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RELATIVE ATTACHED PRIMES AND COREGULAR SEQUENCES

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Abstract. We extend the existing concepts of secondary representation of a module, coregular sequence and attached prime ideals to the more general setting of any hereditary torsion theory. We prove that any τ-artinian module is τ-representable and that such a representation has some sort of unicity in terms of the set of τ-attached prime ideals associated to it. Then we use τ-coregular sequences to find a nice way to compute the relative width of a module. Finally we give some connections with the relative local homology.

1. Introduction

Since 1950's authors like Auslander, Buchsbaum, Serre and Kaplansky used regular sequences to find homological characterizations of some interesting rings in Algebraic Geometry.

Coregular sequences, as well as the width of an artinian module, were introduced by Matlis ([6]) in 1960, and later, in 1976 Ooishi ([9]) gave the concept of the cograde of a module.

Recently, a characterization of the width of a module by means of local homology modules has been given ([8]), and these local homology modules have been proved to be very close to left derived functors of the I-adic completion ([2]).

In this paper we use a torsion theory to extend the concept of a coregular sequence, attached prime ideal to a module and the width of a module. We first study conditions for a module to admit a (relative) secondary representation (in the sense of [5]) and prove that, if this is the case, the set of prime ideals associated to a secondary decomposition of the module is unique and coincides with the set of attached prime ideals to the module and to the set of attached prime ideals to any of its coprime quotients.

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Then, we study the existing relation between coregular sequences and the torsion of some Tor modules. From this relation we deduce a nice way to compute the width of a module in terms of the torsion of Tor_n^R. This also makes possible to find a relation between the width of the three modules of a short exact sequence.

Finally we introduce a functor ${}_\tau H_i^I$ for any $i \geq 0$ that, over any module M, acts as $\lim\limits_{\leftarrow} \mathcal{Q}_\tau \left(\mathrm{Tor}_i^R(M, R/I^n)\right)$. The previous results suggest that these modules ${}_\tau H_i^I(M)$ should be related is some sense with the existence of (relative) coregular sequences. We show that this is indeed the case and that actually, using these ${}_\tau H_i^I(M)$, τ-$Width_I(M)$ can be computed as the projective or injective dimension of a module can be computed using Ext.

Finally, we prove that the functors ${}_\tau H_i^I$ are indeed derived functors of $\lim\limits_{\leftarrow} \mathcal{Q}_\tau$ $(M/I^n M)$.

Throughout the paper R will be a commutative ring with identity, τ an idempotent kernel functor in the category R-Mod, $\mathcal{T}_\tau$ and $\mathcal{F}_\tau$ the classes of all τ-torsion and all τ-torsion free modules respectively, and $\mathcal{L}(\tau)$ the Gabriel filter associated to τ. By a torsion or torsion free module we shall mean a τ-torsion or τ-torsion free module respectively. The localization functor associated to τ will be denoted by $\mathcal{Q}_\tau$.

Recall that given a module M and a submodule $N \leq M$, the τ-closure of N in M is defined as

$$Cl_\tau^M(N) = \{m \in M;\ \exists I \in \mathcal{L}(\tau) \text{ such that } Im \subseteq N\}.$$

N is said to be τ-dense in M if $M/N \in \mathcal{T}_\tau$, that is, if $Cl_\tau^M(N) = M$. On the other hand, N is τ-closed in M if $M/N \in \mathcal{F}_\tau$, that is, $Cl_\tau^M(N) = N$.

A module M is said to be τ-noetherian (τ-artinian) provided that any nonempty set of τ-closed submodules posses a maximal (minimal) element. A ring R is τ-noetherian if it is as an R-module.

Further information on torsion theories can be found in [1, 10] or [4].

2. Attached Primes and Secondary Representations

The aim of this section is the study of relative secondary decompositions. We will prove that over any ring, every τ-artinian module admits a relative secondary decomposition, or equivalently, a minimal relative secondary decomposition. In addition, these minimal decompositions have some sort of unicity that involves the set of relative attached prime ideals to the module.

Definition 2.1. For any module M we define the set of τ-attached primes to M as

$$\tau\text{-}Att(M) = \{J \in Spec(R);\ J = \mathrm{Ann}_R(M/N);\ N \lneq M\ \tau\text{-closed}\}.$$

Proposition 2.2. *Any maximal element of the set*

$$\mathcal{A} = \left\{ \mathrm{Ann}_R \left(\frac{M}{N} \right) ; \ 0 \neq \frac{M}{N} \in \mathcal{F}_\tau \right\}$$

is τ-attached to M.

Proof. If $J = \mathrm{Ann}(M/N)$ is any maximal ideal of $\mathcal{A}$ and $ab \in J$, $a \notin J$, then $N \subsetneq N_1 = N + aM$. Since $M/Cl_\tau^M(N_1)$ is torsion free and $J \subseteq \mathrm{Ann}\left(M/Cl_\tau^M(N_1)\right)$, if $Cl_\tau^M(N_1) \neq M$ then $J = \mathrm{Ann}\left(M/Cl_\tau^M(N_1)\right)$ by the maximality of J in $\mathcal{A}$. Therefore, from the chain

$$J \subseteq \mathrm{Ann}(M/N_1) \subseteq \mathrm{Ann}\left(M/Cl_\tau^M(N_1)\right)$$

we immediately get that $J = \mathrm{Ann}(M/N_1)$. But then $a \in \mathrm{Ann}(M/N_1)$ implies $a \in J$, a contradiction.

Thus we get that $Cl_\tau^M(N_1) = M$, and then, for any $m \in M$ there exists $I \in \mathcal{L}(\tau)$ such that $Im \subseteq N_1$ and so that $bIm \subseteq bN_1 = N$, that is, $bm \in Cl^M(N) = N$ for all $m \in M$. This means that $b \in J$ and so that J is prime. ∎

Now, every ideal in the set $\mathcal{A} = \{\mathrm{Ann}(M/N);\ N \lneq M \ \tau\text{-closed}\}$ is τ-closed in R, so if the ring is τ-noetherian then $\mathcal{A}$ has a maximal element which is τ-attached to M by the above proposition. Thus we have the following.

Corollary 2.3. *If R is τ-noetherian then τ-$Att(M) = \emptyset$ if and only if M is a torsion module.*

Definition 2.4. A nonzero torsion free module is said to be τ-coprime if its annihilator coincides with the annihilator of any of its nonzero torsion free quotients.

It is clear from the definition (and Proposition 2.2) that if M is τ-coprime then τ-$Att_R(M) = \{\mathrm{Ann}(M)\}$.

Proposition 2.5. *If R is τ-noetherian and $M \notin \mathcal{T}_\tau$, then M has a nonzero τ-coprime quotient.*

Proof. The set $\mathcal{A} = \{\mathrm{Ann}(M/N);\ N \lneq M \ \tau\text{-closed}\}$ is nonempty since $M \notin \mathcal{T}_\tau$ (and so τ-$Att_R(M) \neq \emptyset$). Thus, since every element of $\mathcal{A}$ is a τ-closed ideal of R, $\mathcal{A}$ has a maximal element $\mathrm{Ann}(M/N)$. If M/N is not τ-coprime we find a τ-closed submodule L/N (so $\mathrm{Ann} M/L \in \mathcal{A}$) such that $\mathrm{Ann}(M/N) \subsetneq \mathrm{Ann}(L/N)$, a contradiction. ∎

Definition 2.6. We say that an R-module M is τ-secondary provided that $M \notin \mathcal{T}_\tau$ and that, for any $r \in R$, the endomorphism $r\cdot$ has a torsion cokernel or $r^n M \in \mathcal{T}_\tau$ for some $n \geq 1$.

It is clear that last two conditions on $r\cdot$ cannot occur at the same time on a τ-secondary module since $M/rM \in \mathcal{T}_\tau$ implies $M/r^nM \in \mathcal{T}_\tau$ for all $n \geq 1$, so if $r^nM \in \mathcal{T}_\tau$, the exact sequence

$$0 \to r^nM \to M \to \frac{M}{r^nM} \to 0$$

forces M to be torsion, so M would not be τ-secondary.

We then see that M being a τ-secondary module means that it makes the ring to partition in two sets: $\mathrm{Rad}\,(\mathrm{Ann}\,(M/\tau(M)))$, and the set of all those r with rM τ-dense in M. Moreover, it is easy to prove that if M is τ-secondary, then $\mathrm{Rad}\,(\mathrm{Ann}\,(M/\tau(M)))$ is actually a prime ideal. For if we let $a^nb^nM \subseteq \tau(M)$ but $b^mM \nsubseteq \tau(M)$ for every m (so $Cl_\tau^M(bM) = M$ since M is τ-secondary), we get

$$a^nM = a^nCl_\tau^M(b^nM) \subseteq Cl_\tau^M(a^nb^nM) \subseteq Cl_\tau^M(\tau(M)) = \tau(M),$$

that is, $a \in \mathrm{Rad}\,(\mathrm{Ann}\,(M/\tau(M)))$.

