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ASYMPTOTIC BEHAVIOR AND BLOW-UP OF SOLUTIONS FOR A NONLINEAR VISCOELASTIC WAVE EQUATION WITH BOUNDARY DISSIPATION

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Abstract. We study the nonlinear viscoelastic wave equation

$$u_{tt} - k_0\Delta u + \int_0^t g(t-s)\mathrm{div}\big[a(x)\nabla u(s)\big]ds + \big(k_1 + b(x)|u_t|^{m-2}\big)u_t = |u|^{p-2}u$$

with dissipative boundary conditions. Under some restrictions on the initial data and the relaxation function and without imposing any restrictive assumption on $a(x)$, we show that the rate of decay is similar to that of g. We also prove the blow-up results for certain solutions in two cases. In the case $k_1 = 0$, $m = 2$, we show that the solutions blow up in finite time under some restrictions on initial data and for arbitrary initial energy. In another case, $k_1 \geq 0$, $m \geq 2$, we prove a nonexistence result when the initial energy is less than potential well depth.

1. Introduction

In this article, we investigate the following initial value problem:

$$u_{tt} - k_0\Delta u + \int_0^t g(t-s)\mathrm{div}\big[a(x)\nabla u(s)\big]ds + \big(k_1 + b(x)|u_t|^{m-2}\big)u_t = |u|^{p-2}u \quad \text{in } \Omega\times[0,\infty), \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \Gamma_0\times(0,\infty), \tag{1.2}$$

$$k_0\frac{\partial u}{\partial \nu} - \int_0^t g(t-s)\big(a(x)\nabla u(s)\big)\,.\,\nu ds + h(u_t) = 0, \quad (x,t)\in\Gamma_1\times(0,\infty), \tag{1.3}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x\in\Omega, \tag{1.4}$$

where Ω is a bounded domain in $R^n(n \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1, \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, Γ_0 and Γ_1 are closed with positive measures, ν is the unit outward

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normal to $\partial\Omega$, $k_0 > 0, k_1 \geq 0, m \geq 2, p > 2$, g denotes the memory kernel and a, b, and h are real valued functions which satisfy appropriate conditions.

This problem arises in the study of motion of viscoelastic materials. We refer to [6, 22] for mathematical analysis on the motions of materials with memory.

The above problem with dirichlet boundary conditions has been considered by many authors. Cavalcanti et al. [3] considered,

$$(1.5)\qquad \begin{aligned} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + b(x)u_t + |u|^p u = 0, \quad \text{in } \Omega \times [0,\infty),\\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,\infty),\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{aligned}$$

for $b : \Omega \to R^+$ is a function which may vanish on any part of Ω. Assuming $b(x) \geq b_0 > 0$ on $\omega \subset \Omega$ and imposing geometry restrictions on the boundary, they established an exponential decay result for the energy function when g decays exponentially. In another work, Cavalcanti and Oquendo [4] considered a more general problem than (1.5) for the equation

$$(1.6)\qquad u_{tt} - k_0\Delta u + \int_0^t g(t-s)\mathrm{div}[a(x)\nabla u(s)]ds + b(x)h(u_t) + f(u) = 0.$$

Under the same conditions on g in [3] and for $a(x) + b(x) \geq \delta > 0$, they proved an exponential stability when g decays exponentially and h is linear and a polynomial stability when g decays polynomially and h is nonlinear. For the same kernel g and without considering the boundary geometric constraints, Messaoudi and Berrimi [2] extended these results to a nonlinear damping case in (1.5) by the use of the perturbed energy technique. In fact, they allowed b to vanish on any part of Ω (including Ω itself). Later, this last result improved by Liu [7], where a larger class of relaxation functions have been considered. When $a(x) = 1$ and $b(x) = 1$, Messaoudi [15] studied (1.6) for $f(u) = -|u|^{p-2}u$ and $h(u_t) = |u_t|^{m-2}u_t$ and proved a global existence result for $2 \leq p \leq m$ and a nonexistence result for $p > m \geq 2$. In this regard, see [16, 17, 23, 24] and references therein for more related studies in connecting with the existence, finite time blow-up and asymptotic properties of solutions for nonlinear wave equations.

In [8], Li and Zhao studied the problem

$$(1.7)\qquad \begin{aligned} u_{tt} &- k_0\Delta u + \int_0^t g(t-s)\mathrm{div}\big[a(x)\nabla u(s)\big]ds\\ &+ b(x)h(u_t) = 0, \quad (x,t) \in \Omega \times (0,\infty),\\ &u(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,\infty),\\ &-k_0\frac{\partial u}{\partial\nu} + \int_0^t g(t-s)\big(a(x)\nabla u(s)\big)\,.\nu ds = f(u),\ (x,t) \in \Gamma_1 \times (0,\infty),\\ &u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{aligned}$$

and proved exponential and polynomial decay results under weaker assumptions on g which improved [20]. In fact, in [20], the authors studied problem (1.7) with nonlinear boundary damping when $f(u) = |u|^\gamma u$ and $b(x) = 1$ on Ω. Assuming that the kernel g in the memory term decays exponentially, they showed exponential energy decay by using the perturbed energy method provided that $\|g\|_{L^1[0,\infty)}$ is small enough. In [9] Li et al. considered a related problem with nonlinear boundary dissipation (1.3). Under suitable conditions on the initial data and the relaxation function, they established existence and uniqueness of global solutions by means of Galerkin method and showed that the energy decays exponentially if the decay rate of the memory kernel is also exponential. Recently, these results have been improved by Wu and Chen [25] where the authors considered (1.1)-(1.4) with $k_1 = 0$ and $m = 2$. The authors used Lyapunov functions to establish general decay rate of solution energy which is not necessarily of exponential or polynomial type. However, conditions on initial data have not been given to ensure nonexistence results in these works. In this regard, we refer to a recent work by Ma and Geng [14] in which authors considered (1.7) with $b(x) = 1$ and $h(s) = s$ and showed the nonexistence of global solutions with arbitrary initial energy by exploiting the concavity arguments.

For more related studies about the boundary stabilization and blow-up results, we can refer to Cavalcanti et al. [5], Liu and Yu [10], Lu et al. [11] and Messaoudi and Soufyane [18].

Motivated by the above works, we study the problem (1.1)-(1.4). We first show that for a certain class of relaxation functions, the decay rates are similar to those of the relaxation function provided that the initial data are small enough. We note that, in the case $k_1 = 0$ and $m = 2$, our results are in the line with the ones obtained in [25]. The ingredients of our proof are based on an inequality (Lemma 3.1) given by Martinez [19]. In this way, the result is obtained without imposing any restrictive assumption on $a(x)$ (see $(A3)$ in [25]). Moreover, we allow $b(x)$ to vanish on any part of Ω (including Ω itself). We also prove the blow-up results for certain solutions in two cases: In the case $k_1 = 0$, $m = 2$, we show that the solutions blow up in finite time under some restrictions on initial data and for arbitrary initial energy. In another case, $k_1 \geq 0$, $m \geq 2$, we prove a nonexistence result when the initial energy is less than the mountain pass level value.

The outline of this paper is as follows. In section 2 we present some notations, assumptions and lemmas needed throughout our proofs. Section 3 is devoted to the establishment of uniform decay rates of solutions: Theorem 3.2. The blow-up results are given in section 4: Theorems 4.1 and 4.8.

