TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 4, pp. 1441-1472, August 2013
DOI: 10.11650/tjm.17.2013.2864
This paper is available online at http://journal.taiwanmathsoc.org.tw

HYBRID STEEPEST-DESCENT METHODS FOR TRIPLE HIERARCHICAL VARIATIONAL INEQUALITIES

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Abstract. In this paper, we consider a triple hierarchical variational inequality defined over the common solution set of minimization and mixed equilibrium problems. Combining the hybrid steepest-descent method, viscosity approximation method and averaged mapping approach to the gradient-projection algorithm, we propose two iterative methods: implicit one and explicit one, to compute the approximate solutions of our problem. The convergence analysis of the sequences generated by the proposed methods is also established.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $T : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T and by $\mathbf{R}$ the set of all real numbers. A mapping $T : C \to C$ is called L-Lipschitzian if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$, then T is called a nonexpansive mapping; if $L \in [0, 1)$, then T is called a contractive mapping. A mapping $A : C \to H$ is called α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Received January 16, 2013, accepted February 5, 2013.
Communicated by Chong Li.
2010 *Mathematics Subject Classification*: 49J40, 47J20, 47H10, 65K05, 47H09.
Key words and phrases: Triple hierarchical variational inequality, Minimization problem, Mixed equilibrium problem, Implicit iterative algorithm, Explicit iterative algorithm, Averaged mapping approach.
[1]This research was partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133) and Ph.D. Program Foundation of Ministry of Education of China (20123127110002).
[2]This research was partially supported by the grant from the NSC 101-2115-M-037-001.
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$A : C \to H$ is called η-strongly monotone, if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C.$$

Let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the minimization problem (MP) of minimizing f over the constraint set C

$$\min_{x\in C} f(x) \tag{1.1}$$

where we denote by Γ the set of minimizers of MP (1.1) which is assumed to be nonempty.

For a given mapping $A : C \to H$, the classical variational inequality (VI) is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.2}$$

It is well-known that variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework; see, e.g., [1-9] and the references therein. The solution set of the VI (1.2) is denoted by $\mathrm{VI}(C, A)$. The recent research work shows that variational inequalities cover several topics, for example, monotone inclusions, convex optimization and quadratic minimization over fixed point sets; see e.g., [1, 10-17] and the references therein for more details.

Let $\Theta : C\times C \to \mathbf{R}$ be a bifunction and $\varphi : C \to \mathbf{R}$ be a function. Then, consider the following mixed equilibrium problem (MEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which was studied by Ceng and Yao [14]. The solution set of MEP (1.3) is denoted by $\mathrm{MEP}(\Theta, \varphi)$.

In this paper, we introduce and study the following triple hierarchical variational inequality (THVI) defined over the common solution set of minimization and mixed equilibrium problems:

THVI. Let $S : H \to H$ be a nonexpansive mapping, $V : H \to H$ be a ρ-contractive mapping with constant $\rho \in [0, 1)$ and $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa, \eta > 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \leq \tau$ where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Consider the following triple hierarchical variational inequality (THVI): find $x^* \in \Xi$ such that

$$\langle(\mu F - \gamma V)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \Xi, \tag{1.4}$$

which Ξ denotes the solution set of the following hierarchical variational inequality (HVI): find $z^* \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ such that

$$\langle (\mu F - \gamma S)z^*, z - z^* \rangle \geq 0, \quad \forall z \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma, \tag{1.5}$$

where the solution set Ξ is assumed to be nonempty.

We combine the viscosity approximation method, hybrid steepest-descent method and averaged mapping approach to the gradient-projection method to propose an implicit iterative algorithm that generates a sequence in an implicit way, and study its strong convergence to a unique solution of the THVI (1.4). We also introduce an explicit iterative algorithm that generates a sequence in an explicit way and prove that this sequence converges strongly to a unique solution of the THVI (1.4).

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1. *For given $x \in H$ and $z \in C$:*
(i) $z = P_C x \;\Leftrightarrow\; \langle x - z, y - z \rangle \leq 0,\ \forall y \in C$;
(ii) $z = P_C x \;\Leftrightarrow\; \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;
(iii) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2,\ \forall y \in H$.

Consequently, P_C is nonexpansive and monotone.

