TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 6, pp. 2083-2100, December 2013
DOI: 10.11650/tjm.17.2013.3185
This paper is available online at http://journal.taiwanmathsoc.org.tw

SOME INEQUALITIES ON SCREEN HOMOTHETIC LIGHTLIKE HYPERSURFACES OF A LORENTZIAN MANIFOLD

Mehmet Gülbahar*, Erol Kılıç and Sadık Keleş

Abstract. In this paper, we establish some inequalities involving k-Ricci curvature, k-scalar curvature, the screen scalar curvature on a screen homothetic lightlike hypersurface of a Lorentzian manifold. We compute Chen-Ricci inequality and Chen inequality on a screen homothetic lightlike hypersurface of a Lorentzian manifold. We give an optimal inequality involving the $\delta(n_1, \ldots, n_k)$-invariant and some characterizations (totally umbilicity, totally geodesicity, minimality, etc.) for lightlike hypersurfaces.

1. Introduction

In 1993, B.-Y. Chen [4] introduced a new Riemannian invariant for a Riemannian manifold M as follows:

$$\delta_M = \tau(p) - \inf(K)(p), \tag{1.1}$$

where $\tau(p)$ is scalar curvature of M and

$$\inf(K)(p) = \inf\{K(\Pi) : K(\Pi) \text{ is a plane section of } T_pM\}.$$

In [5], B.-Y. Chen established the following general optimal inequality involving the new intrinsic invariant δ_M, the squared mean curvature $\|H\|^2$ for an n-dimensional submanifold M in a real space form $R(c)$ of constant sectional curvature c:

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)c. \tag{1.2}$$

In [6], B.-Y. Chen proved a basic inequality involving the Ricci curvature and squared mean curvature of a submanifold in a real space form. In [16], S. Hong, M.

Received April 13, 2013, accepted May 25, 2013.
Communicated by Bang-Yen Chen.
2010 *Mathematics Subject Classification*: 53C40, 53C42, 53C50.
Key words and phrases: Curvature, Lightlike hypersurface, Lorentzian manifold.

M. Tripathi studied this inequality and they presented a general theory for submanifolds of Riemann manifolds by proving a basic inequality as follows:

(a) For $X \in T_p^1 M$,

$$\mathrm{Ric}\,(X) \leq \frac{1}{4}\, n^2 \|H\|^2 + \widetilde{\mathrm{Ric}}_{(T_p M)}\,(X)\,, \tag{1.3}$$

where M is an n-dimensional submanifold of $\widetilde{M}$, $\widetilde{\mathrm{Ric}}_{(T_p M)}\,(X)$ is the n-Ricci curvature of $T_p M$ at $X \in T_p^1 M$ with respect to the ambient manifold $\widetilde{M}$ and $T_p^1 M$ is the set of unit vectors in $T_p M$.

(b) The equality case of (1.3) is satisfied by $X \in T_p^1 M$ if and only if

$$\begin{cases} \sigma\,(X,Y) = 0, \text{ for all } Y \in T_p M \text{ orthogonal to } X, \\ \sigma\,(X,X) = \frac{n}{2} H(p). \end{cases} \tag{1.4}$$

(c) The equality case of (1.3) holds for all $X \in T_p^1 M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

In [20], the inequality (1.3) is named Chen-Ricci inequality by M. M. Tripathi.

Later, B. Y. Chen and some authors found inequalities for non-degenerate submanifolds of different spaces such as in [9, 15, 17, 18, 20, 21].

In degenerate submanifolds, M. Gülbahar, E. Kılıç and S. Keleş introduced k-Ricci curvature, k-scalar curvature, k-degenerate Ricci curvature, k-degenerate scalar curvature and they established some inequalities that characterize lightlike hypersurface of a Lorentzian manifold in [14]. However, as it is well known, since the sectional curvature and the induced Ricci curvature are not symmetric on lightlike manifolds, establishing Chen-like inequalities on lightlike submanifolds are more difficult than establishing such inequalities on non-degenerate submanifolds. Thus, due to above mentioned difficulties, they couldn't compute some Chen-like inequalities (Chen-Ricci inequality, Chen-inequality etc.).

In this paper, we introduce screen homothetic lightlike hypersurfaces. Since the sectional curvature and the screen Ricci curvature of screen homothetic lightlike hypersurface are symmetric therefore we are able to establish Chen's inequalities on screen homothetic lightlike hypersurface of a Lorentzian manifold and we give some characterizations using these inequalities.

The paper is organized as follows. Section 2 is concerned with preliminaries. In section 3, we compute Chen-Ricci inequality on screen homothetic lightlike hypersurfaces. In section 4, we establish some inequalities and we give some characterizations on screen homothetic lightlike hypersurfaces of a Lorentzian manifold.