Definition 2.7. We shall indicate that $\mathfrak{p} = \mathrm{Rad}\,(\mathrm{Ann}\,(M/\tau(M)))$ by saying that M is $(\tau, \mathfrak{p})$-secondary.

We now prove some properties of the class of all $(\tau, \mathfrak{p})$-secondary modules.

Proposition 2.8. *Let R be any ring. The following statements hold.*

(i) *Any finite direct sum of $(\tau, \mathfrak{p})$-secondary modules is $(\tau, \mathfrak{p})$-secondary.*

(ii) *If M is $(\tau, \mathfrak{p})$-secondary and $K \leq M$ is such that $M/K \notin \mathcal{T}_\tau$, then M/K is $(\tau, \mathfrak{p})$-secondary.*

(iii) *If M is a $(\tau, \mathfrak{p})$-secondary module then $\mathrm{Ann}\,(M/\tau(M))$ is a $\mathfrak{p}$-primary ideal.*

(iv) *If K is any torsion submodule of M and M/K $(\tau, \mathfrak{p})$-secondary, then M is (τ, p)-secondary.*

(v) *If M_i, $i = 1, \ldots, n$ are $(\tau, \mathfrak{p})$-secondary submodules of a given module, then $M_1 + \cdots + M_n$ is also $(\tau, \mathfrak{p})$-secondary.*

(vi) *If $0 \to K \to M \to T \to 0$ is exact with K is $(\tau, \mathfrak{p})$-secondary and $T \in \mathcal{T}_\tau$, then M is $(\tau, \mathfrak{p})$-secondary.*

Proof. (i) Let A and B two $(\tau, \mathfrak{p})$-secondary modules, call $M = A \oplus B$ and suppose there exists $r \in R$ such that $M/rM \notin \mathcal{T}_\tau$. Let us prove that $r^nM \in \mathcal{T}_\tau$ for some $n \geq 1$.

$M/rM \notin \mathcal{T}_\tau$ means $Cl_\tau^M(rM) \neq M$, so either $Cl_\tau^A(rA) \neq A$ or $Cl_\tau^B(rB) \neq B$ since $Cl_\tau^{A\oplus B}(r(A \oplus B)) = Cl_\tau^A(rA) \oplus Cl_\tau^B(rB)$.

If $Cl_{\tau}^{A}(rA) \neq A$ we have that $r^n A \in \mathcal{T}_\tau$ for some $n \geq 1$ since A is τ-secondary. Thus, $r \in \mathrm{Rad}(\mathrm{Ann}(A/\tau(A))) = \mathfrak{p}$, and then there exists $k \geq 1$ such that $r^k B \in \mathcal{T}_\tau$. Therefore, letting $m = \max\{n, k\}$ we have that $r^m M \in \mathcal{T}_\tau$.

To see that $\mathfrak{p} = \mathrm{Rad}(\mathrm{Ann}(M/\tau(M)))$ take any $x \in \mathrm{Rad}(\mathrm{Ann}(M/\tau(M)))$ and any $k \geq 1$ with $x^k M \in \mathcal{T}_\tau$. Then $x^k A \oplus x^k B \in \mathcal{T}_\tau$ so clearly $x^k A \in \mathcal{T}_\tau$ and then $x \in \mathfrak{p}$. The converse is clear.

The general case is a trivial extension of the case $n = 2$ using induction.

(ii) For any $r \in R$ we have that either $\dfrac{M}{rM} \in \mathcal{T}_\tau$ or $r^n M \in \mathcal{T}_\tau$ for some $n \geq 1$.

If $\dfrac{M}{rM} \in \mathcal{T}_\tau$ then also $\dfrac{M}{rM + K} \in \mathcal{T}_\tau$, but

$$\frac{M}{rM + K} \cong \frac{M/K}{(rM + K)/K} = \frac{M/K}{r \cdot \ M/K},$$

so $\dfrac{M/K}{r \cdot \ M/K} \in \mathcal{T}_\tau$.

On the other hand, we have an epimorphism $r^n M \to r^n \cdot \dfrac{M}{K}$, so if $r^n M \in \mathcal{T}_\tau$ then also $r^n \dfrac{M}{K} \in \mathcal{T}_\tau$. This also shows that $\mathfrak{p} \subseteq \mathrm{Rad}\left(\mathrm{Ann}\left(\dfrac{M/K}{\tau(M/K)}\right)\right)$.

Finally, if $x \in \mathrm{Rad}\left(\mathrm{Ann}\left(\dfrac{M/K}{\tau(M/K)}\right)\right)$ then there exists $n \geq 1$ such that $x^n \cdot \dfrac{M}{K} \in \mathcal{T}_\tau$ and therefore $\dfrac{M/K}{x \cdot \ M/K}$ cannot be a torsion module.

Thus, M/xM cannot be a torsion module either since otherwise $\dfrac{M/K}{x \cdot \ M/K}$ would be torsion by the above.

But M is τ-secondary, so necessarily $x^n M \in \mathcal{T}_\tau$ for some n, that is, $x \in \mathfrak{p}$.

(iii) Clear.

(iv) Let $r \in R$ be any element. If $r^n \dfrac{M}{K} \in \mathcal{T}_\tau$ then $\dfrac{r^n M}{r^n M \cap K} \in \mathcal{T}_\tau$. But $r^n M \cap K \in \mathcal{T}_\tau$ since K is torsion, so necessarily $r^n M \in \mathcal{T}_\tau$.

If, on the other hand, the torsion module is $\dfrac{M/K}{r \ M/K} \cong \dfrac{M}{rM + K}$, since $\dfrac{rM + K}{rM}$ is torsion because K is, from the exact sequence

$$0 \to \frac{rM + K}{rM} \to \frac{M}{rM} \to \frac{M}{rM + K} \to 0$$

we immediately see that $M/rM \in \mathcal{T}_\tau$.

Therefore M is a τ-secondary module.

Finally we have

$$\mathrm{Rad}\left(\mathrm{Ann}\left(\frac{M/K}{\tau(M/K)}\right)\right) = \mathrm{Rad}\left(\mathrm{Ann}\left(\frac{M/K}{\tau(M)/K}\right)\right) = \mathrm{Rad}\left(\mathrm{Ann}\left(\frac{M}{\tau(M)}\right)\right).$$

(v) Call $V = \oplus_i M_i,\ S = \sum_i M_i$ and $K = \ker(V \to S)$, so $S \cong V/K$.

By (i) V is $(\tau, \mathfrak{p})$-secondary, and by ii) so is $V/Cl_\tau^V(K)$. But $Cl_\tau^V(K)/K$ is a torsion module and $\dfrac{V/K}{Cl_\tau^V(K)/K} \cong \dfrac{V}{Cl_\tau^V(K)}$, so V/K (that is, S) is $(\tau, \mathfrak{p})$-secondary by iv).

(vi) Given any $r \in R$ either $K/rK \in \mathcal{T}_\tau$ or $r^n K \in \mathcal{T}_\tau$ for some $n \geq 1$.

If $K/rK \in \mathcal{T}_\tau$ then $K/(K \cap rM) \in \mathcal{T}_\tau$ since $rK \leq K \cap rM$, so $(K+rM)/rM$ is a torsion module. But $T \in \mathcal{T}_\tau$ means $\dfrac{M/K}{(K+rM)/K} \in \mathcal{T}_\tau$, that is, $M/(K+rM) \in \mathcal{T}_\tau$, so from the exactness of the sequence

$$0 \to \frac{K+rM}{rM} \to \frac{M}{rM} \to \frac{M}{K+rM} \to 0$$

we get that $M/rM \in \mathcal{T}_\tau$.

On the other hand, $M/K \in \mathcal{T}_\tau$ implies $r^n M/r^n K \in \mathcal{T}_\tau$ for every $n \geq 1$ since we have an epimorphism

$$\begin{array}{ccc} \dfrac{M}{K} & \to & \dfrac{r^n M}{r^n K} \\ m + K & \mapsto & r^n m + r^n K \end{array}$$

Therefore, if $r^n K \in \mathcal{T}_\tau$ necessarily $r^n M \in \mathcal{T}_\tau$ since the sequence

$$0 \to r^n K \to r^n M \to \frac{r^n M}{r^n K} \to 0$$

is exact. ■

Definition 2.9. When a module M can be written as $M = Cl_\tau^M\left(\sum_{i=1}^n N_i\right)$ where each N_i is a $(\tau, \mathfrak{p}_i)$-secondary module, we say that $Cl_\tau^M\left(\sum_{i=1}^n N_i\right)$ is a τ-secondary representation of M. If this representation is such that all prime ideals $\mathfrak{p}_i$ are different, it will be called a minimal τ-secondary representation.

