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SOLVABILITY FOR A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. This article is concerned with the coupled system of fractional differential equations with nonlocal integral boundary conditions. The existence results are obtained by applying some standard fixed point theorems. Finally, an example is also provided to illustrate the availability of our main results.

1. INTRODUCTION

The fractional calculus, an active branch of mathematical analysis, is as old as the classical calculus which we know today. In recent years, fractional differential equations have been studied by many researchers, ranging from the theoretical aspects of existence and uniqueness to the numerical methods for finding solutions. It is well known that fractional differential equations provide an excellent tool for the description of memory and hereditary properties of various materials and processes. With these advantages, the fractional models become more practical and realistic than the classical integer-order ones, such effects in the latter are not taken into account. As a result, the subject of fractional differential equations is gaining more and more attention and importance. For more details on this branch of differential equations, please refer to the recent monographs of Miller and Ross [18], Kilbas et al. [13], Lakshmikantham [15], Podlubny[19] , Hilfer [9], and the papers of [1-3, 5-7, 11, 16, 17, 27, 28, 30].

Recently, many researchers paid much attention to the coupled system of fractional differential equations due to its applications in different fields, for instance, see [10, 22-26, 29] and references therein. Wang, Ahmad, et al. [23] discussed a coupled

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system of nonlinear fractional differential equations with m-point boundary conditions on an unbounded domain.

$$\begin{cases} D^p u(t) + f(t, v(t)) = 0, \ D^q v(t) + g(t, u(t)) = 0, \ 2 < p, q < 3, \\ u(0) = u'(0) = 0, \ D^{p-1} u(+\infty) = \sum\limits_{i=1}^{m-2} \beta_i u(\xi_i), \\ v(0) = v'(0) = 0, \ D^{q-1} v(+\infty) = \sum\limits_{i=1}^{m-2} \gamma_i v(\xi_i), \end{cases}$$

where $t \in J = [0, +\infty), f, g \in C(J \times \mathbb{R}, \mathbb{R}), \ 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, D^p and D^q denote the Riemann-Liouville fractional derivatives of order p and q, respectively.

Yang [25] considered the boundary value problem (BVP) for a coupled system of nonlinear fractional differential equations as follows:

$$\begin{cases} D^\alpha u(t) + a(t) f(t, v(t)) = 0, \ D^\beta v(t) + b(t) g(t, u(t)) = 0, \ 0 < t < 1, \\ u(0) = v(0) = 0, \ u(1) = \int_0^1 \phi(t) u(t) dt, v(1) = \int_0^1 \psi(t) v(t) dt, \end{cases}$$

where $1 < \alpha, \beta < 2$, $a, b \in C((0,1), [0, +\infty))$, $\phi, \psi \in L^1[0,1]$ are nonnegative and $f, g \in C([0,1] \times [0, +\infty), [0, +\infty))$. D is the standard Riemann-Liouville fractional derivative.

However, in these above works, the existence results of boundary value problem for nonlinear fractional differential equations were all obtained under the condition that the nonlinear terms f, g were independent of the fractional derivative of unknown functions v, u, respectively. But the opposite case is more complicated and difficult. To the best of our knowledge, only few papers can be found to study these coupled systems currently where the fractional derivatives of the unknown functions were involved in the nonlinear terms explicitly ([4, 12, 22]). In the present work, we attempt to discuss the coupled system where the nonlinear terms contain the fractional derivatives of the unknown functions. More precisely, this paper deals with the following coupled system of fractional differential equations involving integral boundary conditions:

$$\text{(1.1)} \quad \begin{cases} {}^c D_{0^+}^\alpha u(t) = f(t, v(t), {}^c D_{0^+}^p v(t)), {}^c D_{0^+}^\beta v(t) = g(t, u(t), {}^c D_{0^+}^q u(t)), 0 < t < 1, \\ a u'(0) + u(\eta_1) = \int_0^1 \phi(s, v(s)) ds, \ u(\eta_2) + b u'(1) = \int_0^1 \psi(s, v(s)) ds, \\ c v'(0) + v(\xi_1) = \int_0^1 \varphi(s, u(s)) ds, \ v(\xi_2) + d v'(1) = \int_0^1 \rho(s, u(s)) ds, \end{cases}$$

where $1 < \alpha, \beta < 2, \ 0 < p, q < 1$ and $\alpha - p - 1 \geq 0, \beta - q - 1 \geq 0$. $0 \leq \eta_1 < \eta_2 \leq 1, \ 0 \leq \xi_1 < \xi_2 \leq 1$. $f, g, \phi, \psi, \varphi, \rho$ are given functions satisfying some assumptions that will be specified later. The coupled system of Yang ([25]) can be recovered by choosing $a = b = c = d = 0, \eta_1 = \xi_1 = 0, \eta_2 = \xi_2 = 1$, and consequently, this result is a generalization of [25].

It is worthwhile to mention that $u(t)$ and $v(t)$ always equal to zero at $t = 0$ if the derivative is in the Riemann-Liouville sense ([22, 23, 25]). And many applied problems require the definitions of fractional derivative which physically interpret the initial or boundary value conditions. At this point, the fraction derivative in the sense of Caputo satisfies this requirement. On the other hand, integral boundary conditions come up as the values of functions on the boundary are connected to their values inside the domain, and they have physical significations such as blood flow problems, underground water flow, population dynamics, etc.. Sometimes, it is better to impose integral conditions for more precise measures than the local conditions. And the boundary conditions in (1.1) arise in the study of heat flow problems.