2. Preliminaries

Let (M, g) be a lightlike hypersurface of an $(n+2)$-dimensional semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. *The radical space* or *the null space* of $T_p M$, at each point $p \in M$, is a one dimensional subspace $Rad\ T_pM$ defined by

$$Rad\ T_pM = \{\xi \in T_pM : g_p(\xi, X) = 0 \text{ for all } X \in T_pM\}. \tag{2.1}$$

The complementary non-degenerate vector bundle $S(TM)$ of $Rad\,TM$ in TM is called *the screen bundle* of M. Thus, we have

$$TM = Rad\ TM \oplus_{orth} S(TM), \tag{2.2}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. For any null section ξ of $Rad\ TM$ on a coordinate neighborhood $U \subset M$, there exits a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^{\perp}$ satisfying

$$\widetilde{g}(N, X) = \widetilde{g}(N, N) = 0, \quad \widetilde{g}(N, \xi) = 1,\ \forall X \in \Gamma(S(TM)). \tag{2.3}$$

Then the tangent bundle $T\widetilde{M}$ of $\widetilde{M}$ is decomposed as follows:

$$T\widetilde{M} = TM \oplus tr(TM) = \{TM^{\perp} \oplus tr(TM)\} \oplus_{orth} S(TM), \tag{2.4}$$

where $tr(TM)$ is called *transversal vector bundle* of M.

Let $\widetilde{\nabla}$ be Levi-Civita connection of $\widetilde{M}$ and P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$. The Gauss and Weingarten formulas are given

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + B(X, Y)N, \\ \widetilde{\nabla}_X N &= -A_N X + \omega(X)N, \\ \nabla_X PY &= \nabla^*_X PY + C(X, PY)\xi, \\ \nabla_X \xi &= -A^*_{\xi}(X) - \omega(X)\xi. \end{aligned} \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are the induced linear connection on TM and $S(TM)$, respectively; B and C are the local second fundamental forms on TM and $S(TM)$, respectively; A_N and A^*_{ξ} are the shape operators on TM and $S(TM)$, respectively; and ω is a 1-form on TM [11, 12].

From the fact that $B(X, Y) = \widetilde{g}(\widetilde{\nabla}_X Y, \xi)$, it is known that B is independent of the choice of a screen distribution and

$$B(X, \xi) = 0,\ \forall X, Y \in \Gamma(TM). \tag{2.6}$$

The local second fundamental forms B and C of M and $S(TM)$, respectively, are related to their shape operators A^*_{ξ} and A_N by

$$B(X, Y) = g(A^*_{\xi} X, Y), \tag{2.7}$$

$$C(X, PY) = g(A_N X, PY). \tag{2.8}$$

If $B = 0$, then the lightlike hypersurface M is called *totally geodesic* in $\widetilde{M}$. A point $p \in M$ is said to be *umbilical* if

$$B(X,Y)_p = Hg_p(X,Y), \qquad X, Y \in T_pM,$$

where $H \in R$. The lightlike hypersurface M is called *totally umbilical* in $\widetilde{M}$ if every points of M is umbilical [11].

The mean curvature μ of M with respect to an orthonormal basis $\{e_1, \dots, e_n\}$ of $\Gamma(S(TM))$ is defined in [3] as follows:

$$\mu = \frac{1}{n}\, tr(B) = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i B(e_i, e_i), \qquad g(e_i, e_i) = \varepsilon_i. \tag{2.9}$$

A lightlike hypersurface (M, g) of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ is called *screen locally conformal* if the shape operators A_N and A^*_ξ of M and $S(TM)$, respectively, are related by

$$A_N = \varphi A^*_\xi, \tag{2.10}$$

where φ is a non-vanishing smooth function on a neighborhood U on M. In particular, M is called *screen homothetic* if φ is a non-zero constant [1].

We denote the Riemann curvature tensors of $\widetilde{M}$ and M by $\widetilde{R}$ and R, respectively. The Gauss-Codazzi type equations for M are given as follows:

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z, PU) &= g(R(X,Y)Z, PU) + B(X,Z)C(Y,PU) \\ &- B(Y,Z)C(X,PU), \end{aligned} \tag{2.11}$$

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z, \xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ B(Y,Z)w(X) - B(X,Z)w(Y), \end{aligned} \tag{2.12}$$

$$\begin{aligned} \widetilde{g}(\widetilde{R}(X,Y)Z, N) &= g(R(X,Y)Z, N), \\ \widetilde{g}(\widetilde{R}(X,Y)PZ, N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \end{aligned} \tag{2.13}$$

$$+ w(Y)C(X,PZ) - w(X)C(Y,PZ), \tag{2.14}$$

where

$$(\nabla_X B)(Y,Z) = XB(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), \tag{2.15}$$

and

$$(\nabla_X C)(Y,PZ) = XC(Y,PZ) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ), \tag{2.16}$$

for all $X, Y, Z, U \in \Gamma(TM)$ [11].

Furthermore, from (2.12) if $\widetilde{M}(c)$ is a Lorentzian space form, then it is known that

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = B(X,Z)w(Y) - B(Y,Z)w(X). \tag{2.17}$$

Let $\Pi = sp\{e_i, e_j\}$ be 2-dimensional non-degenerate plane of the tangent space T_pM at $p \in M$. Then the number

$$K_{ij} = \frac{g(R(e_j, e_i)e_i, e_j)}{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)^2}$$

is called *the sectional curvature* of the section Π at $p \in M$. Since the screen second fundamental form C is symmetric on a screen homothetic lightlike hypersurface, the sectional curvature K_{ij} is symmetric, that is, $K_{ij} = K_{ji}$. But, in general, the sectional curvature need not be symmetric for a lightlike hypersurface of a semi-Riemannian manifold [12].