A module M is said to be τ-representable if it has a minimal τ-secondary representation.

By Proposition 2.8 we see that a module is τ-representable if and only if it has a τ-secondary representation.

Thus, associated to each τ-representable module there is a family of distinct prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Our next goal will be to prove that any τ-artinian module is τ-representable, and that the last family of prime ideals does not depend on the choice of the minimal τ-secondary representation of M.

We introduce some notation.

Definition 2.10. Given any ideal $I \leq R$ and any R-module M, we write $M(I)$ to denote the set of all $m \in M$ annihilated by I. Thus, if $a \in R$ is any element, $M(a)$ is the kernel of the map $M \xrightarrow{a\cdot} M$. Similarly, if $b \in R$ is any other element then $M(a, b)$ is the kernel of $M(a) \xrightarrow{b\cdot} M(a)$.

Proposition 2.11. *For any ring R, any τ-artinian R-module M with the following two properties is τ-secondary.*

(1) $M \notin \mathcal{T}_\tau$.

(2) If $N_1, N_2 \leq M$ are such that $M/N_1,\ M/N_2 \notin \mathcal{T}_\tau$ then $M/(N_1 + N_2) \notin \mathcal{T}_\tau$.

Proof. Suppose there exists $r \in R$ such that $r^n M \notin \mathcal{T}_\tau\ \forall\ n$ and that $M/rM \notin \mathcal{T}_\tau$.

M is τ-artinian so there is k such that $Cl_\tau^M(r^k M) = Cl_\tau^M(r^{k+n} M)\ \forall\ n$. If we call $N_1 = M(r^k)$ and $N_2 = r^k M$ we have $M/N_1 \cong r^k M \notin \mathcal{T}_\tau$ and $M/N_2 \notin \mathcal{T}_\tau$ by our hypotheses.

However $Cl_\tau^M(r^k M) = Cl_\tau^M(r^{2k} M)$, so for any $m \in M$ we find $I \in \mathcal{L}(\tau)$ such that $I r^k m \subseteq r^{2k} M$, that is, for any $y \in I$ we have $r^k(ym - r^k m_y) = 0$. This means that $ym - r^k m_y \in N_1$ and so that $ym = r^k m_y + n_y$ with $n_y \in N_1$. Thus $m \in Cl_\tau^M(N_1 + N_2)$ and then $M/(N_1 + N_2)$ is torsion, contradicting 2). ∎

Theorem 2.12. *Every τ-artinian module M over any ring is τ-representable.*

Proof. Suppose this is not the case and consider the set $\mathcal{A}$ of all not torsion τ-closed and not τ-representable submodules of M. $\mathcal{A}$ is not empty since $M \in \mathcal{A}$, so $\mathcal{A}$ has a minimal element, say N. But N τ-closed and not τ-representable implies N is not τ-secondary, and since N is τ-artinian, condition 2) of Proposition 2.11 must fail.

Thus, there are two submodules $N_1,\ N_2 \leq N$ such that $N/N_i \notin \mathcal{T}_\tau$ but $N/(N_1 + N_2) \in \mathcal{T}_\tau$. Then we have

$$N = Cl_\tau^N(N_1 + N_2) \subseteq Cl_\tau^N\left(Cl_\tau^N(N_1) + Cl_\tau^N(N_2)\right),$$

so $N = Cl_\tau^N\left(Cl_\tau^N(N_1) + Cl_\tau^N(N_2)\right)$.

If $Cl_\tau^N(N_i) \notin \mathcal{T}_\tau$ then $Cl_\tau^N(N_i)$ must be τ-representable by the minimality of N in $\mathcal{A}$, and then it is easy to check that $Cl_\tau^N(N_1) = Cl_\tau^N\left(\sum A_i\right)$, being each A_i a τ-secondary module. Thus, if both $Cl_\tau^N(N_i),\ i = 1, 2$ are torsion we have

$$N = Cl^N\left(Cl_\tau^N\left(\sum A_i\right) + Cl_\tau^N\left(\sum B_i\right)\right) = Cl_\tau^N\left(\sum A_i + \sum B_i\right),$$

that is, N τ-representable, a contradiction.

Therefore we see some of the $Cl_\tau^N(N_i)$ must be a torsion module, so some of the N_i is torsion.

The two of them cannot be torsion because in that case N_1+N_2 would be torsion too, which, in addition to the fact that $N/(N_1+N_2)$ is torsion, would force N to be torsion, a contradiction.

Thus we can suppose N_1 to be torsion but not N_2. Then

$$N = Cl_\tau^N\left(Cl_\tau^N(0)+Cl_\tau^N(N_2)\right) = Cl_\tau^N\left(Cl_\tau^N(N_2)\right) = Cl_\tau^N(N_2),$$

that is, N/N_2 is torsion, a contradiction. ■

Theorem 2.13. *Let R be any τ-noetherian ring, M any τ-representable R-module, $\mathcal{A}=\{\mathfrak{p}_1,\ldots,\mathfrak{p}_n\}$ the set of associated prime ideals to a minimal τ-secondary representation of M and $\mathcal{B}=\{\mathrm{Ann}(M/N);\ M/N$ τ-coprime$\}$. Then $\mathcal{A}=\mathcal{B}=\tau$-$Att_R(M)$.*

Proof. $\mathcal{B}\subseteq\tau$-$Att_R(M)$ is clear.

If $\mathfrak{p}\in\tau$-$Att_R(M)$ then $\mathfrak{p}=\mathrm{Ann}(M/N)$ for some nonzero torsion free quotient M/N, but since $\mathfrak{p}$ is actually prime, the equality $\mathfrak{p}=\mathrm{Rad}\left(\mathrm{Ann}(M/N)\right)$ holds.

Let $M=Cl_\tau^M\left(\sum_{i=1}^n N_i\right)$ be a minimal τ-secondary representation of M (so each N_i is $(\tau,\mathfrak{p}_i)$-secondary). Then $Cl_\tau^M\left(\sum_{i=1}^n N_i\right)/N$ is torsion free, and then so is the submodule $\left(\sum_{i=1}^n N_i+N\right)/N$. If we order the N_i 's so that $N_i\nsubseteq N$ $i=1,\ldots,r$ and $N_i\subseteq N,\ i=r+1,\ldots,n$, we have

$$\frac{\displaystyle\sum_{i=1}^n N_i+N}{N}\cong\sum_{i=1}^r\frac{N_i}{N_i\cap N}.$$

Thus, the quotient $\dfrac{M/N}{\left(\sum_{i=1}^r N_i\right)/(N_i\cap N)}$ is torsion since

$$\frac{Cl_\tau^M\left(\displaystyle\sum_{i=1}^n N_i\right)}{\displaystyle\sum_{i=1}^n N_i+N}\cong\frac{M/N}{\left(\displaystyle\sum_{i=1}^r N_i\right)/\left(N_i\cap N\right)}.$$

Therefore we get $M/N=Cl_\tau^{M/N}\left(\sum_{i=1}^r\left(N_i/(N_i\cap N)\right)\right)$ and so that the module $Cl_\tau^{M/N}\left(\sum_{i=1}^r\left(N_i/(N_i\cap N)\right)\right)$ is torsion free. But it is not hard to prove that if the closure of a module is torsion free then its annihilator coincides with the annihilator of the module itself. Therefore we have

$$\mathfrak{p} = \operatorname{Rad}\left(\operatorname{Ann}\left(\frac{M}{N}\right)\right) = \operatorname{Rad}\left(\operatorname{Ann}\left(\sum_{i=1}^{r}\frac{N_i}{N_i \cap N}\right)\right)$$
$$= \bigcap_{i=1}^{r} \operatorname{Rad}\left(\operatorname{Ann}\left(\frac{N_i}{N_i \cap N}\right)\right) = \bigcap_{i=1}^{r} \mathfrak{p}_i,$$

where the last equality holds because each $N_i/(N_i \cap N)$ is a torsion free quotient of a $(\tau, \mathfrak{p}_i)$-secondary module (see ii) of Proposition 2.8).

But $\mathfrak{p} = \cap\, \mathfrak{p}_i$ clearly implies that $\mathfrak{p}_j = \mathfrak{p}$ for some j, so $\tau\text{-}Att_R(M) \subseteq \mathcal{A}$.