The rest of this paper is organized as follows. In Section 2, some preliminary definitions, notations, and lemmas are listed that will be used in the sequel. In Section 3, the existence results of solutions for the coupled system (1.1) are obtained by means of standard fixed point theorems. Finally, an example is given to illustrate the effectiveness of the main results.

2. Preliminaries

In this section, we present some basic knowledge and definitions about fractional calculus theory, which can be found in [13, 18, 20].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is defined by

$$I_{0+}^{\alpha} y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.2. The Caputo fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is defined by

$$({}^{c}D_{0+}^{\alpha} y)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds,$$

where $n = -[-\alpha]$.

Definition 2.3. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is defined by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

where $n = -[-\alpha]$, provided the right-hand side is point-wise defined on $(0, \infty)$.

Lemma 2.1. ([15]). *For $\alpha > 0$, the general solution of the fractional differential equation ${}^cD_{0+}^{\alpha}x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1 (n = -[-\alpha])$.

In view of Lemma 2.1, it turns out that

$$I_{0+}^{\alpha}\, {}^cD_{0+}^{\alpha}x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1} \tag{2.1}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1 (n = -[-\alpha])$.

Lemma 2.2. *For given $x, y \in C[0,1]$, $\delta, \sigma, \gamma, \kappa \in C[0,1]$, the unique solution of the following boundary value problem*

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t) = x(t), \quad {}^cD_{0+}^{\beta}v(t) = y(t), \quad 0 < t < 1, \\ au'(0) + u(\eta_1) = \int_0^1 \delta(t)dt, \quad u(\eta_2) + bu'(1) = \int_0^1 \sigma(t)dt, \\ cv'(0) + v(\xi_1) = \int_0^1 \gamma(t)dt, \quad v(\xi_2) + dv'(1) = \int_0^1 \kappa(t)dt, \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} u(t) = & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds + \frac{b+\eta_2-t}{b-a+\eta_2-\eta_1}\Bigg[\int_0^1 \delta(t)dt - \frac{1}{\Gamma(\alpha)} \\ & \int_0^{\eta_1}(\eta_1-s)^{\alpha-1}x(s)ds\Bigg] - \frac{a+\eta_1-t}{b-a+\eta_2-\eta_1}\Bigg[\int_0^1 \sigma(t)dt - \frac{1}{\Gamma(\alpha)} \\ & \int_0^{\eta_2}(\eta_2-s)^{\alpha-1}x(s)ds - \frac{b}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}x(s)ds\Bigg] \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} v(t) = & \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}y(s)ds + \frac{d+\xi_2-t}{d-c+\xi_2-\xi_1}\Bigg[\int_0^1 \gamma(t)dt - \frac{1}{\Gamma(\beta)} \\ & \int_0^{\xi_1}(\xi_1-s)^{\beta-1}y(s)ds\Bigg] - \frac{c+\xi_1-t}{d-c+\xi_2-\xi_1}\Bigg[\int_0^1 \kappa(t)dt - \frac{1}{\Gamma(\beta)} \\ & \int_0^{\xi_2}(\xi_2-s)^{\beta-1}y(s)ds - \frac{d}{\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2}y(s)ds\Bigg]. \end{aligned} \tag{2.4}$$

Proof. Since (u, v) satisfies the BVP(2.2), we can see, from (2.1), that

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds + c_0 + c_1 t, \tag{2.5}$$

$$(2.6) \qquad v(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds + c_0' + c_1' t,$$

then the first order derivatives of u and v are given by

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} x(s) ds + c_1,$$

$$v'(t) = \frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-1} y(s) ds + c_1'.$$

Substituting the boundary value conditions into the above expressions yields that

$$\begin{cases} ac_1 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^{\eta_1} (\eta_1 - s)^{\alpha-1} x(s) ds + c_0 + c_1 \eta_1 = \int_0^1 \delta(t) dt, \\ bc_1 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^{\eta_2} (\eta_2 - s)^{\alpha-1} x(s) ds + c_0 + c_1 \eta_2 \\ \qquad + \dfrac{b}{\Gamma(\alpha-1)} \displaystyle\int_0^1 (1-s)^{\alpha-2} x(s) ds = \int_0^1 \sigma(t) dt \end{cases}$$

and

$$\begin{cases} cc_1' + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^{\xi_1} (\xi_1 - s)^{\beta-1} y(s) ds + c_0' + c_1' \xi_1 = \int_0^1 \gamma(t) dt, \\ dc_1' + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^{\xi_2} (\xi_2 - s)^{\beta-1} y(s) ds + c_0' + c_1' \xi_2 \\ \qquad + \dfrac{d}{\Gamma(\beta-1)} \displaystyle\int_0^1 (1-s)^{\beta-2} y(s) ds = \int_0^1 \kappa(t) dt. \end{cases}$$

Solving the above two equations , we have

$$\begin{aligned} c_0 = &\frac{b+\eta_2}{b-a+\eta_2-\eta_1} \left[\int_0^1 \delta(t) dt - \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} (\eta_1 - s)^{\alpha-1} x(s) ds \right] \\ &- \frac{a+\eta_1}{b-a+\eta_2-\eta_1} \left[\int_0^1 \sigma(t) dt - \frac{1}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2 - s)^{\alpha-1} x(s) ds \right. \\ &\left. - \frac{b}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} x(s) ds \right], \\ c_0' = &\frac{d+\xi_2}{d-c+\xi_2-\xi_1} \left[\int_0^1 \gamma(t) dt - \frac{1}{\Gamma(\beta)} \int_0^{\xi_1} (\xi_1 - s)^{\beta-1} y(s) ds \right] \\ &- \frac{c+\xi_1}{d-c+\xi_2-\xi_1} \left[\int_0^1 \kappa(t) dt - \frac{1}{\Gamma(\beta)} \int_0^{\xi_2} (\xi_2 - s)^{\beta-1} y(s) ds \right. \\ &\left. - \frac{d}{\Gamma(\beta-1)} \int_0^1 (1-s)^{\beta-2} y(s) ds \right]. \end{aligned}$$