2. Preliminaries

In this section, we present some notations and materials needed throughout the paper. First, we introduce

$$\|u\|_s = \|u\|_{L^s(\Omega)}, \qquad \|u\|_{s,\Gamma_1} = \|u\|_{L^s(\Gamma_1)}, \qquad 0 \le s \le \infty,$$

and the Hilbert space

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) \ : \ u = 0 \text{ on } \Gamma_0\},$$

endowed with the norm $\|\nabla u\|_2$. Now, we present the following hypotheses on problem (1.1)–(1.4).

(A1) $a, b : \Omega \to R$ are positive functions so that $a, b \in L^\infty(\Omega)$.

(A2) $h : R \to R$ is a nondecreasing function, such that for some positive constants α and β, satisfies

$$h(s)s \ge \alpha|s|^2, \qquad |h(s)| \le \beta|s|, \qquad \forall s \in R. \tag{2.1}$$

(A3) $g : [0, \infty) \to [0, \infty)$ is a non-increasing C^1 function such that

$$g(0) > 0, \qquad k_0 - \|a\|_\infty \int_0^{+\infty} g(s)ds = l > 0, \tag{2.2}$$

and there exists a non-increasing positive differentiable function ξ such that

$$g'(t) \le -\xi(t)g(t), \qquad \forall t \ge 0, \qquad \int_0^{+\infty} \xi(s)ds = \infty. \tag{2.3}$$

(A4) For nonlinear terms we assume

$$0 \le m, p \le \frac{2n}{n-2}, \quad \text{if} \ \ n > 2, \qquad m, p \ge 0 \quad \text{if} \ \ n = 1, 2. \tag{2.4}$$

In the sequel, we use the following Sobolev embedding

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Omega), \ 2 \le q \le \overline{q}, \ \text{where } \overline{q} = \begin{cases} 2n/(n-2) & \text{if } n \ge 3, \\ +\infty & \text{if } n = 1, 2, \end{cases} \tag{2.5}$$

with optimal embedding constant B, and the following trace Sobolev embedding

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^q(\Gamma_1), \ 2 \le q \le \overline{q}, \ \text{where } \overline{q} = \begin{cases} 2(n-1)/(n-2) & \text{if } n \ge 3, \\ +\infty & \text{if } n = 1, 2 \end{cases} \tag{2.6}$$

with the embedding constant B_1 (cf. [1]).

Next, we define the following functionals

$$I(t) := I(u(t)) = \int_\Omega \left(k_0 - a(x) \int_0^t g(s)ds \right) |\nabla u(t)|^2 dx + (g \circ \nabla u)(t) - \|u\|_p^p, \tag{2.7}$$

$$J(t) := J(u(t))$$

$$= \frac{1}{2}\int_\Omega\left(k_0 - a(x)\int_0^t g(s)ds\right)|\nabla u(t)|^2 dx + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|_p^p, \tag{2.8}$$

$$E(t) := E(u(t)) = \frac{1}{2}\|u_t\|_2^2 + J(t), \tag{2.9}$$

on $H^1_{\Gamma_0}(\Omega)$ where

$$(g \circ \nabla u)(t) = \int_\Omega\int_0^t g(t-s)a(x)|\nabla u(t) - \nabla u(s)|^2 ds dx.$$

Lemma 2.1. *$E(t)$ is a non-increasing function for $t \geq 0$ and*

$$\begin{aligned} E'(t) = &-\int_{\Gamma_1} u_t h(u_t) d\Gamma + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\int_\Omega a(x)|\nabla u(t)|^2 dx \\ &- \int_\Omega\left(k_1 + b(x)|u_t|^{m-2}\right)|u_t|^2 dx \leq 0. \end{aligned} \tag{2.10}$$

Proof. Multiplying (1.1) by u_t, integrating over Ω and using (1.2)-(1.4), we obtain (2.10). ■

Referring to [9] and [20], we state the following existence and uniqueness theorem.

Theorem 2.2. *If $(u_0, u_1) \in \left(H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)\right) \times H^1_{\Gamma_0}(\Omega)$, then the problem (1.1)-*(1.4) *has a unique solution satisfying*

$$u \in L^\infty\left([0,T); H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)\right),\ u_t \in L^\infty\left([0,T); H^1_{\Gamma_0}(\Omega)\right),\ u_{tt} \in L^\infty\left([0,T); L^2(\Omega)\right).$$

Moreover, if $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, then the problem (1.1)-(1.4) *has a weak solution satisfying*

$$u \in C\left([0,T); H^1_{\Gamma_0}(\Omega)\right), \qquad u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^m_b\left(\Omega \times [0,T)\right),$$

for some $T > 0$, where L^m_b is the weighted Lebesgue space.

Finally, we define:

$$d(t) = \inf_{u \in H^1_{\Gamma_0}(\Omega),\ u|_{\Gamma_0} \neq 0} \sup_{\lambda \geq 0} J(\lambda u).$$

Then, using (A1)-(A2), (2.2) and by the arguments in [12, 21] and [26], we can prove the following two lemmas.

Lemma 2.3. *For $t \geq 0$ we have*

$$0 < d_1 \leq d(t) \leq d_2(u) = \sup_{\lambda \geq 0} J(\lambda u)$$

where

$$d_1 = \left(\frac{p-2}{2p}\right)\left(\frac{l}{B^2}\right)^{\frac{p}{p-2}},$$

and

$$d_2(u) = \left(\frac{p-2}{2p}\right)\left(\frac{k_0\|\nabla u(t)\|_2^2 - \|\sqrt{a(x)}\nabla u(t)\|_2^2\int_0^t g(s)ds + (g\circ\nabla u)(t)}{\|u(t)\|_p^2}\right)^{\frac{p}{p-2}}.$$

Lemma 2.4. *Suppose that assumptions of Theorem* 2.2 *hold and* $E(0) < d_1$. *We have*

(i) *If* $I(0) > 0$, *then* $I(t) > 0$, $\forall t \in [0, T)$, *and the solution of* (1.1) – (1.4) *is bounded and global in time so that*

$$\|u_t(t)\|_2^2 + l\|\nabla u(t)\|_2^2 \le \frac{2p}{p-2}E(t) \le \frac{2p}{p-2}E(0). \tag{2.11}$$

(ii) *If* $I(0) < 0$, *then* $I(t) < 0$, $\forall t \in [0, T)$, *and*

$$\|\nabla u(t)\|_2^2 \ge \frac{2pd_1}{(p-2)l}. \tag{2.12}$$

3. Energy Decay

In this section we shall prove the energy decay of solutions of the problem (1.1)–(1.4). First, we present the following lemma by Martinez [19] which plays a critical role in our proof.

Lemma 3.1. *Let* $E : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-increasing function and* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a* C^2 *increasing function such that* $\psi(0) = 0$ *and* $\lim_{t\to+\infty}\psi(t) = +\infty$. *Assume that there exists* $c > 0$ *for which*

$$\int_t^{+\infty} \psi'(s)E(s)ds \le cE(t), \qquad \forall t \ge 0, \tag{3.1}$$

then

$$E(t) \le \lambda E(0)e^{-\omega\psi(t)}, \tag{3.2}$$

for some positive constants ω *and* λ.