Let ξ be a null vector of T_pM. A plane Π of T_pM is called a *null plane* if it contains ξ and e_i such that $\widetilde{g}(\xi, e_i) = 0$ and $\widetilde{g}(e_i, e_i) \neq 0$. *The null sectional curvature* of Π be given in [2] as follows:

$$K_i^{null} = \frac{g(R_p(e_i, \xi)\xi, e_i)}{g_p(e_i, e_i)}.$$

Let M be a lightlike hypersurface of a Lorentzian manifold $\widetilde{M}$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. The Ricci tensor $\widetilde{Ric}$ of $\widetilde{M}$ and the induced Ricci type tensor $R^{(0,2)}$ of M are defined by

$$R^{(0,2)}(X,Y) = \sum_{j=1}^{n} g(R(e_j, X)Y, e_j) + \widetilde{g}(R(\xi, X)Y, N), \tag{2.18}$$

and scalar curvature τ be given by

$$\tau = \sum_{i,j=1}^{n} K_{ij} + \sum_{i=1}^{n} K_i^{null} + K_{iN}, \tag{2.19}$$

where $K_{iN} = \widetilde{g}(R(\xi, e_i)e_i, N)$ for $i \in \{1, \ldots, n\}$ [13].

3. Chen-ricci Inequality

Let M be an $(n+1)$-dimensional lightlike hypersurface of a Lorentzian manifold $\widetilde{M}$ and $\{e_1, \ldots, e_n, \xi\}$ be a basis of $\Gamma(TM)$ where $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\Gamma(S(TM))$. For $k \leq n$, we set $\pi_{k,\xi} = sp\{e_1, ..., e_k, \xi\}$ is a $(k+1)$-dimensional degenerate plane section and $\pi_k = sp\{e_1, \ldots, e_k\}$ is k-dimensional non-degenerate plane section. Define k-degenerate Ricci curvature and k-Ricci curvature at a unit vector $X \in \Gamma(TM)$ as follows:

$$(3.1)\quad Ric_{\pi_{k,\xi}}(X) = R^{(0,2)}(X,X) = \sum_{j=1}^{k} g(R(e_j,X)X,e_j) + \widetilde{g}(R(\xi,X)X,N),$$

$$(3.2)\quad Ric_{\pi_k}(X) = R^{(0,2)}(X,X) = \sum_{j=1}^{k} g(R(e_j,X)X,e_j),$$

respectively. Furthermore, k-degenerate scalar curvature and k-scalar curvature at $p \in M$ are given as follows:

$$(3.3)\quad \tau_{\pi_{k,\xi}}(p) = \sum_{i,j=1}^{k} K_{ij} + \sum_{i=1}^{k} K_i^{null} + K_{iN},$$

$$(3.4)\quad \tau_{\pi_k}(p) = \sum_{i,j=1}^{k} K_{ij},$$

respectively. For $k = n$, $\pi_n = sp\{e_1,\ldots,e_n\} = \Gamma(S(TM))$, then

$$(3.5)\quad Ric_{S(TM)}(e_1) = Ric_{\pi_n}(e_1) = \sum_{j=1}^{n} K_{1j} = K_{12} + \ldots + K_{1n},$$

and

$$(3.6)\quad \tau_{S(TM)}(p) = \sum_{i,j=1}^{n} K_{ij}.$$

$Ric_{S(TM)}(e_1)$ and $\tau_{S(TM)}(p)$ are called *screen Ricci curvature* and *screen scalar curvature*, respectively [14]. From (2.11) we can write

$$(3.7)\quad \tau_{S(TM)}(p) = \widetilde{\tau}_{S(TM)}(p) + \sum_{i,j=1}^{n} B_{ii}C_{jj} - B_{ij}C_{ji},$$

where $B_{ij} = B(e_i,e_j)$, $C_{ij} = C(e_i,e_j)$ for $i,j \in \{1,\ldots,n\}$, $\widetilde{\tau}_{T_pM}(p)$ is scalar curvature of n-plane section (screen distribution) of $\widetilde{M}$ given by [15]

$$\widetilde{\tau}_{S(TM)}(p) = \sum_{i,j=1}^{n} \widetilde{g}(\widetilde{R}(e_i,e_j)e_j,e_i).$$

We now recall the following lemma:

Lemma 3.1. [12]. *Let M be a screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$ with constant curvature c. Then*

$$(3.8)\quad 2\varphi w(\xi)B(X,PZ) = -cg(X,PZ).$$

Let M be a screen homothetic lightlike hypersurface of an $(n+2)$-dimensional Lorentzian space form $\widetilde{M}(c)$. From the Gauss-Codazzi type equations, equation (2.15) and applying Lemma 3.1 we have the following equations:

$$\tau_{S(TM)}(p) = n(n-1)c + \varphi n^2\mu^2 - \varphi\sum_{i,j}^{n}(B_{ij})^2, \tag{3.9}$$

$$\begin{aligned}\sum_{i=1}^{n} K_i^{null} &= \sum_{i=1}^{n} g(R(e_i,\xi)\xi, e_i) \\ &= \sum_{i=1}^{n} \widetilde{g}(\widetilde{R}(\xi, e_i)e_i, \xi) \\ &= \sum_{i=1}^{n}\{(\nabla_\xi B)(e_i,e_i) - (\nabla_{e_i}B)(\xi, e_i) + B(e_i,e_i)\omega(\xi)\} \\ &= \sum_{i=1}^{n}\{-B(e_i,e_i)\omega(\xi) + B(e_i,e_i)\omega(\xi)\} = 0,\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\textstyle\sum_{i=1}^{n} K_i^N &= \sum_{i=1}^{n} g(R(\xi,e_i)e_i, N) \\ &= \sum_{i=1}^{n} \widetilde{g}(\widetilde{R}(\xi, e_i)e_i, N) \\ &= \sum_{i=1}^{n}\varphi\{(\nabla_\xi B)(e_i,e_i) - (\nabla_{e_i}B)(\xi, e_i) - B(e_i,e_i)\omega(\xi)\} \\ &= -2\varphi n\mu\omega(\xi) \\ &= nc.\end{aligned} \tag{3.11}$$

From (2.19), (3.9), (3.10) and (3.11) the induced scalar curvature $\tau(p)$ of M becomes

$$\tau(p) = n^2c + \varphi n^2\mu^2 - \varphi\sum_{i,j}^{n}(B_{ij})^2. \tag{3.12}$$

Using (3.12) we obtain the following theorem immediately:

Theorem 3.2. *Let M be a screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$. Then we have*

$$\frac{1}{\varphi}(\tau(p) - n^2c) \leq n^2\mu^2. \tag{3.13}$$

The equality of (3.13) holds for $p \in M$ if and only if p is a totally geodesic point.

Now, we shall need the following lemma:

Lemma 3.3. [19]. *If $a_1, \ldots, a_n$ are $n(n > 1)$ real numbers then*

$$\frac{1}{n}(\sum_{i=1}^{n} a_i)^2 \leq \sum_{i=1}^{n} a_i^2, \tag{3.14}$$

with equality if and only if $a_1 = \cdots = a_n$.

From (3.12) and Lemma 3.3 we have the following theorem immediately:

Theorem 3.4. *Let M be a screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian space form $\widetilde{M}(c)$. Then we*

$$\tau(p) \leq n^2 c + \varphi\{n(n-1)\mu^2\}. \tag{3.15}$$

The equality of (3.15) holds for $p \in M$ if and only if p is a totally umbilical point.

Proof. By Lemma 3.3 we can write

$$\varphi \sum_{i=1}^{n} (B_{ij})^2 \geq \varphi \mu^2. \tag{3.16}$$

If we put (3.16) in (3.12) then we get (3.15).
The equality of (3.15) holds for $p \in M$ if and only if

$$B_{11} = \cdots = B_{nn}.$$

Thus p is a totally umbilical point.
We now recall the following lemma:

Lemma 3.5. [10]. *Let $a_1, \ldots, a_n$ be n-real numbers and define $A = \sum_{i<j}(a_i - a_j)^2$. Then*
(***1***) *$A \geq \frac{n}{2}(a_1 - a_2)^2$ and equality holds if and only if*

$$\frac{1}{2}(a_1 + a_2) = a_3 = \ldots = a_n.$$

(***2***) *Let k, ℓ be integers such that $1 \leq k < \ell \leq n$ and $(k, \ell) \neq (1, 2)$. If $A = \frac{n}{2}(a_1 - a_2)^2 = \frac{n}{2}(a_k - a_1)^2$ then $a_1 = a_2 = \ldots = a_n$.*

Since the sectional curvature of screen homothetic lightlike hypersurface is symmetric, we can denote the screen scalar curvature by $r_{S(TM)}$ as follows:

$$r_{S(TM)}(p) = \sum_{1\le i<j\le n} K_{ij} = \frac{1}{2}\sum_{i,j=1}^{n} K_{ij} = \frac{1}{2}\tau_{S(TM)}(p). \tag{3.17}$$

By (3.17), (3.9) equality become

$$2r_{S(TM)}(p) = 2\widetilde{r}_{S(TM)}(p) + \varphi n^2\mu^2 - \varphi\sum_{i,j}^{n}(B_{ij})^2. \tag{3.18}$$

Using (3.18) and Lemma 3.5 we get the following theorem:

Theorem 3.6. *Let M be a screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian manifold $\widetilde{M}$. Then we have*

$$2r_{S(TM)}(p) \le 2\widetilde{r}_{S(TM)}(p) + \frac{n+2}{n+1}\varphi n^2\mu^2 + \frac{\varphi n}{2(n+1)}(B_{11} - B_{22})^2. \tag{3.19}$$