$$
\begin{aligned}
c_1 =& \frac{1}{b-a+\eta_2-\eta_1}\Big[\int_0^1 \sigma(t)dt - \int_0^1 \delta(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}x(s)ds \\
&-\frac{1}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-s)^{\alpha-1}x(s)ds - \frac{b}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}x(s)ds\Big], \\
c_1' =& \frac{1}{d-c+\xi_2-\xi_1}\Big[\int_0^1 \kappa(t)dt - \int_0^1 \gamma(t)dt + \frac{1}{\Gamma(\beta)}\int_0^{\xi_1}(\xi_1-s)^{\beta-1}y(s)ds \\
&-\frac{1}{\Gamma(\beta)}\int_0^{\xi_2}(\xi_2-s)^{\beta-1}y(s)ds - \frac{d}{\Gamma(\beta-1)}\int_0^1(1-s)^{\beta-2}y(s)ds\Big].
\end{aligned}
$$

Substituting the values of c_0, c_1, c_0', c_1' into (2.5) and (2.6), we can obtain the solution of BVP (2.2) given by (2.3) and (2.4). □

Denote $I = [0,1]$. Let $X_1 = \{u | u \in C[0,1], {}^cD^q u \in C[0,1]\}$ endowed with the norm $||u||_{X_1} = \max_{t\in I}|u(t)| + \max_{t\in I}|{}^cD^q u(t)|$. Then X_1 is a Banach space. Also define the Banach space $X_2 = \{v | v \in C[0,1], {}^cD^p v \in C[0,1]\}$ with the norm $||v||_{X_2} = \max_{t\in I}|v(t)| + \max_{t\in I}|{}^cD^p v(t)|$. For $(u,v) \in X_1 \times X_2 \triangleq X$, let $||(u,v)||_X = ||(u,v)||_{X_1\times X_2} = \max\{||u||_{X_1}, ||v||_{X_2}\}$. Clearly, $\{X, ||\cdot||_X\}$ is a Banach space.

Now we define an operator $T : X_1 \times X_2 \to X_2 \times X_1$ by $T(u,v) = (T_1 v, T_2 u)$, where

$$
\begin{aligned}
T_1 v(t) =& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, v(s), {}^cD^p v(s))ds \\
&+\frac{b+\eta_2-t}{b-a+\eta_2-\eta_1}\Big[\int_0^1 \phi(s, v(s))dt \\
&-\frac{1}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1-s)^{\alpha-1} f(s, v(s), {}^cD^p v(s))ds\Big] \\
&-\frac{a+\eta_1-t}{b-a+\eta_2-\eta_1}\Big[\int_0^1 \psi(s, v(s))dt \\
&-\frac{1}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-s)^{\alpha-1} f(s, v(s), {}^cD^p v(s))ds \\
&-\frac{b}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f(s, v(s), {}^cD^p v(s))ds\Big],
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
T_2 u(t) =& \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, u(s), {}^cD^q u(s))ds \\
&+\frac{d+\xi_2-t}{d-c+\xi_2-\xi_1}\Big[\int_0^1 \varphi(s, u(s))ds \\
&-\frac{1}{\Gamma(\beta)}\int_0^{\xi_1}(\xi_1-s)^{\beta-1} g(s, u(s), {}^cD^q u(s))ds\Big]
\end{aligned} \tag{2.8}
$$

$$-\frac{c+\xi_1-t}{d-c+\xi_2-\xi_1}\left[\int_0^1 \rho(s,u(s))ds\right.$$
$$-\frac{1}{\Gamma(\beta)}\int_0^{\xi_2}(\xi_2-s)^{\beta-1}g(s,u(s),{}^cD^q u(s))ds$$
$$\left.-\frac{d}{\Gamma(\beta-1)}\int_0^1(1-s)^{\beta-2}g(s,u(s),{}^cD^q u(s))ds\right].$$

Thus, the existence of the BVP(1.1) has been translated into the fixed point problem of the operator T. So in the next section, we shall discuss the fixed point of T.

3. Main Results

In this section, we shall obtain the existence results of the BVP(1.1). To this end, we need the following hypotheses:

(H_1) There exist positive functions $l_i (i=1,2,\ldots,8)$ such that

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\le l_1(t)|x_1-x_2|+l_2(t)|y_1-y_2|,$$

$$|g(t,x_1,y_1)-g(t,x_2,y_2)|\le l_3(t)|x_1-x_2|+l_4(t)|y_1-y_2|,$$

$$|\phi(t,v_1)-\phi(t,v_2)|\le l_5(t)|v_1-v_2|,\quad |\psi(t,v_1)-\psi(t,v_2)|\le l_6(t)|v_1-v_2|,$$

$$|\varphi(t,u_1)-\varphi(t,u_2)|\le l_7(t)|u_1-u_2|,\quad |\rho(t,u_1)-\rho(t,u_2)|\le l_8(t)|u_1-u_2|.$$