Our main result reads as follows:

Theorem 3.2. *Assume that* (A1) – (A4) *hold. Let* $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *be given and satisfying*

$$I(0) > 0, \qquad E(0) < d_1. \tag{3.3}$$

Then, the solution of problem (1.1)–(1.4) *satisfies*

$$E(t) \leq KE(0)e^{-k\int_0^t \xi(s)ds}, \tag{3.4}$$

for some positive constants K and k. Proof. Multiplying (1.1) by $\xi(t)u(t)$ and integrating over $\Omega \times [t_1, t_2]$ $(0 \leq t_1 \leq t_2)$, we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \xi(t) \int_\Omega uu_{tt}dxdt + k_0 \int_{t_1}^{t_2} \xi(t) \int_\Omega |\nabla u(t)|^2 dxdt \\
&+ \int_{t_1}^{t_2} \xi(t) \int_{\Gamma_1} uh(u_t)d\Gamma dt + k_1 \int_{t_1}^{t_2} \xi(t) \int_\Omega uu_t dxdt \\
&+ \int_{t_1}^{t_2} \xi(t) \int_\Omega b(x)uu_t|u_t|^{m-2}dxdt \\
&- \int_{t_1}^{t_2} \xi(t) \int_\Omega \nabla u(t) \,. \int_0^t g(t-s)\big(a(x)\nabla u(s)\big)dsdxdt \\
&= \int_{t_1}^{t_2} \xi(t)\|u(t)\|_p^p dt.
\end{aligned} \tag{3.5}$$

For the last term in the left hand side of (3.5) we have

$$\begin{aligned}
&\int_\Omega \nabla u(t) \,. \int_0^t g(t-s)\big(a(x)\nabla u(s)\big)dsdx \\
&= \int_\Omega \nabla u(t) \,. \int_0^t g(t-s)a(x)\big(\nabla u(s) - \nabla u(t)\big)dsdx \\
&+ \int_0^t g(s)ds \int_\Omega a(x)|\nabla u(t)|^2 dx.
\end{aligned} \tag{3.6}$$

Using (3.6) and (2.7), the equality (3.5) takes the form

$$\begin{aligned}
&2\int_{t_1}^{t_2} \xi(t)\Big(E(t) - \frac{p-2}{2p}\|u\|_p^p\Big)dt \\
&= -\int_{t_1}^{t_2} \xi(t) \int_\Omega uu_{tt}dxdt - k_1 \int_{t_1}^{t_2} \xi(t) \int_\Omega uu_t dxdt - \int_{t_1}^{t_2} \xi(t) \int_{\Gamma_1} uh(u_t)d\Gamma dt \\
&- \int_{t_1}^{t_2} \xi(t) \int_\Omega b(x)uu_t|u_t|^{m-2}dxdt + \int_{t_1}^{t_2} \xi(t)\|u_t\|_2^2 dt + \int_{t_1}^{t_2} \xi(t)(g \circ \nabla u)(t)dt \\
&+ \int_{t_1}^{t_2} \xi(t) \int_\Omega \nabla u(t) \,. \int_0^t g(t-s)a(x)\big(\nabla u(s) - \nabla u(t)\big)dsdxdt.
\end{aligned} \tag{3.7}$$

Integrating by parts, for the first term in the right-hand side of (3.7), we have

$$\begin{aligned}
&- \int_{t_1}^{t_2} \xi(t) \int_\Omega uu_{tt}dxdt \\
&= -\int_\Omega \xi(t)uu_t dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \xi'(t) \int_\Omega uu_t dxdt + \int_{t_1}^{t_2} \xi(t)\|u_t\|_2^2 dt.
\end{aligned} \tag{3.8}$$

By Young's inequality, (2.5) and (2.11), we get

$$
\begin{aligned}
\left| -\int_\Omega \xi(t) u u_t dx \Big|_{t_1}^{t_2} \right| &\le \sum_{i=1}^{2} \left| \xi(t) \int_\Omega u u_t dx \right|_{t=t_i} \\
&\le \sum_{i=1}^{2} \left[\xi(t) \left(\frac{B^2}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 \right) \right]_{t=t_i} \\
&\le \sum_{i=1}^{2} \left[\left(\frac{p}{p-2} \right) \left(\frac{B^2}{l} + 1 \right) \xi(t) E(t) \right]_{t=t_i} \\
&\le \left(\frac{2p}{p-2} \right) \left(\frac{B^2 + l}{l} \right) \xi(0) E(t_1).
\end{aligned} \tag{3.9}
$$

Similarly,

$$
\begin{aligned}
\left| \int_{t_1}^{t_2} \xi'(t) \int_\Omega u u_t dx dt \right| &\le \int_{t_1}^{t_2} |\xi'(t)| \left(\frac{B^2}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 \right) \\
&\le -\left(\frac{p}{p-2} \right) \left(\frac{B^2 + l}{l} \right) \int_{t_1}^{t_2} \xi'(t) E(t) dt.
\end{aligned} \tag{3.10}
$$

For the last term in the right hand side of (3.8), we use (2.10) to get

$$
\int_{t_1}^{t_2} \xi(t) \|u_t\|_2^2 dt \le -\frac{1}{k_1} \int_{t_1}^{t_2} \xi(t) E'(t) dt. \tag{3.11}
$$

By using Young's inequality, (2.5), (2.11) and (3.11), for the second term in the right hand side of (3.7), we have

$$
\begin{aligned}
&\left| -\int_{t_1}^{t_2} \xi(t) \int_\Omega u u_t dx dt \right| \\
&\le \int_{t_1}^{t_2} \xi(t) \left(\frac{\delta B^2}{2} \|\nabla u\|_2^2 + \frac{1}{2\delta} \|u_t\|_2^2 \right) dt \\
&\le \frac{\delta}{l} \left(\frac{p}{p-2} \right) B^2 \int_{t_1}^{t_2} \xi(t) E(t) dt - \frac{1}{2\delta k_1} \int_{t_1}^{t_2} \xi(t) E'(t) dt.
\end{aligned} \tag{3.12}
$$

By (2.10) and the first inequality in (2.1), we have

$$
\|u_t\|_{2,\Gamma_1}^2 \le -\frac{1}{\alpha} E'(t). \tag{3.13}
$$

Taking the second inequality in (2.1) and (3.13) into account, using Young's inequality and the trace embedding (2.6), the third term in the right hand side of (3.7) can be estimated in the form

$$
\begin{aligned}
&\left|-\int_{t_1}^{t_2}\xi(t)\int_{\Gamma_1}uh(u_t)d\Gamma dt\right| \\
&\le \beta\int_{t_1}^{t_2}\xi(t)\left(\frac{\delta}{2}B_1^2\|\nabla u\|_2^2+\frac{1}{2\delta}\|u_t\|_{2,\Gamma_1}^2\right)dt \qquad (3.14)\\
&\le \frac{\delta}{l}\left(\frac{\beta p}{p-2}\right)B_1^2\int_{t_1}^{t_2}\xi(t)E(t)dt-\frac{\beta}{2\delta\alpha}\int_{t_1}^{t_2}\xi(t)E'(t)dt.
\end{aligned}
$$