The equality of (3.19) *holds at $p \in M$ if and only if the mean curvature of M is equal to $\frac{n}{2}(B_{11} + B_{22})$, that is, $\mu = \frac{n}{2}(B_{11} + B_{22})$.*

Proof. From the Binomial theorem we can write

$$(B_{11} - B_{22})^2 + \ldots + (B_{11} - B_{nn})^2 + (B_{22} - B_{33})^2 + \ldots + (B_{22} - B_{nn})^2 + \ldots + (B_{n-1n-1} - B_{nn})^2 = n\sum_{i=1}^{n}(B_{ii})^2 - 2\sum_{1\le i\ne j\le n} B_{ii}B_{jj}.$$

By Lemma 3.5 we have

$$\sum_{i=1}^{n}(B_{ii})^2 \ge \frac{1}{n}\sum_{i\ne j} B_{ii}B_{jj} + \frac{1}{2}(B_{11} - B_{22})^2. \tag{3.20}$$

On the other hand, we can write

$$\frac{1}{n}\sum_{i\ne j} B_{ii}B_{jj} = n\mu^2 - \frac{1}{n}\sum_{i=1}^{n}(B_{ii})^2. \tag{3.21}$$

Using (3.20) and (3.21) we get

$$\sum_{i=1}^{n}(B_{ii})^2 \ge \frac{n^2}{n+1}\mu^2 + \frac{1}{2(n+1)}(B_{11} - B_{22})^2. \tag{3.22}$$

Finally, by (3.18) and (3.22), we obtain (3.19).

The equality case of (3.19) holds then taking consideration of the case (1) of Lemma 3.5 we get $\mu = \frac{1}{2}(B_{11} + B_{22})$. The converse part of the theorem is straightforward.

From the Binomial theorem there is such as the following equation between the components of the second fundamental form:

$$\begin{aligned}\sum_{i,j=1}^{n}(B_{ij})^2 &= \frac{1}{2}n^2\mu^2 + \frac{1}{2}(B_{11} - B_{22} - \ldots - B_{nn})^2 \\ &+ 2\sum_{j=2}^{n}(B_{1j})^2 - 2\sum_{2\le i<j\le n} B_{ii}B_{jj} - (B_{ij})^2.\end{aligned} \tag{3.23}$$

Now, we shall introduce Chen-Ricci inequality on screen homothetic lightlike hypersurfaces.

Theorem 3.7. *Let M be a screen homothetic lightlike hypersurface with $\varphi > 0$ of a Lorentzian manifold $\widetilde{M}$. Then, the following statements are true.*

(***a***) *For $X \in S^1(TM) = \{X \in S(TM) : \langle X, X\rangle = 1\}$*

$$\frac{1}{4}n^2\mu^2 \le \frac{1}{\varphi}(Ric_{S(TM)}(X) - \widetilde{Ric}_{S(TM)}(X)). \tag{3.24}$$

(***b***) *The equality case of* (3.24) *is satisfied by $X \in T_p^1M$ if and only if*

$$\begin{cases} B(X,Y) = 0, \text{ for all } Y \in T_pM \text{ orthogonal to } X, \\ B(X,X) = \frac{n}{2}\mu. \end{cases} \tag{3.25}$$

(***c***) *The equality case of* (3.24) *holds for all $X \in T_p^1M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. From (3.18) and (3.23) we get

$$\begin{aligned}\frac{1}{4}\varphi n^2\mu^2 &= r(p) - \widetilde{r}_{S(TM)}(p) + \frac{\varphi}{4}(B_{11} - \ldots - B_{nn})^2 \\ &+ \varphi\sum_{j=2}^{n}(B_{1j})^2 - \varphi\sum_{2\le i<j\le n} B_{ii}B_{jj} - (B_{ij})^2.\end{aligned} \tag{3.26}$$

From (3.18) we also have

$$\varphi\sum_{2\le i<j\le n} B_{ii}B_{jj} - (B_{ij})^2 = \sum_{2\le i<j\le n}(K_{ij} - \widetilde{K}_{ij}). \tag{3.27}$$

Since

$$\begin{aligned}\sum_{2\le i<j\le n} K_{ij} &= r_{S(TM)}(p) - Ric_{S(TM)}(e_1), \\ \sum_{2\le i<j\le n} \widetilde{K}_{ij} &= \widetilde{r}_{S(TM)}(p) - \widetilde{Ric}_{S(TM)}(e_1),\end{aligned} \tag{3.28}$$

and using (3.26) we get

$$\frac{1}{4}\varphi n^2\mu^2 \leq Ric_{S(TM)}(X) - \widetilde{Ric}_{S(TM)}(X). \tag{3.29}$$

If we put $e_1 = X$ as any vector of $T_p^1 M$ in (3.29) we obtain (3.24).

The equality case of (3.24) holds for $X \in T_p^1 M$ if and only if

$$B_{12} = \cdots = B_{1n} = 0 \;\; and \;\; B_{11} = B_{22} + \cdots + B_{nn}.$$

So we have

$$n\mu = B_{11} + \cdots + B_{nn} = 2B_{11},$$

which is equivalent to (3.25).