(H_2) Assume that $\triangle=\max\{\triangle_1,\triangle_2\}$ satisfies $0<\triangle<1$ with $\triangle_1=\triangle_{11}+\triangle_{12}, \triangle_2=\triangle_{21}+\triangle_{22}$, where

$$\begin{aligned}\triangle_{11} &= \max\{I^\alpha l_1(1)+\lambda_1 I^\alpha l_1(\eta_1)+\lambda_2 I^\alpha l_1(\eta_2)+|b|\lambda_2 I^{\alpha-1}l_1(1)+\lambda_1 I^1 l_5(1)\\ &\quad +\lambda_2 I^1 l_6(1), I^\alpha l_2(1)+\lambda_1 I^\alpha l_2(\eta_1)+\lambda_2 I^\alpha l_2(\eta_2)+|b|\lambda_2 I^{\alpha-1}l_2(1)\},\end{aligned}$$

$$\begin{aligned}\triangle_{12} &= \max\{I^{\alpha-p}l_1(1)+\lambda_3(I^\alpha l_1(\eta_1)+I^\alpha l_1(\eta_2)+|b|I^{\alpha-1}l_1(1)+I^1 l_5(1)\\ &\quad +I^1 l_6(1)), I^{\alpha-p}l_2(1)+\lambda_3(I^\alpha l_2(\eta_1)+I^\alpha l_2(\eta_2)+|b|I^{\alpha-1}l_2(1))\},\end{aligned}$$

$$\begin{aligned}\triangle_{21} &= \max\{I^\beta l_3(1)+\lambda_4 I^\beta l_3(\xi_1)+\lambda_5 I^\beta l_3(\xi_2)+|d|\lambda_5 I^{\beta-1}l_3(1)+\lambda_4 I^1 l_7(1)\\ &\quad +\lambda_5 I^1 l_8(1), I^\beta l_4(1)+\lambda_4 I^\beta l_4(\xi_1)+\lambda_5 I^\beta l_4(\xi_2)+|d|\lambda_5 I^{\beta-1}l_4(1)\},\end{aligned}$$

$$\begin{aligned}\triangle_{22} &= \max\{I^{\beta-q}l_3(1)+\lambda_6(I^\beta l_3(\xi_1)+I^\beta l_3(\xi_2)+|d|I^{\beta-1}l_3(1)+I^1 l_7(1)\\ &\quad +I^1 l_8(1)), I^{\beta-q}l_4(1)+\lambda_6(I^\beta l_4(\xi_1)+I^\beta l_4(\xi_2)+|d|I^{\beta-1}l_4(1))\}.\end{aligned}$$

$$\lambda_1 = \sup_{t\in I}\Big|\frac{b+\eta_2-t}{b-a+\eta_2-\eta_1}\Big|, \qquad \lambda_2 = \sup_{t\in I}\Big|\frac{a+\eta_1-t}{b-a+\eta_2-\eta_1}\Big|,$$
$$\lambda_3 = \sup_{t\in I}\Big|\frac{t^{1-p}}{\Gamma(2-p)(b-a+\eta_2-\eta_1)}\Big|, \qquad \lambda_4 = \sup_{t\in I}\Big|\frac{d+\xi_2-t}{d-c+\xi_2-\xi_1}\Big|,$$
$$\lambda_5 = \sup_{t\in I}\Big|\frac{c+\xi_1-t}{d-c+\xi_2-\xi_1}\Big|, \qquad \lambda_6 = \sup_{t\in I}\Big|\frac{t^{1-q}}{\Gamma(2-q)(d-c+\xi_2-\xi_1)}\Big|,$$

(H_3) Assume that there exist functions $M_\phi(t), M_\psi(t), M_\varphi(t), M_\rho(t)$ such that

$$|\phi(t,v)| \le M_\phi(t), |\psi(t,v)| \le M_\psi(t), |\varphi(t,u)| \\ \le M_\varphi(t), |\rho(t,u)| \le M_\rho(t), \text{ for } t\in I, (u,v)\in X$$

Theorem 3.1. *Suppose that $f,\ g$ are continuous functions satisfying the assumption (H_1), and the condition (H_1)-(H_3) holds. Then the BVP (1.1) has a unique solution.*

Proof. Since $f,\ g$ are continuous, there exist M_f, M_g such that $|f(t,x,y)| \le M_f,\ |g(t,x,y)| \le M_g$. Choosing

$$r_{11} = \frac{M_f}{\Gamma(\alpha+1)}(1+\lambda_1\eta_1^\alpha+\lambda_2\eta_2^\alpha+|b|\alpha\lambda_2)+(\lambda_1 M_\phi+\lambda_2 M_\psi),$$

$$r_{12} = M_f\left[\frac{1}{\Gamma(\alpha-p+1)}+\frac{\lambda_3(\eta_1^\alpha+\eta_2^\alpha+|b|\alpha)}{\Gamma(\alpha+1)}\right]+\lambda_3(M_\phi+M_\psi),$$

$$r_{21} = \frac{M_g}{\Gamma(\beta+1)}(1+\lambda_4\xi_1^\beta+\lambda_5\xi_2^\beta+|d|\beta\lambda_5)+(\lambda_4 M_\varphi+\lambda_5 M_\rho),$$

$$r_{22} = M_g\left[\frac{1}{\Gamma(\beta-q+1)}+\frac{\lambda_6(\xi_1^\beta+\xi_2^\beta+|d|\beta)}{\Gamma(\beta+1)}\right]+\lambda_6(M_\varphi+M_\rho),$$

where $M_\tau = \int_0^1 M_\tau(t)dt, \tau\in\{\phi,\psi,\varphi,\rho\}$.

Let $\Omega_r = \{(u,v)|(u,v)\in X, ||(u,v)||_X \le r\}$, where $r=\max\{r_1,r_2\}$ and $r_1 = r_{11}+r_{12},\ r_2 = r_{21}+r_{22}$. We shall show that $T\Omega_r \subset \Omega_r$.