Finally, let $\mathfrak{p} \in \mathcal{A}$, for instance, $\mathfrak{p} = \mathfrak{p}_j = \operatorname{Rad}(\operatorname{Ann}(N_j/\tau(N_j)))$.

If we call $L_j = Cl_\tau^M\left(\sum_{i \neq j} N_i\right)$ we clearly have $M = Cl_\tau^M(N_j + L_j)$, so the quotient M/L_j is torsion free. Then, the submodule

$$\frac{N_j + L_j}{L_j} \cong \frac{N_j}{N_j \cap L_j}$$

is also torsion free and so $(\tau, \mathfrak{p})$-secondary since N_j is $(\tau, \mathfrak{p})$-secondary. But then the exactness of the sequence

$$0 \to \frac{N_j + L_j}{L_j} \to \frac{M}{L_j} \to \frac{M}{N_j + L_j} \to 0$$

forces M/L_j to be $(\tau, \mathfrak{p})$-secondary by vi) of Proposition 2.8 since $M/(N_j + L_j) = Cl_\tau^M(N_j + L_j)/(N_j + L_j) \in \mathcal{T}_\tau$.

Now, M/L_j torsion free implies there exists a τ-coprime quotient M/N (Proposition 2.5), and then, again by Proposition 2.8, M/N is $(\tau, \mathfrak{p})$-secondary, that is, $\mathfrak{p} = \operatorname{Rad}(\operatorname{Ann}(M/N))$.

But M/N τ-coprime means that $\tau\text{-}Att_R(M/N) = \{\operatorname{Ann}(M/N)\}$, so we know $\operatorname{Ann}(M/N)$ is a prime ideal, and then of course $\operatorname{Rad}(\operatorname{Ann}(M/N)) = \operatorname{Ann}(M/N)$, so $\mathfrak{p} \in \mathcal{B}$. ■

The following is now immediate from Theorems 2.12 and 2.13

Corollary 2.14. *If R is τ-noetherian and M is τ-representable (so for instance if M is τ-artinian), then $\tau\text{-}Att_R(M)$ is finite.*

3. Coregular Sequences and the Width of a Module

In this section we introduce the concept of relative coregular M-sequences and relate them with the torsion of some Tor modules. This connection will end up in a very nice way to compute the relative width of a module, and gives a relation between the (relative) width of the modules of a short exact sequence.

Definition 3.1. If M is an R-module, a sequence $\{a_1, \ldots, a_n\} \subseteq R$ is said to be a weak τ-coregular M-sequence if the cokernel of each one of the maps

$$M(a_1, \ldots, a_i) \xrightarrow{a_{i+1}\cdot} M(a_1, \ldots, a_i), \ i = 1, \ldots, n-1,$$

is a torsion module. If in addition $M(a_1, \ldots, a_n) \notin \mathcal{T}_\tau$ then sequence is called a τ-coregular M-sequence.

An element $x \in R$ is (M, τ)-coregular if $\{x\}$ is a weak τ-coregular M-sequence.

Lemma 3.2. *Let $I \leq R$ be an ideal, M an R-module and $\{a_1, \ldots, a_n\} \subseteq I$ a weak τ-coregular M-sequence. If* $\operatorname{Tor}_i^R(M, R/I) \in \mathcal{T}_\tau$ $\forall i < n$ *then*

$$\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(M, R/I)\right) \cong \mathcal{Q}_\tau\left(\operatorname{Tor}_{i-j}^R(M(a_1, \ldots, a_j), R/I)\right) \ \forall j < i \leq n$$

(so $\operatorname{Tor}_{i-j}^R(M(a_1, \ldots, a_j), R/I) \in \mathcal{T}_\tau$ $\forall j < i < n-j-1$*).*

Proof. From the exact sequences

$$0 \to M(a_1) \longrightarrow M \xrightarrow{\cdot a_1} a_1 M \to 0$$

and

$$0 \to a_1 M \hookrightarrow M \to \frac{M}{a_1 M} \to 0$$

we get the long exact sequences

$$\operatorname{Tor}_i^R\left(M, \frac{R}{I}\right) \xrightarrow{a_1\cdot} \operatorname{Tor}_i^R\left(a_1 M, \frac{R}{I}\right) \to \operatorname{Tor}_{i-1}^R\left(M(a_1), \frac{R}{I}\right) \to \operatorname{Tor}_{i-1}^R\left(M, \frac{R}{I}\right)$$

and

$$\operatorname{Tor}_{i+1}^R\left(\frac{M}{a_1 M}, \frac{R}{I}\right) \to \operatorname{Tor}_i^R\left(a_1 M, \frac{R}{I}\right) \to \operatorname{Tor}_i^R\left(M, \frac{R}{I}\right) \to \operatorname{Tor}_i^R\left(\frac{M}{a_1 M}, \frac{R}{I}\right)$$

Since $a_1 \in I$ we have $\cdot a_1 = 0$ and then we get from the first long exact sequence that $\operatorname{Tor}_i^R(a_1 M, R/I)$ is a submodule of $\operatorname{Tor}_{i-1}^R(M(a_1), R/I)$ for all i, so $\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(a_1 M, R/I)\right) \leq \mathcal{Q}_\tau\left(\operatorname{Tor}_{i-1}^R(M(a_1), R/I)\right)$ for all i. But $\operatorname{Tor}_i^R(M, R/I)$ is torsion for all $i < n$ so again from the first exact sequence we get indeed that $\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(a_1 M, R/I)\right) \cong \mathcal{Q}_\tau\left(\operatorname{Tor}_{i-1}^R(M(a_1), R/I)\right)$ for all $i \leq n$.

Now, $M/a_1 M \in \mathcal{T}_\tau$ so $\operatorname{Tor}_i^R(M/a_1 M, R/I) \in \mathcal{T}_\tau$ $\forall i$, and then the second long exact sequence shows that $\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(a_1 M, R/I)\right) \cong \mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(M, R/I)\right)$ for all i.

Therefore $\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(M, R/I)\right) \cong \mathcal{Q}_\tau\left(\operatorname{Tor}_{i-1}^R(M(a_1), R/I)\right)$ for all $i \leq n$.

But then $\operatorname{Tor}_i^R(M(a_1), R/I) \in \mathcal{T}_\tau$ $\forall i < n-2$, and applying the previous argument, $\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(M(a_1), R/I)\right) \cong \mathcal{Q}_\tau\left(\operatorname{Tor}_{i-1}^R(M(a_1, a_2), R/I)\right)$ $\forall i \leq n-1$, that is, $\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(M, R/I)\right) \cong \mathcal{Q}_\tau\left(\operatorname{Tor}_{i-2}^R(M(a_1, a_2), R/I)\right)$ $\forall i \leq n$.

The result follows repeating this procedure. ■

The next definition is inspired by [9, Definition 2.4].

Definition 3.3. For any R-module M we call $\tau\text{-}W_R(M)$ the set of all non (M,τ)-coregular elements of R, that is, $\tau\text{-}W_R(M) = \{a \in R;\ Cl_\tau^M(aM) \neq M\}$.

Proposition 3.4. *If R is τ-noetherian then* $\tau\text{-}W_R(M) = \bigcup_{J \in \tau\text{-}Att_R(M)} J.$

Proof. We first see that

$$\tau\text{-}W_R(M) = \emptyset \Leftrightarrow M/rM \in \mathcal{T}_\tau\ \forall r \in R \Leftrightarrow \tau\text{-}Att_R(M) = \emptyset.$$

On the other hand, if $a \in \tau\text{-}W_R(M)$ then $M/aM \notin \mathcal{T}_\tau$, so by the last corollary there exists $J = \mathrm{Ann}(M/N)$ with $aM \leq N \lneq M$ $(M/N \in \mathcal{F}_\tau)$. But then $a \in J \in \tau\text{-}Att_R(M)$, that is, $a \in \bigcup_{J \in \tau\text{-}Att_R(M)} J$.

Conversely, if $a \in \mathrm{Ann}(M/N)$ with $M/N \in \mathcal{F}_\tau$, then $aM \subseteq N$ and then $Cl_\tau^M(aM) \subseteq Cl_\tau^M(N) \neq M$, so $a \in \tau\text{-}W_R(M)$. ■

Theorem 3.5. *Let I be an ideal of a τ-noetherian ring R and M a τ-representable module such that $M(I) \notin \mathcal{T}_\tau$. Then $\mathrm{Tor}_i^R(M, R/I)$ is a torsion module for all $i < n$ if and only if there exists a τ-coregular M-sequence $\{a_1, \ldots, a_n\} \subseteq I$.*

Moreover, if M is τ-artinian then the sequence $\{a_1, \ldots, a_n\}$ is maximal if and only if $\mathrm{Tor}_n^R(M, R/I) \notin \mathcal{T}_\tau$.