We now suppose that the equality case of (3.24) holds for all $X \in T_p^1 M$ then we have

$$B_{ij} = 0, \qquad i \neq j. \tag{3.30}$$

$$2B_{ii} = B_{11} + \cdots + B_{nn}, \quad i \in \{1, \ldots, n\}. \tag{3.31}$$

From (3.31) we have

$$2B_{11} = 2B_{22} = \cdots = 2B_{nn} = \sum_{i=1}^{n} B_{ii},$$

which implies that

$$(n-2)\sum_{i=1}^{n} B_{ii} = 0.$$

Thus, either $\sum_{i=1}^{n} B_{ii} = 0$ or $n = 2$. If $\sum_{i=1}^{n} B_{ii} = 0$, then in view of (3.31), we get $B_{ii} = 0$ for all $i \in \{1, \ldots, r\}$. This together with (3.30) gives $B_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$, that is, p is a totally geodesic point. If $n = 2$, then form (3.31) we have

$$2B_{11} = 2B_{22} = B_{11} + B_{22},$$

which shows that p is a totally umbilical point. The proof of the converse part is straightforward.

From Theorem 3.7 we get the following corollary immediately:

Corollary 3.8. *Let M be a screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$. Then, the following statements are true.*

(***a***) *For* $X \in S^1(TM) = \{X \in S(TM) : \langle X, X\rangle = 1\}$

$$\frac{1}{4}n^2\mu^2 \leq \frac{1}{\varphi}(Ric_{S(TM)}(X) - (n-1)c). \tag{3.32}$$

(***b***) *The equality case of* (3.32) *is satisfied by* $X \in T_p^1M$ *if and only if*

$$\begin{cases} B(X,Y) = 0, \text{ for all } Y \in T_pM \text{ orthogonal to } X, \\ B(X,X) = \frac{n}{2}\mu. \end{cases} \tag{3.33}$$

(***c***) *The equality case of* (3.32) *holds for all* $X \in T_p^1M$ *if and only if either* p *is a totally geodesic point or* $n = 2$ *and* p *is a totally umbilical point.*

4. Chen-like Inequalities on Screen Conformal Lightlike Hypersurfaces

We begin this section with the following lemma:

Lemma 4.1. [5]. *If* $n \geq 2$ *and* $a_1, \ldots, a_n, a$ *are real numbers such that*

$$(\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + a), \tag{4.1}$$

then

$$2a_1a_2 \geq a,$$

with equality holding if and only if

$$a_1 + a_2 = a_3 = \cdots = a_n.$$

Theorem 4.2. *Let* M *be a screen homothetic lightlike hypersurface of a Lorentzian manifold* $\widetilde{M}$, $\varphi > 0$, $\Pi = Span\{e_1, e_2\}$ *be a 2-dimensional non-degenerate plane section of* T_pM, $p \in M$. *Then*

$$\begin{aligned} \tau_{S(TM)}(p) - \tau(\Pi) \leq\ & \widetilde{\tau}_{S(TM)}(p) - \widetilde{\tau}(\Pi) \\ & - \varphi\frac{n^2(n-2)}{n-1}\mu^2 + \varphi\sum_{i=3}^{n}(B_{ii})^2. \end{aligned} \tag{4.2}$$

Equality of (4.2) *holds at* $p \in M$ *then* M *is minimal and the shape operator of* M *take the form:*

$$A_\xi^* = \begin{pmatrix} B_{11} & B_{12} & \ldots 0 \\ B_{21} & -B_{11} & \ldots 0 \\ \vdots & & \\ 0 & 0 & \ldots 0 \end{pmatrix}. \tag{4.3}$$

Proof. If we put

$$\delta = \tau_{S(TM)}(p) - \varphi\frac{n^2(n-2)}{n-1}\mu^2 - \widetilde{\tau}_{S(TM)}(p),$$

in (3.9) we have

$$\delta = \varphi\frac{n^2}{n-1}\mu^2 - \varphi\sum_{i,j=1}^{n}(B_{ij})^2.$$

Therefore, we can write

$$(\sum_{i=1}^{n} B_{ii})^2 = (n-1)\left(\frac{\delta}{\varphi} + \sum_{i=1}^{n}(B_{ii})^2 + \sum_{i\neq j=1}^{n}(B_{ij})^2\right).$$

From Lemma 4.1 we get

$$2B_{11}B_{22} \geq \frac{\delta}{\varphi} + \sum_{i\neq j=1}^{n}(B_{ij})^2.$$

Now, let us choose a non-degenerate plane section Π that is spanned by e_1 and e_2. Then we obtain

$$\begin{aligned}
\tau(\Pi) &= \widetilde{\tau}(\Pi) + \varphi\sum_{i,j}^{2} B_{ii}B_{jj} - (B_{ij})^2 \\
&\geq \widetilde{\tau}(\Pi) + \delta + \varphi\sum_{i\neq j=1}^{n}(B_{ij})^2 - \varphi\sum_{i\neq j=1}^{2}(B_{ij})^2 \\
&\geq \widetilde{\tau}(\Pi) + \delta + \varphi\sum_{i,j=1}^{n}(B_{ij})^2 - \varphi\sum_{i=1}^{n}(B_{ii})^2 - \varphi\sum_{i\neq j=1}^{2}(B_{ij})^2 \\
&\geq \widetilde{\tau}(\Pi) + \delta - \varphi\sum_{i=3}^{n}(B_{ii})^2.
\end{aligned} \tag{4.4}$$

From (4.4) we finally have (4.2) and (4.3). Therefore, M is minimal.