For the fourth term we use (2.4) and (2.10) to obtain

$$
\begin{aligned}
&\left|-\int_{t_1}^{t_2}\xi(t)\int_{\Omega}b(x)uu_t|u_t|^{m-2}dxdt\right| \\
&\le \int_{t_1}^{t_2}\xi(t)\int_{\Omega}b(x)\left(\delta|u(t)|^m+c(\delta)|u_t(t)|^m\right)dxdt \\
&\le \delta\|b\|_\infty\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\int_{t_1}^{t_2}\xi(t)\|\nabla u\|_2^2dt \qquad (3.15)\\
&\quad+c(\delta)\int_{t_1}^{t_2}\xi(t)\int_{\Omega}b(x)|u_t|^mdxdt \\
&\le \frac{\delta}{l}\|b\|_\infty\left(\frac{2p}{p-2}\right)\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{m-2}{2}}\int_{t_1}^{t_2}\xi(t)E(t)dt \\
&\quad-c(\delta)\int_{t_1}^{t_2}\xi(t)E'(t)dt.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&\int_{\Omega}a(x)\nabla u(t)\,.\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)dsdx \\
&\le \delta\int_{\Omega}a(x)|\nabla u(t)|^2dx+\frac{1}{4\delta}\int_{\Omega}a(x)\bigg(\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)ds\bigg)^2dx \qquad (3.16)\\
&\le \delta\|a\|_\infty\|\nabla u\|_2^2+\frac{1}{4\delta}\int_0^t g(s)ds\int_{\Omega}\int_0^t g(t-s)a(x)|\nabla u(s)-\nabla u(t)|^2dsdx \\
&\le \frac{\delta}{l}\left(\frac{2p}{p-2}\right)\|a\|_\infty E(t)+\left(\frac{k_0-l}{4\delta\|a\|_\infty}\right)(g\circ\nabla u)(t).
\end{aligned}
$$

Combining (3.7)-(3.16) and using

$$\xi(t)(g\circ\nabla u)(t)\le-(g'\circ\nabla u)(t)\le-2E'(t),$$

we arrive at

$$
\begin{aligned}
&2\int_{t_1}^{t_2}\xi(t)\Big(E(t)-\frac{p-2}{2p}\|u\|_p^p\Big)dt\\
&\le \frac{\delta}{l}\Big(\frac{p}{p-2}\Big)\Big\{k_1B^2+\beta B_1^2+2\|a\|_\infty+2\|b\|_\infty\Big(\frac{2p}{l(p-2)}E(0)\Big)^{\frac{m-2}{2}}\Big\}\int_{t_1}^{t_2}\xi(t)E(t)dt\\
&\quad+\Big(\frac{2p}{p-2}\Big)\Big(\frac{B^2+l}{l}\Big)\xi(0)E(t_1)-\Big(\frac{p}{p-2}\Big)\Big(\frac{B^2+l}{l}\Big)\int_{t_1}^{t_2}\xi'(t)E(t)dt\\
&\quad-\Big(\frac{2}{k_1}+\frac{1}{2\delta}+\frac{\beta}{2\delta\alpha}+c(\delta)\Big)\int_{t_1}^{t_2}\xi(t)E'(t)dt-\Big(\frac{k_0-l}{2\delta\|a\|_\infty}+2\Big)\int_{t_1}^{t_2}E'(t)dt.\\
&\le \frac{\delta}{l}\Big(\frac{p}{p-2}\Big)\Big\{k_1B^2+\beta B_1^2+2\|a\|_\infty+2\|b\|_\infty\Big(\frac{2p}{l(p-2)}E(0)\Big)^{\frac{m-2}{2}}\Big\}\int_{t_1}^{t_2}\xi(t)E(t)dt\\
&\quad+\Big\{\Big[\Big(\frac{3p}{p-2}\Big)\Big(\frac{B^2+l}{l}\Big)+\frac{2}{k_1}+\frac{1}{2\delta}+\frac{\beta}{2\delta\alpha}+c(\delta)\Big]\xi(0)+\frac{k_0-l}{2\delta\|a\|_\infty}+2\Big\}E(t_1).
\end{aligned}
\tag{3.17}
$$

On the other hand, by the use of (2.4) and (2.11), we have

$$
\begin{aligned}
\frac{p-2}{2p}\int_{t_1}^{t_2}\xi(t)\|u(t)\|_p^p dt &\le B^p\Big(\frac{p-2}{2p}\Big)\Big(\frac{2p}{l(p-2)}E(0)\Big)^{\frac{p-2}{2}}\int_{t_1}^{t_2}\xi(t)\|\nabla u(t)\|_2^2 dt\\
&\le \Big(\frac{E(0)}{d_1}\Big)^{\frac{p-2}{2}}\int_{t_1}^{t_2}\xi(t)E(t)dt,
\end{aligned}
$$

which implies

$$
\int_{t_1}^{t_2}\xi(t)\Big(E(t)-\frac{p-2}{2p}\|u\|_p^p\Big)dt \ge \Big[1-\Big(\frac{E(0)}{d_1}\Big)^{\frac{p-2}{p}}\Big]\int_{t_1}^{t_2}\xi(t)E(t)dt. \tag{3.18}
$$

Using the fact that $E(0) < d_1$ and choosing δ sufficiently small, (3.17) and (3.18) imply that

$$
\int_{t_1}^{t_2}\xi(t)E(t)dt \le cE(t_1),
$$

for some $c > 0$. Letting t_2 go to infinity, assumptions of lemma 3.1 satisfy with $\psi(t) = \int_0^t \xi(s)ds$. Therefore, (3.4) is now established and the proof of Theorem 3.2 is complete. ■

4. Blow-up Property

In this section, we consider the blow-up properties for the solutions of (1.1)-(1.4) in two cases. First, we suppose that $k_1 = 0$, $m = 2$ and $h(s) = s$. We show that the solutions blow-up in a finite time T^* when the initial energy lies in different ranges.

Secondly, we will obtain a nonexistence result in the case that $k_1 \geq 0$ and $2 \leq m < p$ with positive initial energy less than potential well depth.

4.1. Blow-up with different ranges of initial energy: the case $k_1 = 0$, $m = 2$

Theorem 4.1. *Suppose that* (2.2) *hold,* $k_1 = 0$, $m = 2$, $h(s) = s$ *and*

$$a_1 = k_0(p-2) - (p-2+1/p)\|a\|_\infty \int_0^{+\infty} g(s)ds > 0. \tag{4.1}$$

Assume that $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *and that either one of the following conditions is satisfied:*

(1) $E(0) < 0$,

(2) $E(0) = 0$ *and* $\displaystyle\int_\Omega u_0 u_1 dx > 0$,

(3) $0 < E(0) < \dfrac{a_1 d_1}{l(p-2)}$ $\left(one\ can\ verify\ that\ \dfrac{a_1}{l(p-2)} < 1\right)$ *and* $I(u_0) < 0$,

(4) $$\frac{a_1 d_1}{l(p-2)} \leq E(0) < \min\left\{ \frac{\left(\int_\Omega u_0 u_1 dx\right)^2}{2\left[\|u_0\|_2^2 + T_1(\|u_0\|^2_{2,\Gamma_1} + \|\sqrt{b(x)}u_0\|_2^2)\right]}, \right.$$
$$\left. \frac{p+2}{2p}\left(r_1 \int_\Omega u_0 u_1 dx - \left(\|u_0\|_2^2 + \|u_0\|^2_{2,\Gamma_1} + \|\sqrt{b(x)}u_0\|_2^2\right)\right)\right\},$$

where $r_1 = 1 + \sqrt{p-2}/\sqrt{p+2}$. *Then, the solution* $u(t)$ *blows up at finite time* T^* *in the sense of*

$$\lim_{t \nearrow T^{*-}} \|\nabla u(t)\|_2^2 = +\infty.$$

To prove the above theorem, we will use the following lemmas.