Proof. If there is no $a \in I$ with $Cl_\tau^N(aM) = M$ then $I \subseteq \tau\text{-}W_R(M) = \bigcup_{\tau\text{-}Att_R(M)} J$ and then I is contained in some $J = \mathrm{Ann}(M/N) \in \tau\text{-}Att_R(M)$ since $\tau\text{-}Att_R(M)$ is finite.

Now, $\mathrm{Tor}_0^R(M, R/I) \in \mathcal{T}_\tau$ by hypothesis, that is $Cl_\tau^M(IM) = M$, but also $Cl_\tau^M(IM) \subseteq Cl_\tau^M(JM) \subseteq Cl_\tau^M(N) = N$ $(M/N \in \mathcal{F}_\tau)$. Thus $M = N$, a contradiction.

Let then $a_1 \in I$ be such that $M/a_1M \in \mathcal{T}_\tau$. Since $\mathrm{Tor}_1^R(M, R/I) \in \mathcal{T}_\tau$ by hypothesis, $M(a_1)/IM(a_1) \in \mathcal{T}_\tau$ by Lemma 3.2. Thus, by the previous argument we find $a_2 \in I$ such that $Cl_\tau^{M(a_1)}(a_2M(a_1)) = M(a_1)$.

We can repeat this procedure and find a sequence $\{a_1, \ldots, a_n\} \subseteq I$ such that $Cl_\tau^M(a_1M) = M$ and $Cl_\tau^{M(a_1,\ldots,a_i)}(a_{i+1}M(a_1, \ldots, a_i)) = M(a_1, \ldots, a_i)$ $\forall i < n-1$.

But clearly $M(a_1, \ldots, a_n) \notin \mathcal{T}_\tau$ since $M(I) \notin \mathcal{T}_\tau$, so $\{a_1, \ldots, a_n\}$ is a τ-coregular M-sequence.

Conversely, if $a \in I$ is such that $Cl_\tau^M(aM) = M$ then

$$M = Cl_\tau^M(aM) \subseteq Cl_\tau^M(IM) \subseteq M$$

so $Cl_\tau^M(IM) = M$ and $\mathrm{Tor}_0^R(M, R/I) \in \mathcal{T}_\tau$.

If $\{a_1, \ldots, a_n\} \subseteq I$ is now a τ-coregular M-sequence, by induction hypothesis $\operatorname{Tor}_i^R(M, R/I) \in \mathcal{T}_\tau$ $\forall i < n-1$. But again $Cl_\tau^{M(a_1,\ldots,a_{n-1})}(a_n M(a_1, \ldots, a_{n-1})) = M(a_1, \ldots, a_{n-1})$ implies $Cl_\tau^{M(a_1,\ldots,a_{n-1})}(IM(a_1, \ldots, a_{n-1})) = M(a_1, \ldots, a_{n-1})$, so $\operatorname{Tor}_0^R(M(a_1, \ldots, a_{n-1}), R/I) \in \mathcal{T}_\tau$.

Now $\operatorname{Tor}_i^R(M, R/I) \in \mathcal{T}_\tau$ $\forall i < n-1$ implies by Lemma 3.2 that

$$\mathcal{Q}_\tau\left(\operatorname{Tor}_{n-1}^R(M, R/I)\right) \cong \mathcal{Q}_\tau\left(\operatorname{Tor}_0^R(M(a_1, \ldots, a_{n-1}), R/I)\right),$$

so $\operatorname{Tor}_{n-1}^R(M, R/I) \in \mathcal{T}_\tau$.

Finally, if $\operatorname{Tor}_n^R(M, R/I) \in \mathcal{T}_\tau$, Lemma 3.2 would say that $\dfrac{M(a_1, \ldots, a_n)}{IM(a_1, \ldots, a_n)} \in \mathcal{T}_\tau$ and then that $\{a_1, \ldots, a_n\}$ would not be maximal, for otherwise I would be contained in τ-$W_R(M(a_1, \ldots, a_n))$, and since $M(a_1, \ldots, a_n)$ is τ-artinian, $I \subseteq J = \operatorname{Ann}((M(a_1, \ldots, a_n)/N)$ ($N \lneq M(a_1, \ldots, a_n)$ with $M(a_1, \ldots, a_n)/N \in \mathcal{F}_\tau$). But then

$$M(a_1, \ldots, a_n) = Cl^{M(a_1,\ldots,a_n)}(IM(a_1, \ldots, a_n)) \subseteq$$
$$Cl_\tau^{M(a_1,\ldots,a_n)}(JM(a_1, \ldots, a_n)) \subseteq Cl_\tau^{M(a_1,\ldots,a_n)}(N) = N,$$

a contradiction.

Conversely, if $\{a_1, \ldots, a_n\}$ were not maximal, we could find $b \in I$ with $\dfrac{M(a_1, \ldots, a_n)}{bM(a_1, \ldots, a_n)} \in \mathcal{T}_\tau$, so also $\dfrac{M(a_1, \ldots, a_n)}{IM(a_1, \ldots, a_n)} \in \mathcal{T}_\tau$. But we know by Lemma 3.2 that $\mathcal{Q}_\tau\left(\operatorname{Tor}_n^R(M, R/I)\right) \cong \mathcal{Q}_\tau\left(M(a_1, \ldots, a_n) \otimes R/I\right)$ so $\operatorname{Tor}_n^R(M, R/I) \in \mathcal{T}_\tau$, a contradiction. ■

We are now able to give the announced way for computing the (relative) width of a module.

Definition 3.6. If $I \leq R$ is any ideal, τ-$Width_I(M)$ is defined as the length of the longest τ-coregular M-sequence contained in I or ∞ if such a sequence is infinite.

Corollary 3.7. *If R is τ-noetherian, M is τ-representable and $M(I) \notin \mathcal{T}_\tau$, then τ-$Width_I(M)$ may be computed as the minimum of the set*

$$\left\{n \geq 0;\ \operatorname{Tor}_n^R(M, R/I) \notin \mathcal{T}_\tau\right\}.$$

If furthermore M is τ-artinian then τ-$Width_I(M)$ is always finite.

Proof. The first assertion is clear.

If $|\tau\text{-}Width_I(M)| = \infty$ there exists an infinite τ-coregular M-sequence $\{a_1, a_2, \ldots\} \subseteq I$, and since M is τ-artinian, the sequence

$$Cl_\tau^M(M(a_1) \supseteq Cl_\tau^M(M(a_1, a_2) \supseteq \cdots$$

becomes constant after some n. But then

$$Cl_{\tau}^{M(a_1,\ldots,a_n)}\left(M(a_1,\ldots,a_{n+1})\right) = Cl_{\tau}^{M}\left(M(a_1,\ldots,a_{n+1})\right) \cap M(a_1,\ldots,a_n)$$
$$= Cl_{\tau}^{M}\left(M(a_1,\ldots,a_n)\right) \cap M(a_1,\ldots,a_n) = M(a_1,\ldots,a_n).$$

If we then choose any $m \in M(a_1,\ldots,a_n)$, we find $J \in \mathcal{L}(\tau)$ such that $Im \subseteq M(a_1,\ldots,a_{n+1})$, so $a_{n+1}IM = 0$ and then $I \subseteq \operatorname{Ann}(a_{n+1}m)$, that is, $\operatorname{Ann}(a_{n+1}m) \in \mathcal{L}(\tau)$. This means that $a_{n+1}m \in \tau\left(a_{n+1}M(a_1,\ldots,a_n)\right)$. Thus $a_{n+1}M(a_1,\ldots,a_n) \in \mathcal{T}_\tau$.

On the other hand $Cl_{\tau}^{M(a_1,\ldots,a_n)}\left(a_{n+1}M(a_1,\ldots,a_n)\right) = M(a_1,\ldots,a_n)$, that is, $\dfrac{M(a_1,\ldots,a_n)}{a_{n+1}M(a_1,\ldots,a_n)} \in \mathcal{T}_\tau$.