From Theorem 4.2 we have the following corollary:

Corollary 4.3. *Let M be a screen homothetic lightlike hypersurface of a Lorentzian space form $\widetilde{M}(c)$, $\varphi > 0$, $\Pi = Span\{e_1, e_2\}$ be a 2-dimensional non-degenerate plane section of T_pM, $p \in M$. Then*

$$\tau_{S(TM)}(p) - \tau(\Pi) \leq (n+1)(n-2)c - \varphi\frac{n^2(n-2)}{n-1}\mu^2 + \varphi\sum_{i=3}^{n}(B_{ii})^2. \tag{4.5}$$

If the equality of (4.5) *holds at* $p \in M$ *then* M *is minimal and the shape operator of* M *take the form:*

$$A_{\xi}^{*} = \begin{pmatrix} B_{11} & B_{12} & \cdots & 0 \\ B_{21} & -B_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{4.6}$$

Taking into consideration Lemma 3.3 we have the following theorem:

Theorem 4.4. *Let* M *be a screen homothetic lightlike hypersurface and* $\varphi > 0$*. Then we have*

$$\tau_{S(TM)}(p) \leq \widetilde{\tau}_{S(TM)}(p) + \varphi n(n-1)\mu^{2}. \tag{4.7}$$

Equality of (4.7) *holds at* $p \in M$ *if and only if* p *is a totally umbilical point.*

Proof. If we write

$$\varphi \sum_{i,j=1}^{n} (B_{ij})^{2} = \varphi \sum_{i=1}^{n} (B_{ii})^{2} + \varphi \sum_{i \neq j} (B_{ij})^{2},$$

in (3.9) we have

$$\tau_{S(TM)}(p) = \widetilde{\tau}_{S(TM)}(p) + \varphi n^{2}\mu^{2} - \varphi \sum_{i=1}^{n} (B_{ii})^{2} - \varphi \sum_{i \neq j} (B_{ij})^{2}. \tag{4.8}$$

From Lemma 3.3 we get

$$n\mu^{2} \leq \sum_{i=1}^{n} (B_{ii})^{2}. \tag{4.9}$$

Using by (4.8) and (4.9) we obtain (4.7). Equality case of (4.7) holds if and only if

$$B_{11} = \cdots = B_{nn},$$

the shape operator A_n take the form:

$$A_{\xi}^{*} = \begin{pmatrix} B_{11} & 0 & \ldots & 0 & 0 \\ 0 & B_{11} & \ldots & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \ldots & B_{11} & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{4.10}$$

which shows that M is totally umbilical. This completes the proof of the theorem.

Definition 4.5. Let M be a lightlike hypersurface of a Lorentzian manifold. For any given set of mutually orthogonal plane sections π_j with dimensions k-tuples $(n_1, \ldots, n_k)$ such that $n_1 + \cdots + n_k \leq n + 1$, the δ-curvatures in lightlike case is defined by [14]

$$\delta(n_1, \ldots, n_k) = \tau(p) - \inf\{\tau_{\pi_{n_1}} + \cdots + \tau_{\pi_{n_k}}\}. \tag{4.11}$$

We now recall the following Chen's generalized Lemma from [8] for later use.

Lemma 4.6. *Suppose that $a_1, \ldots, a_n$ are n real numbers, k is an integer satisfying $2 \leq k \leq n-1$. Then for any partition $(n_1, \ldots, n_k)$ of n*

$$\begin{aligned} &\sum_{1\leq i_1<j_1\leq n_1} a_{i_1}a_{j_1} + \sum_{n_1+1\leq i_2<j_2\leq n_1+n_2} a_{i_2}a_{j_2} + \cdots \\ &+ \sum_{n_1+\cdots+n_{k-1}+1\leq i_k<j_k\leq n} a_{i_k}a_{j_k} \\ &\geq \frac{1}{2k}\{(a_1+\cdots+a_n)^2 - k(a_1^2+\cdots+a_n^2)\}, \end{aligned} \tag{4.12}$$

with the equality holding if and only if

$$a_1 + \cdots + a_{n_1} = a_{n_1+1} + \cdots + a_{n_1+n_2} = \cdots = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_n. \tag{4.13}$$

Let $C(n_1, \ldots, n_k)$ and $D(n_1, \ldots, n_k)$ be the positive numbers given by

$$C(n_1, \ldots, n_k) = \varphi \frac{n^2\left(n+k-1-\sum_{j=1}^{k} n_j\right)}{n+k-\sum_{j=1}^{k} n_j}, \tag{4.14}$$

$$D(n_1, \ldots, n_k) = n(n-1) - \sum_{j=1}^{k} n_j(n_j-1), \tag{4.15}$$

for each $(n_1, \ldots, n_k) \in S(n)$ the set of all unordered k-tuples with $k \geq 0$.