For $(u,v)\in\Omega_r$, we have

$$\begin{aligned}|T_1 v(t)| \le{}& \frac{M_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds+\lambda_1\left[\int_0^1 M_\phi(s)ds+\frac{M_f}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}ds\right]\\ &+\lambda_2\left[\int_0^1 M_\psi(s)ds+\frac{M_f}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-s)^{\alpha-1}ds+\frac{|b|M_f}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}ds\right]\\ \le{}& \frac{M_f}{\Gamma(\alpha+1)}(1+\lambda_1\eta_1^\alpha+\lambda_2\eta_2^\alpha+|b|\alpha\lambda_2)+(\lambda_1 M_\phi+\lambda_2 M_\psi) = r_{11}\end{aligned}$$

and

$$
\begin{aligned}
|^cD^pT_1v(t)| \leq\ & \frac{M_f}{\Gamma(\alpha-p)}\int_0^t (t-s)^{\alpha-p-1}ds \\
& +\lambda_3\left[\int_0^1 M_\phi(s)ds + \frac{M_f}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}ds\right] \\
& +\lambda_3\left[\int_0^1 M_\psi(s)ds + \frac{M_f}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-s)^{\alpha-1}ds\right. \\
& \left.+\frac{|b|M_f}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}ds\right] \\
\leq\ & M_f\left[\frac{1}{\Gamma(\alpha-p+1)} + \frac{\lambda_3(\eta_1^\alpha+\eta_2^\alpha+|b|\alpha)}{\Gamma(\alpha+1)}\right] + \lambda_3(M_\phi+M_\psi) = r_{12}.
\end{aligned}
$$

Then, we can see that $||T_1v||_{X_2} = \max_{t\in I}|T_1v(t)| + \max_{t\in I}|^cD^pT_1v(t)| \leq r_{11}+r_{12} = r_1$. In the same way, one can obtain that $||T_2u||_{X_1} = \max_{t\in I}|T_2u(t)| + \max_{t\in I}|^cD^qT_2u(t)| \leq r_{21}+r_{22} = r_2$, then $||T(u,v)|| \leq r$.

Next, we show that the operator T is a contraction mapping on Ω_r. For $(u,v)\in\Omega_r$, on one hand,

$$
\begin{aligned}
& |T_1v_2(t) - T_1v_1(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big[l_1(s)|v_2(s)-v_1(s)| + l_2(s)|^cD^pv_2(s) -^c D^pv_1(s)|\big]ds \\
& +\lambda_1\bigg[\int_0^1 l_5(s)|v_2(s)-v_1(s)|ds + \frac{1}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}\big[l_1(s)|v_2(s) \\
& -v_1(s)| + l_2(s)|^cD^pv_2(s) -^c D^pv_1(s)|\big]ds\bigg] \\
& +\lambda_2\bigg[\int_0^1 l_6(s)|v_2(s)-v_1(s)|ds + \frac{1}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-s)^{\alpha-1}[l_1(s)|v_2(s) \\
& -v_1(s)| + l_2(s)|^cD^pv_2(s) -^c D^pv_1(s)|]ds + \frac{|b|}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}[l_1(s)|v_2(s) \\
& -v_1(s)| + l_2(s)|^cD^pv_2(s) -^c D^pv_1(s)|]ds\bigg] \\
\leq\ & [\lambda_1 I^\alpha l_1(\eta_1) + \lambda_2 I^\alpha l_1(\eta_2) + I^\alpha l_1(1) + |b|\lambda_2 I^{\alpha-1}l_1(1) + I^1(\lambda_1 l_5(1) \\
& +\lambda_2 l_6(1))]\max_{t\in I}|v_2(t)-v_1(t)| + [\lambda_1 I^\alpha l_2(\eta_1) + \lambda_2 I^\alpha l_2(\eta_2) + I^\alpha l_2(1) \\
& +|b|\lambda_2 I^{\alpha-1}l_2(1)]\max_{t\in I}|^cD^pv_2(t) -^c D^pv_1(t)| \\
\leq\ & \triangle_{11}||v_2-v_1||_{X_2}.
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
&|^cD^pT_1v_2(t) -^c D^pT_1v_1(t)| \\
\leq\ & \frac{1}{\Gamma(\alpha-p)}\int_0^t (t-s)^{\alpha-p-1}\,[l_1(s)|v_2(s)-v_1(s)| + l_2(s)|^cD^pv_2(s) \\
&-^cD^pv_1(s)|]\,ds + \lambda_3\int_0^1 (l_5(s)+l_6(s))|v_2(s)-v_1(s)|ds \\
&+\frac{\lambda_3}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1-s)^{\alpha-1}[l_1(s)|v_2(s)-v_1(s)| + l_2(s)|^cD^pv_2(s) \\
&-^cD^pv_1(s)|]ds + \frac{\lambda_3}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-s)^{\alpha-1}[l_1(s)|v_2(s)-v_1(s)| \\
&+l_2(s)|^cD^pv_2(s) -^c D^pv_1(s)|]ds + \frac{|b|\lambda_3}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}[l_1(s)|v_2(s) \\
&-v_1(s)| + l_2(s)|^cD^pv_2(s) -^c D^pv_1(s)|]ds \\
\leq\ & [I^{\alpha-p}l_1(1) + \lambda_3(I^\alpha l_1(\eta_1) + I^\alpha l_1(\eta_2) + |b|I^{\alpha-1}l_1(1) + I^1(l_5(1) \\
&+l_6(1)))]\max_{t\in I}|v_2(t)-v_1(t)| + [I^{\alpha-p}l_2(1) + \lambda_3(I^\alpha l_2(\eta_1) + I^\alpha l_2(\eta_2) \\
&+|b|I^{\alpha-1}l_2(1))]\max_{t\in I}|^cD^pv_2(t) -^c D^pv_1(t)| \\
\leq\ & \triangle_{12}||v_2-v_1||_{X_2}.
\end{aligned}$$