Therefore, from the exact sequence

$$0 \to a_{n+1}M(a_1,\ldots,a_n) \to M(a_1,\ldots,a_n) \to \frac{M(a_1,\ldots,a_n)}{a_{n+1}M(a_1,\ldots,a_n)} \to 0$$

we get that $M(a_1,\ldots,a_n) \in \mathcal{T}_\tau$, a contradiction ($M(I) \notin \mathcal{T}_\tau$). ■

Proposition 3.8. *Let R be a τ-noetherian ring, $I \leq R$ an ideal and M a τ-artinian R-module such that $M(I) \notin \mathcal{T}_\tau$. If there exists an (M,τ)-coregular element $x \in I$ then*

$$\tau\text{-}Width_I(M) = \tau\text{-}Width_I(M(x)) + 1.$$

Proof. Suppose $\tau\text{-}Width_I(M(x)) = m$. Then $\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(M(x),R/I)\right) = 0\ \forall i < m$ and $\mathcal{Q}_\tau\left(\operatorname{Tor}_m^R(M(x),R/I)\right) \neq 0$.

As in the proof of Lemma 3.2, since $M/xM \in \mathcal{T}_\tau$ we know

$$\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(M,R/I)\right) \cong \mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(xM,R/I)\right)\ \forall i,$$

and since $\operatorname{Tor}_i^R(M,R/I) \xrightarrow{x\cdot} \operatorname{Tor}_i^R(xM,R/I)$ is the zero map for all i, we see that $\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(xM,R/I)\right) = 0\ \forall i \leq m$. Therefore $\operatorname{Tor}_i^R(M,R/I) \in \mathcal{T}_\tau\ \forall i < m$.

Similarly, the sequence

$$0 \to \operatorname{Tor}_{m+1}^R\left(xM,\frac{R}{I}\right) \to \operatorname{Tor}_m^R\left(M(x),\frac{R}{I}\right) \to \operatorname{Tor}_m^R\left(M,\frac{R}{I}\right)$$

is exact and $\operatorname{Tor}_m^R(M,R/I) \in \mathcal{T}_\tau$, thus

$$\mathcal{Q}_\tau\left(\operatorname{Tor}_{m+1}^R\left(xM,\frac{R}{I}\right)\right) \cong \mathcal{Q}_\tau\left(\operatorname{Tor}_m^R\left(M(x),\frac{R}{I}\right)\right) \neq 0,$$

that is, $\operatorname{Tor}_{m+1}^R\left(M,R/I\right) \notin \mathcal{T}_\tau$.

Therefore $\tau\text{-}Width_I(M) = m + 1$. ■

Proposition 3.9. *Let R be τ-noetherian, $I \leq R$ an ideal and*

$$0 \to M' \to M \to M'' \to 0$$

an exact sequence of τ-artinian R-modules such that $M'(I),\ M(I),\ M''(I) \notin \mathcal{T}_\tau$. Then:

(*1*) $\tau\text{-}Width_I(M) < \tau\text{-}Width_I(M') \Rightarrow \tau\text{-}Width_I(M) = \tau\text{-}Width_I(M'')$.

(*2*) $\tau\text{-}Width_I(M') < \tau\text{-}Width_I(M) \Rightarrow \tau\text{-}Width_I(M'') = \tau\text{-}Width_I(M') + 1$.

(*3*) $\tau\text{-}Width_I(M) = \tau\text{-}Width_I(M') \Rightarrow \tau\text{-}Width_I(M) \leq \tau\text{-}Width_I(M'')$.

Proof. If $\tau\text{-}Width_I(M) = 0$ then obviously $\tau\text{-}Width_I(M) \leq \tau\text{-}Width_I(M'')$ (so 3) holds in this case).

Moreover, $M/xM \in \mathcal{T}_\tau\ \forall x \in I$ so $I \subseteq \tau\text{-}W_R(M)$, that is, $I \subseteq \tau\text{-}Att_R(M) \subseteq \tau\text{-}Att_R(M') \cup \tau\text{-}Att_R(M'')$. Therefore, if $\tau\text{-}Width_I(M') > \tau\text{-}Width_I(M)$ we have $I \subseteq \tau\text{-}Att_R(M'')$ and then $\tau\text{-}Att_R(M'') = 0$ (and 1) holds).

On the other hand, if $\tau\text{-}Width_I(M') = 0$ we only have to prove 2).

In this case $\tau\text{-}Width_I(M) \geq 1$ so $M \otimes R/I \in \mathcal{T}_\tau$ and then $M'' \otimes R/I \in \mathcal{T}_\tau$.

Now if $\mathrm{Tor}_1^R(M'', R/I) \in \mathcal{T}_\tau$ then $\ker(M' \otimes R/I \to M \otimes R/I) \in \mathcal{T}_\tau$ so the sequence

$$0 \to \mathcal{Q}_\tau\left(M' \otimes \frac{R}{I}\right) \to \mathcal{Q}_\tau\left(M \otimes \frac{R}{I}\right)$$

is exact. But $\mathcal{Q}_\tau(M' \otimes R/I) \neq 0$ by Theorem 3.5 and $\mathcal{Q}_\tau(M \otimes R/I) = 0$, a contradiction.

Thus $\mathrm{Tor}_1^R(M'', R/I) \notin \mathcal{T}_\tau$ and Theorem 3.5 says that $\tau\text{-}Width_I(M'') = 1$.

Suppose now that $\tau\text{-}Width_I(M) > 0,\ \tau\text{-}Width_I(M') > 0$. We then have $I \nsubseteq \tau\text{-}W_R(M) \cup \tau\text{-}W_R(M')$ and there exists an (M,τ)-coregular (so (M'',τ)-coregular) and (M',τ)-coregular element $x \in I$. Thus, the modules M/xM', M/xM and M''/xM'' are torsion, and from the snake's lemma we get the exact sequences

$$0 \to M'(x) \to M(x) \to V \to 0 \tag{1}$$

and

$$0 \to V \to M''(x) \to T \to 0$$

with $T \in \mathcal{T}_\tau$, which means that for all i, $\mathrm{Tor}_i^R(V, R/I) \in \mathcal{T}_\tau$ if and only if $\mathrm{Tor}_i^R(M''(x), R/I) \in \mathcal{T}_\tau$. In other words, $\tau\text{-}Width_I(V) = \tau\text{-}Width_I(M''(x))$.

Proposition 3.8 says that

$$\tau\text{-}Width_I(M(x)) = \tau\text{-}Width_I(M) - 1,$$

$$\tau\text{-}Width_I(M'(x)) = \tau\text{-}Width_I(M') - 1,$$

and

$$\tau\text{-}Width_I\left(M''(x)\right) = \tau\text{-}Width_I(M'') - 1.$$

Let us then call $\tau\text{-}Width_I(M'(x)) = n$ and $\tau\text{-}Width_I(M(x)) = m$ and, with the help of the long exact sequence of Tor_i^R associated to (1), let us compute $\tau\text{-}Width_I(V)$.

Case 1. $m < n$

$\mathrm{Tor}_i^R(M'(x), R/I) \in \mathcal{T}_\tau\ \forall i < n$, thus

$$\mathcal{Q}_\tau(\mathrm{Tor}_i^R(M(x), R/I)) \cong \mathcal{Q}_\tau(\mathrm{Tor}_i^R(V, R/I))\ \forall i < n.$$

Therefore, since $\mathrm{Tor}_i^R(M(x), R/I) \in \mathcal{T}_\tau\ \forall i < m$ we have $\mathrm{Tor}_i^R(V, R/I) \in \mathcal{T}_\tau\ \forall i < m$, and since $\mathrm{Tor}_m^R(M(x), R/I) \notin \mathcal{T}_\tau$ we have $\mathrm{Tor}_m^R(V, R/I) \notin \mathcal{T}_\tau$.

This means that $\tau\text{-}Width_I(V) = m = \tau\text{-}Width_I(M(x))$.

The same argument shows cases 2) and 3). ■

With the arguments above it is easy to prove the following.

Proposition 3.10. *Assume the conditions of Proposition 3.9. The following statements hold.*

(1) $\tau\text{-}Width_I(M) < \tau\text{-}Width_I(M'') \Rightarrow \tau\text{-}Width_I(M) = \tau\text{-}Width_I(M')$.

(2) $\tau\text{-}Width_I(M'') < \tau\text{-}Width_I(M) \Rightarrow \tau\text{-}Width_I(M'') = \tau\text{-}Width_I(M') + 1$.

(3) $\tau\text{-}Width_I(M) = \tau\text{-}Width_I(M'') \Rightarrow \tau\text{-}Width_I(M) \leq \tau\text{-}Width_I(M') + 1$.

(4) $\tau\text{-}Width_I(M') < \tau\text{-}Width_I(M'') \Rightarrow \tau\text{-}Width_I(M) \geq \tau\text{-}Width_I(M')$.

(5) $\tau\text{-}Width_I(M'') \leq \tau\text{-}Width_I(M') \Rightarrow \tau\text{-}Width_I(M) = \tau\text{-}Width_I(M'')$.