Lemma 4.2. ([13]). *Let* $\delta > 0$ *and* $B(t) \in C^2(0, \infty)$ *be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0.$$

If

$$B'(0) > r_2 B(0) + K_0,$$

with $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$, *then* $B'(t) > K_0$ *for* $t > 0$, *where* K_0 *is a constant.*

Lemma 4.3. ([13]). *If* $M(t)$ *is a non-increasing function on* $[t_0, \infty)$, $t_0 \geq 0$, *and satisfies the differential inequality*

$$M'(t)^2 \geq \mu_1 + \mu_2 M(t)^{2+\frac{1}{\delta}}, \qquad t \geq t_0,$$

where $\mu_1 > 0, \mu_2 \in \mathbb{R}$. *Then, there exists a finite time* T^* *such that*

$$\lim_{t \longrightarrow T^{*-}} M(t) = 0,$$

and the upper bound of T^* *is estimated respectively by the following cases*:

(1) *If* $\mu_2 < 0$, *then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-\mu_2}} \ln \frac{\sqrt{-\mu_1/\mu_2}}{\sqrt{-\mu_1/\mu_2} - M(t_0)}.$$

(2) *If* $\mu_2 = 0$, *then*

$$T^* \leq t_0 + \frac{M(t_0)}{M'(t_0)}.$$

(3) *If* $\mu_2 < 0$, *then*

$$T^* \leq \frac{M(t_0)}{\sqrt{\mu_1}} \quad or \quad T^* \leq t_0 + 2^{(3\delta+1)/2\delta} \frac{\delta c}{\sqrt{\mu_1}} \left[1 - (1 + cM(t_0))^{-1/2\delta}\right],$$

where $c = (\mu_1/\mu_2)^{2+\frac{1}{\delta}}$.

Lemma 4.4. *Under the conditions of Theorem* 4.1, *for any solution* u *of (1.1)*-(1.4), *we have*

$$\begin{aligned} G''(t) &\geq (p+2)\|u_t\|_2^2 - 2pE(0) + 2p \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds \\ &+2p \int_0^t \|\sqrt{b(x)}u_t(s)\|_2^2 ds, \end{aligned} \tag{4.2}$$

where

$$G(t) = \|u(t)\|_2^2 + \int_0^t \|u(s)\|_{2,\Gamma_1}^2 ds + \int_0^t \|\sqrt{b(x)}u(s)\|_2^2 ds. \tag{4.3}$$

Proof. From (4.3) we have

$$G'(t) = 2\int_\Omega u(t)u_t(t)dx + \int_{\Gamma_1} |u(t)|^2 dx + \int_\Omega b(x)|u(t)|^2 dx. \tag{4.4}$$

$$\begin{aligned} G''(t) &= 2\|u_t(t)\|_2^2 - 2k_0\|\nabla u(t)\|_2^2 \\ &+2\int_\Omega \nabla u(t). \int_0^t g(t-s)a(x)\nabla u(s)dsdx + 2\|u(t)\|_p^p. \end{aligned} \tag{4.5}$$

By the Young's inequality, for any $\eta > 0$, we get

$$\begin{aligned} &\int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)a(x)\nabla u(s)dsdx \\ &\geq \int_\Omega \nabla u(t) \cdot \int_0^t g(t-s)a(x)\big(\nabla u(s) \\ &\quad - \nabla u(t)\big)dsdx + \int_0^t g(s)ds \int_\Omega a(x)|\nabla u(t)|^2 dx \\ &\geq \left(1 - \frac{1}{2\eta}\right) \int_0^t g(s)ds \int_\Omega a(x)|\nabla u(t)|^2 dx - \frac{\eta}{2}(g \circ \nabla u)(t). \end{aligned} \tag{4.6}$$

Using (4.5), (4.6), (2.9) and the fact that

$$E(t) + \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds + \int_0^t \|\sqrt{b(x)}u_t(s)\|^2 ds \leq E(0),$$

we obtain

$$\begin{aligned} G''(t) \geq &(p+2)\|u_t(t)\|_2^2 - 2pE(0) + (p-\eta)(g \circ \nabla u)(t) \\ &+ \int_\Omega \left(k_0(p-2) - (p-2+\frac{1}{\eta}) \int_0^t g(s)ds\ a(x)\right) |\nabla u(t)|^2 dx \\ &+ 2p \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds + 2p \int_0^t \|\sqrt{b(x)}u_t(s)\|^2 dx ds \end{aligned} \tag{4.7}$$

Letting $\eta = p$ and using (4.1), we obtain (4.2). ∎

Lemma 4.5. *Under the conditions of Theorem* 4.1*, for any solution* u *of (1.1)-*(1.4)*, we have*

$$G'(t) > \int_{\Gamma_1} |u_0(x)|^2 d\Gamma + \int_\Omega b(x)|u_0(x)|^2 dx, \quad \forall t \geq t_0, \tag{4.8}$$

where $t_0 = t^*$ *is given in* (4.9) *and* (4.10) *in cases* (1) *and* (3) *and* $t^* = 0$ *in cases* (2) *and* (4).

Proof. To obtain (4.8), we consider different cases on the sign of the initial energy.
(1) If $E(0) < 0$, then from (4.2), we have

$$G'(t) \geq G'(0) - 2pE(0)t.$$

Therefore, we have $G'(t) > \|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2$ for $t > t^*$ where

$$t^* = \max\left\{\frac{G'(0) - \|u_0\|_{2,\Gamma_1}^2 - \|\sqrt{b(x)}u_0\|_2^2}{2pE(0)}, 0\right\}. \tag{4.9}$$

(2) If $E(0) = 0$, then $G''(t) \geq 0$ and so $G'(t) > G'(0)$ for $t \geq 0$. Moreover, if $G'(0) > \|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2$ then $G'(t) > \|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2$ for $t \geq 0$.
(3) If $0 < E(0) < \frac{a_1 d_1}{l(p-2)}$ and $I(u_0) < 0$, by (4.7) and using lemma 2.4-(ii) we have

$$G''(t) \geq a_1\|\nabla u\|_2^2 - 2pE(0) \geq 2p\left(\frac{a_1 d_1}{l(p-2)} - E(0)\right).$$

Thus, we obtain $G'(t) \geq \|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2$ for $t > t^*$ where

$$t^* = \max\left\{\frac{\|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2 - G'(0)}{2p\left(\frac{a_1 d_1}{l(p-2)} - E(0)\right)}, 0\right\}. \tag{4.10}$$

(4) For the case that $E(0) > \frac{a_1 d_1}{l(p-2)}$, we first note that

$$
(4.11)\quad \begin{aligned}
\|u(t)\|_{2,\Gamma_1}^2 - \|u_0\|_{2,\Gamma_1}^2 &= 2\int_0^t \int_{\Gamma_1} u(s)u_t(s)d\Gamma ds,\\
\|\sqrt{b(x)}u(t)\|_2^2 - \|\sqrt{b(x)}u_0\|_2^2 &= 2\int_0^t \int_{\Omega} b(x)u(s)u_t(s)dxds.
\end{aligned}
$$

By the Hölder and Young's inequalities, we obtain

$$
(4.12)\quad \begin{aligned}
&\|u(t)\|_{2,\Gamma_1}^2 \le \|u_0\|_{2,\Gamma_1}^2 + \int_0^t \|u(s)\|_{2,\Gamma_1}^2 ds + \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds,\\
&\|\sqrt{b(x)}u(t)\|_2^2 \le \|\sqrt{b(x)}u_0\|_2^2 + \int_0^t \|\sqrt{b(x)}u(s)\|_2^2 ds + \int_0^t \|\sqrt{b(x)}u_t(s)\|_2^2 ds.
\end{aligned}
$$