Then we have $||T_1v_2-T_1v_1||_{X_2} \leq \triangle_1||v_2-v_1||_{X_2}$. Similarly, we can see that $||T_2u_2-T_2u_1||_{X_2} \leq \triangle_2||u_2-u_1||_{X_1}$. So $||T(u_2,v_2)-T(u_1,v_1)||_X \leq \triangle||(u_2,v_2)-(u_1,v_1)||_X$. Thus we can see, by means of the condition (H_2), that the operator T is a contraction mapping. Using the contraction mapping principle (Banach fixed point theorem), T has a unique fixed point. That is, the BVP (1.1) has a unique solution. ∎

In the sequel, we shall continue to study the existence results of the BVP(1.1) in terms of the following Krasnoselskii fixed point theorem and Schaefer fixed point theorem (see, e.g.[8, 14, 21, 31, 32]), respectively.

Lemma 3.1. (Krasnoselskii fixed point theorem). *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$, wherever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then, there exists $z \in M$ such that $z = Az + Bz$.*

Lemma 3.2. (Schaefer fixed point theorem). *Let E be a Banach space. Assume that $T : E \to E$ be a completely continuous operator and the set $V = \{x \in E | x = \mu Tx, 0 < \mu < 1\}$ be bounded. Then, T has a fixed point in E.*

Denote $T = \Theta_1 + \Theta_2$, where $\Theta_1(u,v) = (\Theta_{11}v, \Theta_{12}u),\ \Theta_2(u,v) = (\Theta_{21}v, \Theta_{22}u)$

and

$$\Theta_{11}v = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, v(s), {}^c D^p v(s))ds,$$

$$\Theta_{12}u = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s, u(s), {}^c D^q u(s))ds,$$

$$\begin{aligned}
\Theta_{21}v \;=\; & \frac{b+\eta_2-t}{b-a+\eta_2-\eta_1}\left[\int_0^1 \phi(s, v(s))dt - \frac{1}{\Gamma(\alpha)}\int_0^{\eta_1} (\eta_1-s)^{\alpha-1} f(s, v(s), {}^c D^p v(s))ds\right] \\
& -\frac{a+\eta_1-t}{b-a+\eta_2-\eta_1}\left[\int_0^1 \psi(s, v(s))dt - \frac{1}{\Gamma(\alpha)}\int_0^{\eta_2} (\eta_2-s)^{\alpha-1} f(s, v(s), {}^c D^p v(s))ds\right. \\
& \left. -\frac{b}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f(s, v(s), {}^c D^p v(s))ds\right],
\end{aligned}$$

$$\begin{aligned}
\Theta_{22}u \;=\; & \frac{d+\xi_2-t}{d-c+\xi_2-\xi_1}\left[\int_0^1 \varphi(s, u(s))ds - \frac{1}{\Gamma(\beta)}\int_0^{\xi_1} (\xi_1-s)^{\beta-1} g(s, u(s), {}^c D^q u(s))ds\right] \\
& -\frac{c+\xi_1-t}{d-c+\xi_2-\xi_1}\left[\int_0^1 \rho(s, u(s))ds - \frac{1}{\Gamma(\beta)}\int_0^{\xi_2} (\xi_2-s)^{\beta-1} g(s, u(s), {}^c D^q u(s))ds\right. \\
& \left. -\frac{d}{\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2} g(s, u(s), {}^c D^q u(s))ds\right].
\end{aligned}$$

Theorem 3.2. *Assume that $f,\ g$ are continuous functions satisfying the assumption* (H_1)*, and the condition* (H_1) *holds. Then the BVP (1.1) has at least one solution provided that* $0 < \triangle' < 1$*, where* $\triangle' = \max\{\triangle'_1, \triangle'_2\},\ \triangle'_1 = \triangle'_{11} + \triangle'_{12}, \triangle'_2 = \triangle'_{21} + \triangle'_{22}$ *and*

$$\begin{aligned}
\triangle'_{11} \;=\; & \max\{\lambda_1 I^\alpha l_1(\eta_1) + \lambda_2 I^\alpha l_1(\eta_2) + |b|\lambda_2 I^{\alpha-1} l_1(1) + \lambda_1 I^1 l_5(1) + \lambda_2 I^1 l_6(1), \\
& \lambda_1 I^\alpha l_2(\eta_1) + \lambda_2 I^\alpha l_2(\eta_2) + |b|\lambda_2 I^{\alpha-1} l_2(1)\},
\end{aligned}$$

$$\begin{aligned}
\triangle'_{12} \;=\; & \max\{\lambda_3(I^\alpha l_1(\eta_1) + I^\alpha l_1(\eta_2) + |b| I^{\alpha-1} l_1(1) + I^1 l_5(1) + I^1 l_6(1)), \\
& \lambda_3(I^\alpha l_2(\eta_1) + I^\alpha l_2(\eta_2) + |b| I^{\alpha-1} l_2(1))\},
\end{aligned}$$