The concept $\tau\text{-}Width_I(M)$ may be thought of as a sort of a dual of the τ-*depth* of M in I, defined in [3] as the length of any maximal τ-regular M-sequence contained in I or ∞.

When M is a τ-finitely generated module over a τ-noetherian ring, its τ-*depth* can be computed as

$$\tau\text{-}depth_I(M) = min\{n \in I\!N;\ \mathcal{Q}_\tau\left(\mathrm{Ext}_R^n(R/I, M)\right) \neq 0\}.$$

The next result gives a relation between both concepts over semilocal rings.

If R is semilocal and Ω denotes the (finite) set of all maximal ideals, we let $E = \oplus_{\mathfrak{M}\in\Omega} E(R/\mathfrak{M})$.

Proposition 3.11. *Let R be a semi-local noetherian ring, $I \leq R$ any ideal, M a finitely generated R-module such that $M(I) \notin \mathcal{T}_\tau$, and τ a stable torsion theory in R-Mod. Then* $\tau\text{-}Width_I\left(\mathrm{Hom}(M, E)\right) = \tau\text{-}depth_I(M)$.

Proof. $\mathrm{Tor}_n^R(R/I, \mathrm{Hom}(M,E)) \cong \mathrm{Hom}\left(\mathrm{Ext}_R^n(R/I,M),E\right)$ for every $n \geq 0$ by [9, Corollary 1.5], and it is not hard to check that a finitely generated R-module M is torsion if and only if $\mathrm{Hom}(M,E)$ is. ■

4. Connections with Relative Local Homology

Throughout this section we assume that τ is a perfect torsion theory. We then have that for any $J \leq R$ and any module M, $J\mathcal{Q}_\tau(M) \cong J\left(M \otimes \mathcal{Q}_\tau(R)\right) = JM \otimes \mathcal{Q}_\tau(R) \cong \mathcal{Q}_\tau(JM)$.

For any module M we let ${}_\tau H_i^I(M) = \varprojlim \mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(M, R/I^n)\right)$. Our first goal in this section will be to find a connection in such a way that the τ-$Width_I$ of a module can be computed using these ${}_\tau H_i^I$'s in a very similar way as the projective or injective dimension using Ext.

We start with the following.

Proposition 4.1. $\cap_s I^s \ {}_\tau H_i^I(M) = 0 \ \forall i$.

Proof. $\cap_s I^s \ {}_\tau H_i^I(M) = \varprojlim_s I^s \left(\varprojlim_n \mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(M, R/I^n)\right)\right)$, which (up to an isomorphism) is a submodule of $\varprojlim_s \varprojlim_n I^s \mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(M, R/I^n)\right)$. But

$$\varprojlim_s \varprojlim_n I^s \mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(M, R/I^n)\right) \cong \varprojlim_n \varprojlim_s \mathcal{Q}_\tau\left(I^s \mathrm{Tor}_i^R(M, R/I^n)\right) = \varprojlim_s 0 = 0. \quad ■$$

Proposition 4.2. *If R is τ-noetherian and M is τ-representable, then there exists an (M,τ)-coregular element $x \in I$ if and only if ${}_\tau H_0^I(M) = 0$.*

Proof. Let $x \in I$ be such that $M = Cl_\tau^M(xM)$.

Since $x^{n-1}M = x^{n-1}Cl_\tau^M(xM) \subseteq Cl_\tau^M(x^nM)$, we get that $Cl_\tau^M(x^{n-1}M) \subseteq Cl_\tau^M(x^nM)$. But by the induction hypothesis $Cl_\tau^M(x^{n-1}M) = M$, so we have $M \subseteq Cl_\tau^M(x^nM) \subseteq M$ and then that $Cl_\tau^M(x^nM) = M$ for every $n \geq 1$.

Therefore $M = Cl_\tau^M(x^nM) \subseteq Cl_\tau^M(I^nM) \subseteq M$, that is, $Cl_\tau^M(I^nM) = M$, and this means $M \otimes R/I^n \in \mathcal{T}_\tau \ \forall n$.

Conversely, if $I \subseteq \tau$-$W_R(M)$ then $I \subseteq J = \mathrm{Ann}_R(M/N)$ with $M/N \in \mathcal{F}_\tau$ since M is τ-artinian. Thus $\mathcal{Q}_\tau(M/N) \neq 0$ and $I^nM \subseteq N$.

If $p_n : M \to M/I^nM$ is the canonical projection for any n, we get a compatible system of morphisms $\{\mathcal{Q}_\tau(p_n);\ n \geq 1\}$, so there exists a unique $h : \mathcal{Q}_\tau(M) \to \varprojlim \mathcal{Q}_\tau(M/I^nM)$ such that $f_ih = \mathcal{Q}_\tau(p_i) \ \forall i$ ($f_i : \varprojlim \mathcal{Q}_\tau(M/I^nM) \to \mathcal{Q}_\tau(M/I^iM)$ are the canonical homomorphisms).

But ${}_\tau H_0^I(M) = 0$ implies that $h = 0$ and then that $\mathcal{Q}_\tau(p_i) = 0 \ \forall i$, so if we call $p : M \to M/N$ and $\pi_n : M/I^nM \to M/N$ the canonical projections, we get

$$0 = \mathcal{Q}_\tau(\pi_n)\mathcal{Q}_\tau(p_n) = \mathcal{Q}_\tau(\pi_np_n) = \mathcal{Q}_\tau(p),$$

a contradiction since $\mathcal{Q}_\tau(p)$ is an epimorphism (τ is perfect) and $\mathcal{Q}_\tau(M/N) \neq 0$. ■

Recall that a module M is τ-finitely generated if there is a finitely generated submodule $N \leq M$ with $M/N \in \mathcal{T}_\tau$. We then prove the following.

Proposition 4.3. *Let R be τ-noetherian, M τ-artinian and N τ-finitely generated. Then $\mathrm{Tor}_n^R(N, M)$ is τ-artinian for every $n \geq 0$.*

Proof. We use induction on n.

Let $N' \leq N$ be finitely generated with $N/N' \in \mathcal{T}_\tau$. We then have an epimorphism $M^n \to N' \otimes M$, so we see $N' \otimes M$ is τ-artinian.

Now, since $N'/N \in \mathcal{T}_\tau$ we know that $\mathcal{Q}_\tau\left(\mathrm{Tor}_1^R(N/N', M)\right) = \mathcal{Q}_\tau(N/N' \otimes M) = 0$ and then that $\mathcal{Q}_\tau(N' \otimes M) \cong \mathcal{Q}_\tau(N \otimes M)$, so $\mathcal{Q}_\tau(N' \otimes M)$ being artinian implies that $\mathcal{Q}_\tau(N \otimes M)$ is artinian, and so that $N \otimes M$ is τ-artinian.

Suppose now $\mathrm{Tor}_{n-1}^R(L, M)$ is τ-artinian for every τ-finitely generated L.

$\mathrm{Tor}_i^R(N/N', M)$ is a torsion module for every i so $\mathcal{Q}_\tau\left(\mathrm{Tor}_n^R(N', M)\right) \cong \mathcal{Q}_\tau \left(\mathrm{Tor}_n^R(N, M)\right)$, that is, $\mathrm{Tor}_n^R(N, M)$ is τ-artinian if and only if $\mathrm{Tor}_n^R(N', M)$ is τ-artinian. But if $N' \cong R^n/L$ we know L is τ-finitely generated since R is τ-noetherian, so we are done since $\mathrm{Tor}_n^R(N', M) \cong \mathrm{Tor}_{n-1}^R(L, M)$. ■

Proposition 4.4. *Let R be a τ-noetherian ring, $I \leq R$ an ideal and M a τ-artinian module such that $M(I) \notin \mathcal{T}_\tau$. Then τ-$Width_I(M) = min\{n \in I\!N;\ {}_\tau H_n^I(M) \neq 0\}$.*

Proof. Let $n = \tau$-$Width_I(M)$. If $n = 0$ then ${}_\tau H_0^I(M) \neq 0$ by Proposition 4.2.

Let$\{x_1, \ldots, x_n\} \subseteq I$ now be a maximal τ-coregular M-sequence. By Proposition 3.8 we know that τ-$Width_I(M(x_1)) = n - 1$, so by the induction hypothesis we get that ${}_\tau H_i^I(M(x_1)) = 0\ \forall i < n-1$ and that ${}_\tau H_{n-1}^I(M(x_1)) \neq 0$.