Definition 2.1. A mapping $T : H \to H$ is said to be:
(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H;$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I+S),$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.2. Let T be a nonlinear operator with domain $D(T) \subseteq H$ and range $R(T) \subseteq H$.

(a) T is said to be monotone if

$$\langle x-y, Tx-Ty\rangle \geq 0, \quad \forall x, y \in D(T).$$

(b) Given a number $\beta > 0$, T is said to be β-strongly monotone if

$$\langle x-y, Tx-Ty\rangle \geq \beta\|x-y\|^2, \quad \forall x, y \in D(T).$$

(c) Given a number $\nu > 0$, T is said to be ν-inverse strongly monotone (ν-ism) if

$$\langle x-y, Tx-Ty\rangle \geq \nu\|Tx-Ty\|^2, \quad \forall x, y \in D(T).$$

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Definition 2.3. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1-\alpha)I + \alpha S$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged maps.

Proposition 2.2. (see [18]). *Let $T : H \to H$ be a given mapping.*

(i) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.

(ii) If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.

(iii) T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.

Proposition 2.3. (see [18]). *Let $S, T, V : H \to H$ be given operators.*

(i) *If* $T = (1-\alpha)S + \alpha V$ *for some* $\alpha \in (0,1)$ *and if* S *is averaged and* V *is nonexpansive, then* T *is averaged.*

(ii) T *is firmly nonexpansive if and only if the complement* $I - T$ *is firmly nonexpansive.*

(iii) *If* $T = (1-\alpha)S + \alpha V$ *for some* $\alpha \in (0,1)$ *and if* S *is firmly nonexpansive and* V *is nonexpansive, then* T *is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings* $\{T_i\}_{i=1}^N$ *is averaged, then so is the composite* $T_1 \cdots T_N$. *In particular, if* T_1 *is* α_1*-averaged and* T_2 *is* α_2*-averaged, where* $\alpha_1, \alpha_2 \in (0,1)$, *then the composite* $T_1 T_2$ *is* α*-averaged, where* $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

(v) *If the mappings* $\{T_i\}_{i=1}^N$ *are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N).$$

The notation $\mathrm{Fix}(T)$ *denotes the set of all fixed points of the mapping* T, *that is,* $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$.

For solving the equilibrium problem for a bifunction $\Theta : C \times C \to \mathbf{R}$, let us assume that Θ and φ satisfy the following conditions (see [7]):

(A1) $\Theta(x,x) = 0$ for all $x \in C$;

(A2) Θ is monotone, that is, $\Theta(x,y) + \Theta(y,x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} \Theta(tz + (1-t)x, y) \le \Theta(x,y)$;

(A4) for each $x \in C$, $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous;

(A5) for each $y \in C$, $x \mapsto \Theta(x,y)$ is weakly upper semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

The following lemmas were given in [7, 19].

Lemma 2.1. (see [19]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $\Theta : C \times C \to \mathbf{R}$ *be a bifunction satisfying (A1)-(A4). Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$\Theta(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.2. (see [7]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\Theta : C \times C \to \mathbf{R}$ *be a bifunction satisfying (A1)-(A5) and* $\varphi : C \to \mathbf{R}$

be a proper lower semicontinuous and convex function. For $r > 0$ *and* $x \in H$*, define a mapping* $Q_r : H \to C$ *as follows:*

$$Q_r x := \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in H$*, (which is called the resolvent of* Θ *and* φ*). Assume that either (B1) or (B2) holds. Then, the following hold:*

(i) *for each* $x \in H$, $Q_r x \neq \emptyset$*;*

(ii) Q_r *is single-valued;*

(iii) Q_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|Q_r x - Q_r y\|^2 \leq \langle Q_r x - Q_r y, x - y\rangle;$$

(iv) $\mathrm{Fix}(Q_r) = \mathrm{MEP}(\Theta, \varphi)$*;*

(v) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

The following lemma plays a key role in proving strong convergence of the sequences generated by our algorithms.

Lemma 2.3. (see [20]). *Let* $\{a_n\}$ *be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \beta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}, \{\beta_n\}$ *are sequences of real numbers such that*

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*, or equivalently,* $\prod_{n=0}^{\infty}(1 - \alpha_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1 - \alpha_k) = 0$*;*

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$*, or* $\sum_{n=0}^{\infty} \alpha_n |\beta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Below we also gather some basic facts that are needed in the sequel.

