 1$, applying Hölder's inequality and if $0 < q < 1$ by using usual inequality, and summing over j', k', j'', k'', we get

$$\begin{aligned} &\left\{ \sum_{j'',k''} 2^{-j'' s_1 q} 2^{-k'' s_2 q} \left\| \sum_{I'',J'',R''} |(\psi_{j'',k''} * T_{\mathcal{Z}} f)(x_{I''}, y_{J''}, z_{R''})| \chi_{I''}\chi_{J''}\chi_{R''} \right\|_p^q \right\}^{1/q} \\ &\leq C \left\{ \sum_{j,k} 2^{-j s_1 q} 2^{-k s_2 q} \left\| \sum_{I,J,R} |(\psi_{j,k} * f)(x_I, y_J, z_R)| \chi_I\chi_J\chi_R \right\|_p^q \right\}^{1/q}, \end{aligned}$$

When $0 < p < 1$, the Fefferman-Stein vector-valued maximal function inequality with $r < p$,we get

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(\sum_{I'',J'',R''} |(\psi_{j'',k''} * T_{\mathcal{Z}} f)|(u'', v'', w'')\chi_{I''}\chi_{J''}\chi_{R''} \right)^p du'' dv'' dw'' \\ &\leq C \sum_{j,k} 2^{-|j-j''|Lp} 2^{-|k-k''|Lp} 2^{4Np(1/r-1)} 2^{\tau p} \\ &\quad \times \int_{\mathbb{R}^3} \left\{ M_{\mathcal{Z}} \left(\sum_{I,J,R} |(\psi_{j,k} * f)| \chi_I\chi_J\chi_R \right)^r \right\}^{p/r} du'' dv'' dw'' \end{aligned}$$

$$\leq C \sum_{j,k} 2^{-|j-j^{''}|Lp} 2^{-|k-k^{''}|Lp} 2^{4Np(1/r-1)} 2^{\tau p}$$
$$\times \int_{\mathbb{R}^3} \left(\sum_{I,J,R} |(\psi_{j,k} * f)| \chi_I \chi_J \chi_R \right)^p du^{''} dv^{''} dw^{''}.$$

So if $q/p \geq 1$, applying Hölder's inequality and if $0 < q/p < 1$ by using usual inequality, we get

$$\left\{ \sum_{j'',k''} 2^{-j^{''} s_1 q} 2^{-k^{''} s_2 q} \left\| \sum_{I'',J'',R''} |(\psi_{j'',k''} * T_{\mathcal{Z}} f)(x_{I''}, y_{J''}, z_{R''})| \chi_{I''} \chi_{J''} \chi_{R''} \right\|_p^q \right\}^{1/q}$$
$$\leq C \left\{ \sum_{j,k} 2^{-js_1 q} 2^{-ks_2 q} \left\| \sum_{I,J,R} |(\psi_{j,k} * f)(x_I, y_J, z_R)| \chi_I \chi_J \chi_R \right\|_p^q \right\}^{1/q},$$

Since $L^2(\mathbb{R}^3)$ is dense in $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$, then T_Z can be extended to be a boundedness operator on $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$. This ends the proof of Theorem 1.7.

We now prove the $\dot{F}^{s,q}_{p,Z}(\mathbb{R}^3)$ and $\dot{B}^{s,q}_{p,Z}(\mathbb{R}^3)$ boundedness of nonconvolution type Ricci-Stein singular integral operator, namely, Theorem 1.8 and Theorem 1.9.

Proof of Theorem 1.8 and Theorem 1.9. We recall that

$$T_{\mathcal{NC}} f(x,y,z) = \int_{\mathbb{R}^3} K(x,y,z,u,v,w) f(u,v,w)\, dudvdw,$$

where K is defined

$$K(x,y,z,u,v,w) = \sum_{j,k\in\mathbb{Z}} 2^{-2(k+j)} \psi_{j,k}\left(\frac{x}{2^k}, \frac{y}{2^j}, \frac{z}{2^{k+j}}, \frac{u}{2^k}, \frac{v}{2^j}, \frac{w}{2^{k+j}}\right),$$

with $\psi_{j,k} \in \mathcal{S}_\infty(\mathbb{R}^3 \times \mathbb{R}^3)$.