We use Hölder and Young's inequalities, (4.3), (4.4), (4.11) and (4.12) to get

$$
(4.13)\quad \begin{aligned}
G'(t) \le & G(t) + \|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2\\
&+ \|u_t(t)\|_2^2 + \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds + \int_0^t \|\sqrt{b(x)}u_t(s)\|_2^2 ds.
\end{aligned}
$$

By (4.7) and (4.13), we find

$$
(4.14)\quad \begin{aligned}
&G''(t) - (p+2)G'(t) + (p+2)G(t) + G_0\\
\ge & (p-2)\int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds + (p-2)\int_0^t \|\sqrt{b(x)}u_t(s)\|_2^2 ds\\
&+ \left(k_0(p-2) - (p-2+\frac{1}{p})\|a\|_\infty \int_0^\infty g(s)ds\right)\|\nabla u(t)\|_2^2,
\end{aligned}
$$

where

$$
G_0 = 2pE(0) + (p+2)\|u_0\|_{2,\Gamma_1}^2 + (p+2)\|\sqrt{b(x)}u_0\|_2^2.
$$

Let

$$
B(t) = G(t) + \frac{G_0}{p+2}.
$$

Then, $B(t)$ satisfies the assumptions of lemma 4.2 for $\delta = \frac{p-2}{4}$. Therefore, if

$$
(4.15)\quad G'(0) \ge r_2\left(\|u_0\|_2^2 + \frac{G_0}{p+2}\right) + \|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2,
$$

then, from the lemma 4.2, we deduce that $G'(t) \ge \|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2$ for $t \ge 0$. This completes the proof. ■

Remark 4.6. By (4.15), one can verify that

$$E(0) < \frac{p+2}{2p}\left(\frac{2}{r_2}\int_\Omega u_0u_1dx - \left(\|u_0\|_2^2 + \|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2\right)\right). \tag{4.16}$$

Proof of Theorem 4.1. Let

$$M(t) = \left[G(t) + (T_1 - t)(\|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2)\right]^{-\delta}, \quad \text{for} \quad t \in [0, T_1], \tag{4.17}$$

where $\delta = \frac{p-2}{4}$ and $T_1 > 0$ is a certain constant which will be specified later. We have

$$\begin{aligned} M'(t) =& -\delta\left[G(t) + (T_1 - t)(\|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2)\right]^{-\delta-1} \\ & \times [G'(t) - (\|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2)] \\ =& -\delta M(t)^{1+\frac{1}{\delta}}[G'(t) - (\|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2)], \end{aligned} \tag{4.18}$$

and

$$M''(t) = -\delta M(t)^{1+\frac{2}{\delta}} V(t), \tag{4.19}$$

where

$$\begin{aligned} V(t) =& G''(t)\left[G(t) + (T_1 - t)(\|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2)\right] \\ & - (\delta+1)[G'(t) - (\|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2)]^2. \end{aligned} \tag{4.20}$$

For simplicity of calculation, we denote

$$\begin{aligned} P_u &= \int_\Omega |u(t)|^2 dx, & Q_u &= \int_0^t \|\sqrt{b(x)}u(s)\|_2^2 ds, \\ R_u &= \int_\Omega |u_t(t)|^2 dx, & S_u &= \int_0^t \|\sqrt{b(x)}u_t(s)\|_2^2 ds, \\ \widehat{Q}_u &= \int_0^t \|u(s)\|_{2,\Gamma_1}^2 ds, & \widehat{S}_u &= \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds. \end{aligned}$$

Using (4.4), (4.11) and Hölder's inequality, we deduce

$$G'(t) \le 2\left(\sqrt{R_uP_u} + \sqrt{Q_uS_u} + \sqrt{\widehat{Q}_u\widehat{S}_u}\right) + \|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2. \tag{4.21}$$

If case (1) or (2) holds, then by (4.7) we get

$$G''(t) \ge 4(\delta+1)(R_u + S_u + \widehat{S}_u) - (4 + 8\delta)E(0). \tag{4.22}$$

Using (4.3), (4.21) and (4.22), from the definition of $V(t)$, we obtain

$$
\begin{aligned}
V(t) \geq & \big[4(\delta+1)(R_u + S_u + \widehat{S}_u) - (4+8\delta)E(0)\big]M^{-\frac{1}{\delta}}(t) \\
& - 4(\delta+1)\left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{\widehat{Q}_u \widehat{S}_u}\right)^2.
\end{aligned} \tag{4.23}
$$

From

$$
G(t) = \int_\Omega u^2 dx + \int_0^t \int_{\Gamma_1} |u(s)|^2 d\Gamma ds + \int_0^t \int_\Omega b(x)|u(s)|^2 dx ds = P_u + \widehat{Q}_u + Q_u,
$$

and (4.17), we have

$$
\begin{aligned}
& V(t) \geq -(4+8\delta)E(0)\big]M^{-\frac{1}{\delta}}(t) \\
& + 4(\delta+1)\big(R_u+S_u+\widehat{S}_u\big)(T_1-t)\big(\|u_0\|^2_{2,\Gamma_1} + \|\sqrt{b(x)}u_0\|_2^2\big) + 4(\delta+1)K(t),
\end{aligned} \tag{4.24}
$$

where

$$
K(t) = \big(R_u + S_u + \widehat{S}_u\big)\big(P_u + Q_u + \widehat{Q}_u\big) - \left(\sqrt{R_u P_u} + \sqrt{Q_u S_u} + \sqrt{\widehat{Q}_u \widehat{S}_u}\right)^2.
$$

By the Schwartz inequality and $K(t)$ being nonnegative, we have

$$
V(t) \geq (-4-8\delta)E(0)M^{-\frac{1}{\delta}}(t).
$$

By (4.19), we get

$$
M''(t) \leq \delta(4+8\delta)E(0)M^{1+\frac{1}{\delta}}(t). \tag{4.25}
$$

By lemma 4.5 and (4.18), we know that $M'(t) < 0$ for $t \geq t_0$. Multiplying (4.25) by $M'(t)$ and integrating it from t_0 to t, we have

$$
M'(t)^2 \geq \mu_1 + \mu_2 M^{2+1/\delta}(t) \quad \text{for} \quad t \geq t_0, \tag{4.26}
$$

where

$$
\begin{aligned}
& \mu_1 = \left(\frac{p-2}{2}\right)^2 M^{\frac{2p+4}{p-2}}(t_0)\left[\left(\int_\Omega (u_0 u_1)dx\right)^2 - 2E(0)M^{\frac{-4}{p-2}}(t_0)\right] > 0, \\
& \mu_2 = \frac{1}{2}(p-2)^2 E(0),
\end{aligned} \tag{4.27}
$$

and

$$
M(t_0) = \big[\|u_0\|_2^2 + T_1(\|u_0\|^2_{2,\Gamma_1} + \|\sqrt{b(x)}u_0\|_2^2)\big]^{-\frac{p-2}{4}}.
$$