$$\begin{aligned}
\triangle'_{21} \;=\; & \max\{\lambda_4 I^\beta l_3(\xi_1) + \lambda_5 I^\beta l_3(\xi_2) + |d|\lambda_5 I^{\beta-1} l_3(1) + \lambda_4 I^1 l_7(1) + \lambda_5 I^1 l_8(1), \\
& \lambda_4 I^\beta l_4(\xi_1) + \lambda_5 I^\beta l_2(\eta_2) + |d|\lambda_5 I^{\beta-1} l_4(1)\},
\end{aligned}$$

$$\begin{aligned}
\triangle'_{22} \;=\; & \max\{\lambda_6(I^\beta l_3(\xi_1) + I^\beta l_3(\xi_2) + |d| I^{\beta-1} l_3(1) + I^1 l_7(1) + I^1 l_8(1)), \\
& \lambda_6(I^\beta l_4(\xi_1) + I^\beta l_4(\xi_2) + |d| I^{\beta-1} l_4(1))\}.
\end{aligned}$$

Proof. Choosing $r = \max\{r_1, r_2\}$, where r_1, r_2 are the same as the ones in the proof of Theorem 3.1. Let $W = \{(u,v) \in X|||(u,v)||_X \leq r\}$. For $(u,v),(x,y) \in W$, we shall show that $\Theta_1(u,v) + \Theta_2(x,y) = (\Theta_{11}v + \Theta_{21}y, \Theta_{12}u + \Theta_{22}x) \in W$. In fact, noting that

$$\begin{aligned}
&|\Theta_{11}v(t) + \Theta_{21}y(t)| \\
\leq\ & \frac{M_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \lambda_1\left[\int_0^1 M_\phi(s)ds + \frac{M_f}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1 - s)^{\alpha-1}ds\right] \\
&+\lambda_2\left[\int_0^1 M_\psi(s)ds + \frac{M_f}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-s)^{\alpha-1}ds + \frac{|b|M_f}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}ds\right] \\
\leq\ & \frac{M_f}{\Gamma(\alpha+1)}(1 + \lambda_1\eta_1^\alpha + \lambda_2\eta_2^\alpha + |b|\alpha\lambda_2) + (\lambda_1 M_\phi + \lambda_2 M_\psi) = r_{11}
\end{aligned}$$

and

$$\begin{aligned}
&|D^p\Theta_{11}v(t) + D^p\Theta_{21}y(t)| \\
\leq\ & \frac{M_f}{\Gamma(\alpha-p)}\int_0^t (t-s)^{\alpha-p-1}ds + \lambda_3\left[\int_0^1 M_\phi(s)ds + \frac{M_f}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1 - s)^{\alpha-1}ds\right] \\
&+\lambda_3\left[\int_0^1 M_\psi(s)ds + \frac{M_f}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2-s)^{\alpha-1}ds + \frac{|b|M_f}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}ds\right] \\
\leq\ & M_f\left[\frac{1}{\Gamma(\alpha-p+1)} + \frac{\lambda_3(\eta_1^\alpha + \eta_2^\alpha + |b|\alpha)}{\Gamma(\alpha+1)}\right] + \lambda_3(M_\phi + M_\psi) = r_{12}.
\end{aligned}$$

Then $||\Theta_{11}v + \Theta_{21}y||_{X_2} \leq r_1 = r_{11} + r_{12}$. Similarly, $||\Theta_{12}u + \Theta_{22}x||_{X_1} \leq r_2 = r_{11}+r_{12}$. So we obtain $||\Theta_1(u,v)+\Theta_2(x,y)||_X \leq r = \max\{r_1, r_2\}$, that is, $\Theta_1(u,v)+\Theta_2(x,y) \in W$.

The continuity of f and g implies that the operator Θ_1 is continuous. Meanwhile, it is obvious that Θ_1 is uniformly bounded.

On the other hand, for $0 \leq t_1 \leq t_2 \leq 1$, we have

$$\begin{aligned}
&|\Theta_{11}v(t_2) - \Theta_{11}v(t_1)| \\
=\ & \frac{1}{\Gamma(\alpha)}|\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,v(s),D^pv(s))ds - \int_0^{t_1}(t_1-s)^{\alpha-1}f(s,v(s),D^pv(s))ds| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}|(t_2-s)^{\alpha-1}f(s,v(s),D^pv(s))|ds + \int_0^{t_1}\big|((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})\right. \\
&\left. f(s,v(s),D^pv(s))\big|ds\right] \\
\leq\ & \frac{M_f}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha)
\end{aligned}$$

and

$$
\begin{aligned}
&|D^p\Theta_{11}v(t_2) - D^p\Theta_{11}v(t_1)| \\
&= \frac{1}{\Gamma(\alpha-p)}\bigg|\int_0^{t_2}(t_2-s)^{\alpha-p-1}f(s,v(s),D^pv(s))ds \\
&\quad - \int_0^{t_1}(t_1-s)^{\alpha-p-1}f(s,v(s),D^pv(s))ds\bigg| \\
&\leq \frac{1}{\Gamma(\alpha-p)}\bigg[\int_{t_1}^{t_2}|(t_2-s)^{\alpha-p-1}f(s,v(s),D^pv(s))|ds \\
&\quad + \int_0^{t_1}|((t_2-s)^{\alpha-p-1}-(t_1-s)^{\alpha-p-1})f(s,v(s),D^pv(s))|ds\bigg] \\
&\leq \frac{M_f}{\Gamma(\alpha-p+1)}(t_2^{\alpha-p}-t_1^{\alpha-p}),
\end{aligned}
$$

which are dependent of v and tend to zero as $t_2 \to t_1$, thus, Θ_{11} is equicontinuous. Similarly, Θ_{21} is equicontinuous. Then Θ_1 is equicontinuous. By using Ascoli-Arzela theorem, Θ_1 is a compact operator on W. Hence, Θ_1 is completely continuous on W.