Thus, by Calderón's identity, we have for any $(u^{''}, v^{''}, w^{''}) \in I^{''} \times J^{''} \times R^{''}$,

$$\begin{aligned}
&(\psi_{j'',k''} * T_{\mathcal{NC}} f)(u^{''}, v^{''}, w^{''}) \\
=&\psi_{j'',k''} * \sum_{j',k'} \sum_{j,k} \sum_{I,J,R} |I||J||R| \int_{\mathbb{R}^3} \phi_{j',k'}(\cdot,\cdot,\cdot,u,v,w)(\widetilde{\psi}_{j,k})(u,v,w,x_I,y_J,z_R)\, dudvdw \\
&\times (u^{''}, v^{''}, w^{''})(\psi_{j,k} * f)(x_I, y_J, z_R) \\
=&\sum_{j',k'} \sum_{j,k} \sum_{I,J,R} |I||J||R| \psi_{j'',k''} * \int_{\mathbb{R}^3} \phi_{j',k'}(\cdot,\cdot,\cdot,u,v,w)(\widetilde{\psi}_{j,k})(u,v,w,x_I,y_J,z_R)\, dudvdw \\
&\times (u^{''}, v^{''}, w^{''})(\psi_{j,k} * f)(x_I, y_J, z_R).
\end{aligned}$$

We note that $\widetilde{\psi}_{j,k}(\cdot,\cdot,\cdot,x_I,y_J,z_R)$ is a function in $\mathcal{S}_Z(\mathbb{R}^3)$ and for any given positive integers L, M_1 and M_2, we have

$$\begin{aligned}F_{j,k,j',k'}(x,y,z,x_I,y_J,z_R) =& \left|\int_{\mathbb{R}^3}\phi_{j',k'}(x,y,z,u,v,w)\widetilde{\psi}_{j,k}(u,v,w,x_I,y_J,z_R)\,dudvdw\right|\\ \le& C2^{-|j-j'|L}2^{-|k-k'|L}\frac{2^{-(j\wedge j')M_1}}{(2^{-(j\wedge j')}+|x-x_I|)^{1+M_1}}\\ &\times\frac{2^{-(k\wedge k')M_2}}{2^{-j*}(2^{-(k\wedge k')}+|y-y_J|+2^{j*}|z-z_R|)^{2+M_2}}\end{aligned}$$

and

$$\int_{\mathbb{R}}F_{j,k,j',k'}(x,y,z,x_I,y_J,z_R)x^\alpha\,dx=\int_{\mathbb{R}^2}F_{j,k,j',k'}(x,y,z,x_I,y_J,z_R)y^\beta z^\gamma\,dydz=0$$

for all nonnegative integers α,β, and γ. This implies $F_{j,k,j',k'}(\cdot,\cdot,\cdot,x_I,y_J,z_R)\in\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$, and

$$\|F_{j,k,j',k'}(\cdot,\cdot,\cdot,x_I,y_J,z_R)\|_{\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)}\le C2^{-|j-j'|L}2^{-|k-k'|L}.$$

Therefore

$$\sum_{j',k'}F_{j,k,j',k'}(\cdot,\cdot,\cdot,x_I,y_J,z_R)\in\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$$

and

$$\|\sum_{j',k'}F_{j,k,j',k'}(\cdot,\cdot,\cdot,x_I,y_J,z_R)\|_{\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)}\le C.$$

Set $\widetilde{\widetilde{\psi}}_{j,k}=\sum_{j',k'}F_{j,k,j',k'}(\cdot,\cdot,\cdot,x_I,y_J,z_R)$, then $\widetilde{\widetilde{\psi}}_{j,k}\in\mathcal{S}_{\mathcal{Z}}(\mathbb{R}^3)$. Repeating the same proof of Theorem 1.6 and Theorem 1.7, we obtain Theorem 1.8 and Theorem 1.9, respectively.

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