We now establish an optimal inequality involving the δ-invariant on screen conformal lightlike hypersurface as follows:

Theorem 4.7. *Let M be a screen homothetic hypersurface of a semi-Riemannian space form $\widetilde{M}$. Then for each point $p \in M$ and for each k-tuple $(n_1, \ldots, n_k) \in S(n)$, we have the following inequality:*

$$\delta(n_1, \ldots, n_k) \leq C(n_1, \ldots, n_k)\mu^2 + D(n_1, \ldots, n_k)c + (2\varphi - 1)n\mu. \tag{4.16}$$

The equality of (4.16) *holds at a point $p \in M$ if and only if the shape operators of M take the form:*

$$A^*_\xi = \begin{pmatrix} A^*_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & A^*_k & \\ & 0 & & \mu_r I \end{pmatrix}, \tag{4.17}$$

where I is an identity matrix and each A^*_j is a symmetric $n_j \times n_j$ submatrix such that

$$trace(A^*_1) = \cdots = trace(A^*_k) = \mu_r. \tag{4.18}$$

Proof. If we put

$$\eta = \tau(p) - n^2 c - \varphi n^2 \mu^2 \frac{(n + k - 1 - \sum_{j=1}^{k} n_j)}{n + k - \sum_{j=1}^{k} n_j}, \tag{4.19}$$

in (3.12) we get

$$\varphi n^2 \mu^2 = \gamma[\eta + \varphi \sum_{i,j=1}^{n} (B_{ij})^2]. \tag{4.20}$$

where $\gamma = n + k - \sum n_j$. We can write the following equality instead of (4.20)

$$(\sum_{i=1}^{n} B_{ii})^2 = \gamma[\frac{\eta}{\varphi} + \sum_{i \neq j}(B_{ij})^2 + \sum_{i=1}^{n}(B_{ii})^2]. \tag{4.21}$$

Equation (4.21) is equivalent to

$$\begin{aligned} (\sum_{i=1}^{\gamma+1} \bar{a}_i)^2 = & \gamma[\frac{\eta}{\varphi} + \sum_{i=1}^{\gamma+1}(\bar{a}_i)^2 + \sum_{i \neq j}(B_{ij})^2 \\ & - \sum_{1 \leq \alpha_1 \neq \beta_1 \leq n_1} a_{\alpha_1} a_{\beta_1} - \cdots - \sum_{\alpha_k \neq \beta_k} a_{\alpha_k} a_{\beta_k}], \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \bar{a}_1 &= a_1,\ \bar{a}_2 = a_2 + \cdots + a_{n_1}, \\ \bar{a}_3 &= a_{n_1+1} + \cdots + a_{n_1 n_2}, \\ &\vdots \\ \bar{a}_{k+1} &= a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k}, \\ &\vdots \\ \bar{a}_{\gamma+1} &= a_n. \end{aligned}$$

and $\alpha_i, \beta_i \in \triangle_i$, $i = \{1, \ldots, k\}$, $\triangle_1 = \{1, \ldots, n_1\}, \ldots, \triangle_k = \{n_1 + \ldots + n_{k-1} + 1, \ldots, n_1 + \ldots + n_k\}$. Since $\varphi > 0$, by applying Lemma 3.3 to (4.22) we have

$$\sum_{\alpha_1 \neq \beta_1} a_{\alpha_1} a_{\beta_1} + \ldots + \sum_{\alpha_k \neq \beta_k} a_{\alpha_k} a_{\beta_k} \geq \frac{\eta}{2\varphi} + \sum_{A<B} (B_{AB})^2. \tag{4.23}$$

In addition to this, from (3.12) we have

$$\tau(\pi_j) = \widetilde{\tau}_{S(TM)}(\pi_j) + 2\varphi \sum_{\alpha_j < \beta_j} B_{\alpha_j \alpha_j} B_{\beta_j \beta_j} - (B_{\alpha_j \beta_\alpha})^2, \tag{4.24}$$

where $dim\pi_j = n_j$. Using (4.23) and (4.24) we get

$$\begin{aligned} \tau(\pi_1) + \cdots + \tau(\pi_k) &\geq \eta + \sum_{j=1}^{k} \widetilde{\tau}_{S(TM)}(\pi_j) + 2\varphi \sum_{A<B} (B_{AB})^2 \\ &\geq \eta + \sum_{j=1}^{k} n_j^2 c. \end{aligned} \tag{4.25}$$

Therefore, by (4.19) and (4.25), we obtain (4.16).

Taking into consideration Lemma 4.6, the equality case of (4.16) holds at $p \in M$ if and only if the shape operators of M take the form as (4.17).

ACKNOWLEDGMENTS

The authors are grateful to the referee for his/her valuable comments and suggestions.

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Mehmet Gülbahar, Erol Kılıç and Sadık Keleş
İnönü University
Science & Arts Faculty
Department of Mathematics
Malatya, Turkey
E-mail: mehmet.gulbahar@inonu.edu.tr
erol.kilic@inonu.edu.tr
sadik.keles@inonu.edu.tr