Finally we can see, by the analogous argument to the proof of Theorem 3.1, that Θ_2 is a contraction mapping for $0 < \triangle' < 1$. The detailed proof is omitted here. Thus all the assumption of Lemma 3.1 are satisfied, which implies that the BVP(1.1) has at least one solution on $I = [0,1]$. ■

Theorem 3.3. *Suppose that the functions $f, g, \phi, \psi, \varphi, \rho$ are continuous, then the BVP (1.1) has at least one solution on $I = [0,1]$.*

Proof. Since the functions $f, g, \phi, \psi, \varphi, \rho$ are continuous, there exist constants $M_f, M_g, M_\phi, M_\psi, M_\varphi, M_\rho$ such that

$$|f(t,v(t),D^pv(t))| \leq M_f,\ |g(t,u(t),D^qu(t))| \leq M_g, |\phi(t,v(t))| \leq M_\phi,$$

$$|\psi(t,v(t))| \leq M_\psi, |\varphi(t,u(t))| \leq M_\varphi, |\rho(t,v(t))| \leq M_\rho, \forall t \in [0,1], (u,v) \in X_1 \times X_2.$$

The operator Θ_1 in Theorem 3.2 is completely continuous from the proof of Theorem 3.2. On the other hand, the operator Θ_2 is also completely continuous provided that $f, g, \phi, \psi, \varphi, \rho$ are continuous. As a result, the operator T here is completely continuous.

We consider the set $\mathcal{P} = \{(u,v) \in X | (u,v) = \lambda T(u,v), 0 < \lambda < 1\}$ and show that it is bounded. Let $(u,v) \in \mathcal{P}$, then $u = \lambda T_1 v,\ v = \lambda T_2 u$, which means that

$$|u(t)| \leq |T_1v(t)|,\ |D^pu(t)| \leq |D^pT_1v(t)|,$$

$$|v(t)| \leq |T_2u(t)|,\ |D^qv(t)| \leq |D^qT_1u(t)|.$$

We can see, from the proof of Theorem 3.1, that $||u||_{X_1},\ ||v||_{X_2}$ are bounded. That is, $\mathcal{P}$ is a bounded set. Hence, by using Lemma 3.2, the BVP(1.1) has at least one solution. ■

4. An Example

Example 4.1. Consider the following boundary value problem of coupled systems

$$
\left\{
\begin{aligned}
{}^cD^{\frac{7}{4}}u(t) &= \frac{t^2}{1+t}v(t) + \frac{t^2}{1+t}{}^cD^{\frac{2}{3}}v(t),\ {}^cD^{\frac{7}{4}}v(t)\\
&= \frac{t^2}{1+t}u(t) + \frac{t^2}{1+t}{}^cD^{\frac{2}{3}}u(t),\quad 0<t<1;\\
\frac{1}{200}u'(0)+u(\frac{1}{3}) &= \frac{1}{20}\int_0^1 t^3\sin v(t)dt,\ u(\frac{2}{3})+\frac{1}{100}u'(1)\\
&= \frac{1}{20}\int_0^1 t^3\sin v(t)dt,\\
\frac{1}{120}v'(0)+v(\frac{1}{4}) &= \frac{1}{30}\int_0^1 t^3\cos u(t)dt,\ v(\frac{1}{2})+\frac{1}{60}v'(1)\\
&= \frac{1}{30}\int_0^1 t^3\cos u(t)dt,
\end{aligned}
\right.
\tag{4.1}
$$

Here $\alpha=\beta=\frac{7}{4},\ p=q=\frac{2}{3};\ \eta_1=\frac{1}{3},\eta_2=\frac{2}{3},\xi_1=\frac{1}{4},\xi_2=\frac{1}{2};\ a=\frac{1}{200},b=\frac{1}{100},c=\frac{1}{120},d=\frac{1}{60}$ with

$$\lambda_1=2,\lambda_2=\frac{397}{203},\lambda_3=\frac{600}{203\Gamma(\frac{4}{3})},\lambda_4=2,\lambda_5=\frac{89}{31},\lambda_6=\frac{120}{31\Gamma(\frac{4}{3})}.$$

Direct calculation shows that

$$\triangle_1=\triangle_{11}+\triangle_{12}=0.188181+0.503581=0.691762,$$

$$\triangle_2=\triangle_{21}+\triangle_{22}=0.194627+0.584286=0.778913,$$

then $\triangle=\max\{\triangle_1,\triangle_2\}=0.778913<1$. That is , the condition (H_2) holds. On the other hand, it is obviously that f,g are continuous, and

$$|\phi(t,v)|=|\psi(t,v)|\le\frac{1}{20}t^3,\ |\varphi(t,u)|=|\rho(t,u)|\le\frac{1}{30}t^3,$$

with

$$l_i=\frac{t^2}{1+t},i=\{1,2,3,4\},\ \ l_j=\frac{1}{20}t^3,j=\{5,6\},\ \ l_k=\frac{1}{30}t^3,k=\{7,8\}.$$

Hence, all the conditions of Theorem 3.1 are satisfied. Therefore, the boundary value problem of coupled system (4.1) has a unique solution.

Remark 4.1. We note that the fractional derivatives of the nonlinear terms are included in the BVP(4.1) explicitly. The earlier methods for existence of BVP, to the best of our knowledge, cannot efficiently be applied to solve this present problem. It turns out that our results extend the previous works, such as [25].

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