In the case (3), from (4.7) and lemma 2.4-(ii), we obtain

$$G''(t) \geq (4+8\delta)c_1 + 4(\delta+1)\left(R_u + S_u + \widehat{S}_u\right),$$

where $c_1 = \frac{a_1 d_1}{l(p-2)} - E(0)$. Following similar procedure in case (1), we find

$$M''(t) \leq -\delta(4+8\delta)c_1 M^{1+\frac{1}{\delta}}(t) \quad \text{for} \quad t \geq t_0,$$

$$M'(t)^2 \geq \mu_1 + \mu_2 M^{2+1/\delta}(t) \quad \text{for} \quad t \geq t_0,$$

where

$$\begin{aligned} \mu_1 &= \left(\frac{p-2}{2}\right)^2 M^{\frac{2p+4}{p-2}}(t_0)\left[\left(\int_\Omega (u_0 u_1\right)^2 + 2c_1 M^{\frac{-4}{p-2}}(t_0)\right] > 0, \\ \mu_2 &= -\frac{c_1}{2}(p-2)^2. \end{aligned} \tag{4.28}$$

For the case (4), by the steps of case (1), we obtain (4.26) with $\mu_1, \mu_2 > 0$ in (4.27) if

$$E(0) < \frac{\left(\int_\Omega u_0 u_1 dx\right)^2}{2\left[\|u_0\|_2^2 + T_1(\|u_0\|_{2,\Gamma_1}^2 + \|\sqrt{b(x)}u_0\|_2^2)\right]}.$$

Then, by lemma 4.3, there exists a finite time T^* so that $\lim_{t\nearrow T^{*-}} M(t) = 0$. This indicates that $\lim_{t\nearrow T^{*-}} \|u(t)\|_2^2 = +\infty$. Using the Poincaré inequality, we obtain $\|\nabla u(t)\|_2^2 \to +\infty$ as $t \to T^{*-}$. This completes the proof. ■

Remark 4.7. By lemma 4.3, the upper bounds of T^* can be estimated respectively according to the sign of $E(0)$. In the case (1)

$$T^* \leq t_0 - \frac{M(t_0)}{M'(t_0)}.$$

Furthermore, if $M(t_0) < \min\{1, \sqrt{-\mu_1/\mu_2}\}$, we have

$$T^* \leq t_0 + \frac{1}{\sqrt{-\mu_2}} \ln \frac{\sqrt{-\mu_1/\mu_2}}{\sqrt{-\mu_1/\mu_2} - M(t_0)},$$

where μ_1 and μ_2 are defined in (4.27). In case (2),

$$T^* \leq t_0 + \frac{M(t_0)}{\sqrt{\mu_1}},$$

where μ_1 is defined in (4.27). In cases (3) and (4),

$$T^* \le \frac{M(t_0)}{\sqrt{\mu_1}}, \quad \text{or} \quad T^* \le t_0 + 2^{\frac{3p-2}{2(p-2)}} \left(\frac{\mu_1^2}{\mu_2^2}\right)^{\frac{p+2}{p-2}} \frac{(p-2)}{4\sqrt{\mu_1}} \left[1 - \left(1 + \left(\frac{\mu_1}{\mu_2}\right)^{\frac{2p}{p-2}} M(t_0)\right)^{-\frac{2}{p-2}}\right].$$

Moreover, in case (3), μ_1 and μ_2 are defined in (4.28) and in case (4), μ_1 and μ_2 are defined in (4.27).

4.2. Blow-up with initial energy less than potential well depth: the case $k_1 \ge 0$, $2 \le m < p$

Theorem 4.8. *Assume that* $2 \le m < p$, $b(x) \ge b_0 > 0$ *and* $(A1)$, $(A2)$ *and* (2.2) *hold. Suppose that* $(u_0, u_1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$ *satisfies,*

$$I(0) < 0, \qquad E(0) < \gamma d_1, \quad \gamma \in (0,1). \tag{4.29}$$

Assume further that

$$\|a\|_\infty \int_0^{+\infty} g(s)ds < \frac{k_0(p-2)(1-\gamma)}{(p-2)(1-\gamma) + 1/[(p-2)(1-\gamma)+2]}. \tag{4.30}$$

Then, the solution of (1.1)-(1.4) *blows up in finite time.*

Proof. On the contrary, suppose that the existence time of solution $u(t)$ can be extended to the whole interval $[0, \infty)$. We define

$$A(t) = \frac{1}{2}\|u(t)\|_2^2. \tag{4.31}$$

By (1.1)-(1.4), we have

$$A'(t) = \int_\Omega u(t)u_t(t)dt,$$

$$\begin{aligned} A''(t) =& \|u_t(t)\|_2^2 - k_0\|\nabla u(t)\|_2^2 + \int_\Omega \nabla u(t) \,.\, \int_0^t g(t-s)a(x)\nabla u(s)dsdx \\ & - \int_{\Gamma_1} uh(u_t)d\Gamma - k_1 \int_\Omega uu_t dx - \int_\Omega b(x)uu_t|u_t|^{m-2}dx + \|u(t)\|_p^p. \end{aligned} \tag{4.32}$$

Using (2.9) to substitute for $\|u(t)\|_p^p$ and (4.6), we obtain

$$\begin{aligned} A''(t) \ge& \left(\frac{p+2}{2}\right)\|u_t(t)\|_2^2 - \int_{\Gamma_1} uh(u_t)d\Gamma - k_1\int_\Omega uu_t dx \\ & + \int_\Omega \left\{k_0\left(\frac{p}{2}-1\right) - \left(\frac{p}{2} - 1 + \frac{1}{2\eta}\right)a(x)\int_0^t g(s)ds\right\}|\nabla u|^2 dx \\ & + \frac{1}{2}(p-\eta)(g\circ\nabla u)(t) - \int_\Omega b(x)uu_t|u_t|^{m-2}dx - pE(t). \end{aligned} \tag{4.33}$$

Now, we set

$$Z(t) = \gamma d_1 - E(t), \qquad \forall t \geq 0. \tag{4.34}$$

Clearly $Z'(t) \geq 0$ and $Z(0) > 0$. Thus,

$$Z(t) \geq Z(0) > 0, \qquad \forall t \geq 0. \tag{4.35}$$

By the lemma 2.3 and lemma 2.4-(ii), we can see

$$0 < d_1 \leq \left(\frac{p-2}{2p}\right)\left\{\int_\Omega \Big(k_0 - a(x)\int_0^t g(s)ds\Big)|\nabla u|^2 dx + (g \circ \nabla u)(t)\right\}. \tag{4.36}$$

Combining (4.33), (4.34) and (4.36), we arrive at

$$\begin{aligned} A''(t) \geq & \left(\frac{p+2}{2}\right)\|u_t(t)\|_2^2 - \int_{\Gamma_1} uh(u_t)d\Gamma \\ & - k_1\int_\Omega uu_t dx + \int_\Omega \Big\{k_0(\frac{p}{2}-1)(1-\gamma) \\ & - \Big((\frac{p}{2}-1)(1-\gamma) + \frac{1}{2\eta}\Big)a(x)\int_0^t g(s)ds\Big\}|\nabla u|^2 dx \\ & + \big[(1-\gamma)(\frac{p}{2}-1) + 1 - \frac{\eta}{2}\big](g \circ \nabla u)(t) \\ & - \int_\Omega b(x)uu_t|u_t|^{m-2}dx + pZ(t). \end{aligned} \tag{4.37}$$