Now, since τ is a perfect torsion theory we have an exact sequence

$$\cdots \to \mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(M, R/I^n)\right) \stackrel{\cdot x_1}{\to} \mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(x_1 M, R/I^n)\right) \to$$

$$\to \mathcal{Q}_\tau\left(\mathrm{Tor}_{i-1}^R(M(x_1), R/I^n)\right) \to \cdots,$$

and M τ-artinian implies that both $M(x_1)$ and $x_1 M$ are τ-artinian, so by Proposition 4.3 we know every $\mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(M, R/I^n)\right)$, $\mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(x_1 M, R/I^n)\right)$ and $\mathcal{Q}_\tau(\mathrm{Tor}_i^R$ $(M(x_1), R/I^n))$ is artinian and then Mittag-Leffler. Therefore, applying $\varprojlim$ we get the exact sequence

$$\cdots \to_\tau H_i^I(M(x_1)) \to_\tau H_i^I(M) \stackrel{\cdot x_1}{\to}_\tau H_i^I(x_1 M) \to_\tau H_{i-1}^I(M(x_1)) \to \cdots$$

Thus, ${}_\tau H_i^I(M(x_1)) = 0\ \forall i < n-1$ implies $x_1 \cdot_\tau H_i^I(M) =_\tau H^I(x_1 M)\ \forall i < n$.

Similarly, we have the exact sequence

$$\cdots \to \mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(x_1 M, R/I^n)\right) \to \mathcal{Q}_\tau\left(\mathrm{Tor}_i^R(M, R/I^n)\right) \to$$

$$\to \mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(M/x_1M, R/I^n))\right) \to \cdots$$

where $\mathcal{Q}_\tau\left(\operatorname{Tor}_i^R(M/x_1M, R/I^n))\right) = 0\ \forall i$ since $M/x_1M \in \mathcal{T}_\tau$ (x_1 is an (M,τ)-coregular element), so we see that ${}_\tau H_i^I(M) \cong_\tau H_i^I(x_1M)\ \forall i$ and then that ${}_\tau H_i^I(M) \cong x_1\ {}_\tau H_i^I(M)\ \forall i < n$. Therefore ${}_\tau H_i^I(M) \cong \cap x_1^t\ {}_\tau H_i^I(M) = 0$ (Proposition 4.1) for all $i < n$.

Now, since ${}_\tau H_{n-1}^I(M) = 0$, the sequence

$${}_\tau H_n^I(M) \to_\tau H_n^I(x_1M) \to_\tau H_{n-1}^I(M(x_1)) \to 0$$

is exact, and we know ${}_\tau H_{n-1}^I(M(x_1)) \neq 0$ so necessarily ${}_\tau H_n^I(x_1M) \neq 0$. But ${}_\tau H_n^I(x_1M) \cong_\tau H_n^I(M)$ so we are done.

Conversely, let $n = min\{n \in I\!N;\ {}_\tau H_n^I(M) \neq 0\}$. If $n = 0$ we immediately get τ-$Width_I(M) = 0$ by Proposition 4.2.

If $n > 0$ then ${}_\tau H_0^I(M) \neq 0$ so there exists an (M,τ)-coregular element $x_1 \in I$. Arguing as in the necessary part, we get that ${}_\tau H_i^I(M) \cong_\tau H_{i-1}^I(M(x_1))\ \forall i < n$, so we see that ${}_\tau H_i^I(M(x_1)) = 0\ \forall i < n-1$ and ${}_\tau H_{n-1}^I(M(x_1)) \neq 0$. Thus, by the induction hypothesis we find a maximal τ-coregular $M(x_1)$-sequence $\{x_2, \ldots, x_n\} \subseteq I$. It is then clear that $\{x_1, \ldots, x_n\}$ is a maximal τ-coregular M-sequence, so τ-$Width_I(M) = n$. ■

We now turn out to prove that the functors ${}_\tau H_i^I$ are indeed derived functors of the relative completion $\varprojlim \mathcal{Q}_\tau(M/I^nM)$. For we extend [2, Proposition 1.1] to this new setting involving torsion theories.

Given any ideal $I \leq R$ and a free resolution X^n of R/I^n, the canonical morphism $R/I^{n+1} \to R/I^n$ induces a morphism of complexes $f^{n+1} : X^{n+1} \to X^n$ in such a way that the family $\{X^n;\ n \geq 1\}$ is an inverse system of complexes.

If we consider the morphism of complexes

$$\begin{array}{rccc} \pi: & \prod_n \mathcal{Q}_\tau\left(X^n \otimes_R M\right) & \to & \prod_n \mathcal{Q}_\tau\left(X^n \otimes_R M\right) \\ & (\ldots, x^n, \ldots, x^1, x^0) & \mapsto & (\ldots, x^n - g^{n+1}(x^{n+1}), \ldots, x^0 - g^1(x^1)) \end{array}$$

where g^n is the induced morphism by f^n, we see that $\ker \pi = \varprojlim \mathcal{Q}_\tau(X^n \otimes_R M)$ and coker $\pi = \varprojlim^1 \mathcal{Q}_\tau(X^n \otimes_R M)$.

Thus, if we let $Mic\left(\mathcal{Q}_\tau(X^n \otimes_R M)\right) = Cone(-\pi)[1]$, the long exact sequence induced by the short exact sequence

$$0 \to \prod \mathcal{Q}_\tau(X^n \otimes_R M)[1] \to Mic\left(\mathcal{Q}_\tau(X^n \otimes_R M)\right) \to \prod \mathcal{Q}_\tau(X^n \otimes_R M) \to 0$$

gives, for any i, a short exact sequence

$$0 \to \varprojlim{}^1 \mathcal{Q}_\tau\left(\operatorname{Tor}_{i+1}^R\left(\frac{R}{I^n}, M\right)\right) \to H_i\left(Mic\left(\mathcal{Q}_\tau(X^n \otimes M)\right)\right) \to$$

$$(1) \qquad \to \varprojlim \mathcal{Q}_\tau \left(\operatorname{Tor}_i^R \left(\frac{R}{I^n}, M \right) \right) \to 0$$

(note that

$$\varprojlim{}^1 H_{i+1} \left(\mathcal{Q}_\tau \left(X^n \otimes_R M \right) \right) \cong \varprojlim{}^1 \mathcal{Q}_\tau \left(\operatorname{Tor}_{i+1}^R \left(\frac{R}{I^n}, M \right) \right),$$

$$\varprojlim H_i \left(\mathcal{Q}_\tau \left(X^n \otimes_R M \right) \right) \cong \varprojlim \mathcal{Q}_\tau \left(\operatorname{Tor}_i^R \left(\frac{R}{I^n}, M \right) \right)$$

since $\mathcal{Q}_\tau$ is an exact functor).

With the use of the last exact sequence we can prove the following.

Theorem 4.5. *If R is τ-noetherian and M is τ-artinian then*

$$H_i \left(\varprojlim \mathcal{Q}_\tau (M/I^n M) \right) \cong {}_\tau H_i^I(M) \ \forall i \geq 0.$$

Proof. By Proposition 4.3 $\mathcal{Q}_\tau \left(\operatorname{Tor}_i^R (R/I^n, M) \right)$ is artinian (and then Mittag-Leffler) so $\varprojlim{}^1 \mathcal{Q}_\tau \left(\operatorname{Tor}_i^R (R/I^n, M) \right) = 0 \ \forall i$. Then, by (1) we get that

$$H_i \left(Mic \left(\mathcal{Q}_\tau (X^n \otimes M) \right) \right) \cong {}_\tau H_i^I(M) \ \forall i \geq 0.$$

Now, clearly $\{H_* \left(Mic \left(\mathcal{Q}_\tau (X^n \otimes -) \right) \right)\}$ and $\left\{ H_* \left(\varprojlim \ \mathcal{Q}_\tau \left(R/I^n \otimes_R - \right) \right) \right\}$ are exact δ-functors, and if we choose any free module F we see that

$$H_i \left(Mic \left(\mathcal{Q}_\tau (X^n \otimes F) \right) \right) = 0 = H_i \left(\varprojlim \ \mathcal{Q}_\tau \left(R/I^n \otimes_R F \right) \right) \ \forall i \geq 1,$$

$$H_0 \left(Mic \left(\mathcal{Q}_\tau (X^n \otimes F) \right) \right) \cong H_0 \left(\varprojlim \ \mathcal{Q}_\tau \left(M/I^n M \right) \right).$$

Therefore, arguing as in [2, page 439] we have

$$H_i \left(Mic \left(\mathcal{Q}_\tau (X^n \otimes M) \right) \right) \cong H_i \left(\varprojlim \mathcal{Q}_\tau (M/I^n M) \right)$$

for every $i \geq 0$ and every τ-artinian module M. ■

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