Choosing η sufficiently small so that

$$(1-\gamma)(\frac{p}{2}-1) + 1 - \frac{\eta}{2} > 0,$$

and using Hölder's inequality, by (4.30), the estimate (4.37) takes the form

$$\begin{aligned} A''(t) \geq & c_1\|u_t(t)\|_2^2 + c_2\|\nabla u(t)\|_2^2 - \int_{\Gamma_1} uh(u_t)d\Gamma - k_1\int_\Omega uu_t dx \\ & + c_3(g \circ \nabla u)(t) - \int_\Omega b(x)uu_t|u_t|^{m-2}dx + pZ(t), \end{aligned} \tag{4.38}$$

for some $c_1, c_2, c_3 > 0$. By using Young's inequality on the third and fourth terms in (4.38) and by (2.1), (2.5) and (2.6), for any $\delta > 0$, we deduce

$$\int_{\Gamma_1} uh(u_t)d\Gamma \leq \frac{\beta}{2}\left(\delta B_1^2\|\nabla u(t)\|_2^2 + \frac{1}{\delta}\|u_t(t)\|_{2,\Gamma_1}^2\right). \tag{4.39}$$

$$\int_\Omega uu_t dx \leq \frac{1}{2}\left(\delta B^2\|\nabla u(t)\|_2^2 + \frac{1}{\delta}\|u_t(t)\|_2^2\right). \tag{4.40}$$

By Hölder's inequality and standard interpolation inequality, we have

$$\begin{aligned}
&\int_\Omega b(x)uu_t|u_t|^{m-2}dx \\
\text{(4.41)} \quad &\le \|b\|_\infty \|u(t)\|_m \|u_t(t)\|_m^{m-1} \le \|b\|_\infty \|u(t)\|_2^k \|u(t)\|_p^{1-k} \|u_t(t)\|_m^{m-1} \\
&\le c_4 \|u(t)\|_p^{\frac{kp}{2}} \|u(t)\|_p^{1-k} \|u_t(t)\|_m^{m-1} \le c_4 \|u(t)\|_p^{\frac{p}{m}} \|u_t(t)\|_m^{m-1},
\end{aligned}$$

where $\frac{k}{2} + \frac{1-k}{p} = \frac{1}{m}$ which gives $k = \frac{2(p-m)}{m(p-2)} > 0$ and $c_4 = \|b\|_\infty (B^2 l^{-1})^{k/2}$ comes from the fact that $\|u\|_2^2 \le B^2 l^{-1} \|u\|_p^p$. By (4.38)-(4.40) and using Young's inequality for the last inequality in (4.41), we get

$$\begin{aligned}
&A''(t) + \frac{1}{2\delta}\big(\beta \|u_t(t)\|_{2,\Gamma_1}^2 + k_1 \|u_t(t)\|_2^2\big) + c_4 c(\delta) \|u_t(t)\|_m^m \\
\text{(4.42)} \quad &\ge c_1 \|u_t(t)\|_2^2 + \Big(c_2 - \frac{\delta}{2}(\beta B_1^2 + k_1 B^2)\Big) \|\nabla u(t)\|_2^2 \\
&\quad + c_3 (g \circ \nabla u)(t) - c_4 \delta \|u(t)\|_p^p + pZ(t).
\end{aligned}$$

Letting $c_5 = \min\{c_1, c_2 k_0^{-1}, c_3, \frac{p}{2}\}$ and decomposing $pZ(t)$ in (4.42) by

$$pZ(t) = 2c_5 Z(t) + (p - 2c_5) Z(t).$$

Therefore, by (4.34) and (2.9), we obtain

$$\begin{aligned}
&A''(t) + \frac{1}{2\delta}\big(\beta \|u_t(t)\|_{2,\Gamma_1}^2 + k_1 \|u_t(t)\|_2^2\big) + c_4 c(\delta) \|u_t(t)\|_m^m \\
\text{(4.43)} \quad &\ge (c_1 - c_5) \|u_t(t)\|_2^2 + \Big(c_2 - c_5 k_0 - \frac{\delta}{2}(\beta B_1^2 + k_1 B^2)\Big) \|\nabla u(t)\|_2^2 \\
&\quad + (c_3 - c_5)(g \circ \nabla u)(t) + \Big(\frac{2c_5}{p} - c_4 \delta\Big) \|u(t)\|_p^p + (p - 2c_5) Z(t).
\end{aligned}$$

Choosing δ sufficiently small so that $\delta \le \min\{\frac{2(c_2 - c_5 k_0)}{\beta B_1^2 + k_1 B^2}, \frac{2c_5}{pc_4}\}$, by (4.35) and (2.12), we arrive at

$$\begin{aligned}
&A''(t) + \frac{1}{2\delta}\big(\beta \|u_t(t)\|_{2,\Gamma_1}^2 + k_1 \|u_t(t)\|_2^2\big) + c_4 c(\delta) \|u_t(t)\|_m^m \\
\text{(4.44)} \quad &\qquad \ge c_6 \|u_t(t)\|_p^p \ge c_6 l \|\nabla u(t)\|_2^2 \ge c_6 \frac{2pd_1}{p-2},
\end{aligned}$$

for some $c_6 > 0$. Integrating (4.44) over $(0, t)$, we deduce

$$\begin{aligned}
&A'(t) + \frac{\beta}{2\delta} \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds + \frac{k_1}{2\delta} \int_0^t \|u_t(s)\|_2^2 ds + c_4 c(\delta) \int_0^t \|u_t(s)\|_m^m ds \\
\text{(4.45)} \quad &\ge c_7 t + \int_\Omega u_0 u_1 dx,
\end{aligned}$$

where $c_7 = 2c_6pd_1/(p-2)$. By lemma 2.1 and the assumptions of Theorem 4.8, for $t \in [0, \infty)$, we have

$$\begin{aligned} \int_0^t \|u_t(s)\|_{2,\Gamma_1}^2 ds &\le \frac{E(0)-E(t)}{\alpha} < \frac{\gamma d_1}{\alpha}, \\ \int_0^t \|u_t(s)\|_2^2 ds &\le \frac{E(0)-E(t)}{k_1} < \frac{\gamma d_1}{k_1}, \\ \int_0^t \|u_t(s)\|_m^m ds &\le \frac{E(0)-E(t)}{b_0} < \frac{\gamma d_1}{b_0}. \end{aligned} \tag{4.46}$$

Inserting (4.46) into (4.45) and integrating over $(0, t)$ once more, we obtain

$$A(t) > \frac{1}{2}c_7t^2 + \left(\int_\Omega u_0u_1dx - \left[\frac{1}{2\delta}\left(\beta\alpha^{-1}+1\right) + \frac{c_4}{b_0}c(\delta)\right]\gamma d_1\right)t + \frac{1}{2}\|u_0\|_2^2, \tag{4.47}$$

which shows that $\|u(t)\|_2^2$ has quadratic growth for $t \ge 0$. On the other hand, similar as in [26], by the use of Hölder's inequality and (4.46), we have

$$\begin{aligned} \|u(t)\|_2 &\le \|u_0\|_2 + \int_0^t \|u_t(s)\|_2 ds \le \|u_0\|_2 + C\int_0^t \|u_t(s)\|_m ds \\ &\le \|u_0\|_2 + C\left(\frac{\gamma d_1}{b_0}\right)^{\frac{1}{m}} t^{\frac{m-1}{m}}, \end{aligned} \tag{4.48}$$

for some $C > 0$. Clearly, (4.48) contradicts (4.47). Therefore, the solution of (1.1)-(1.4) can not be extended to the whole interval $[0, +\infty)$. This completes the proof. ■

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