Lemma 2.4. (see [21, Demiclosedness Principle]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $T : C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$*. If* $\{x_n\}$ *is a sequence in* C *converging weakly to* x *and if* $\{(I - T)x_n\}$ *converges strongly to* y*, then* $(I - T)x = y$*; in particular, if* $y = 0$*, then* $x \in \mathrm{Fix}(T)$.

The following lemma is not hard to prove.

Lemma 2.5. (see [11]). *Let* $V : H \to H$ *be a* ρ*-contraction with* $\rho \in [0, 1)$ *and* $S : H \to H$ *be a nonexpansive mapping. Then,*

(i) $I - V$ *is* $(1 - \rho)$*-strongly monotone:*

$$\langle (I - V)x - (I - V)y, x - y\rangle \geq (1 - \rho)\|x - y\|^2, \quad \forall x, y \in H;$$

(ii) $I - S$ *is monotone:*

$$\langle (I-S)x - (I-S)y, x-y \rangle \geq 0, \quad \forall x, y \in H.$$

The following fact is straightforward but useful.

Lemma 2.6. *There holds the following inequality in an inner product space X:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle, \quad \forall x, y \in X.$$

Lemma 2.7. (see [20, Lemma 3.1]). *Let λ be a number in $(0,1]$ and let $\mu > 0$. Let $F : C \to H$ be an operator on C such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Associating with a nonexpansive mapping $T : C \to C$, define the mapping $T^\lambda : C \to H$ by*

$$T^\lambda x := (I - \lambda\mu F)Tx, \quad \forall x \in C.$$

Then T^λ is a contraction provided $\mu < 2\eta/\kappa^2$, that is,

$$\|T^\lambda x - T^\lambda y\| \leq (1-\lambda\tau)\|x-y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1]$. In particular, if $T = I$ the identity mapping, then

$$\|(I-\lambda\mu F)x - (I-\lambda\mu F)y\| \leq (1-\lambda\tau)\|x-y\|, \quad \forall x, y \in C.$$

3. Main Results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional such that ∇f is an L-Lipschitzian mapping with $L > 0$. Noting that ∇f is L-Lipschitzian, it follows that ∇f is $1/L$-ism, which then implies that $\lambda\nabla f$ is $1/\lambda L$-ism according to Proposition 2.2 (ii). So by Proposition 2.2 (iii), the complement $I - \lambda\nabla f$ is $\lambda L/2$-averaged. Now since the projection P_C is $1/2$-averaged, we see from Proposition 2.3 (iv) that the composite $P_C(I-\lambda\nabla f)$ is $(2+\lambda L)/4$-averaged for $0 < \lambda < 2/L$. Therefore, we can write

$$P_C(I-\lambda\nabla f) = \frac{2-\lambda L}{4}I + \frac{2+\lambda L}{4}T_\lambda = sI + (1-s)T_\lambda,$$

where T_λ is nonexpansive and $s := s(\lambda) = \frac{2-\lambda L}{4} \in (0, \frac{1}{2})$ for each $\lambda \in (0, \frac{2}{L})$. It is easy to see that

$$\lambda \to \frac{2}{L} \ \Leftrightarrow \ s \to 0.$$

Let $S : H \to H$ be a nonexpansive mapping, $V : H \to H$ be a ρ-contractive mapping with constant $\rho \in [0,1)$ and $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa, \eta > 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma \le \tau$ where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. For a given number $r > 0$, we denote the resolvent of Θ and φ by Q_r as defined in Lemma 2.2. Consider the following contraction mapping $G_{s,t}$ on H defined by

$$G_{s,t}(x) := s\gamma(tVx + (1-t)Sx) + (I - s\mu F)T_\lambda Q_r x, \quad \forall x \in H,$$

where $t \in (0,1)$ and $s = \frac{2-\lambda L}{4} \in (0, \frac{1}{2})$ for each $\lambda \in (0, \frac{2}{L})$. Note that this contraction is a self-mapping on H. It is easy to find that the contraction coefficient of $G_{s,t}$ is $1 - (1-\rho)\gamma st$. Indeed, in terms of Lemma 2.7 we know that for each $x, y \in H$

$$\begin{aligned}
&\|G_{s,t}(x) - G_{s,t}(y)\| \\
&= \|s\gamma(tVx + (1-t)Sx) + (I - s\mu F)T_\lambda Q_r x \\
&\quad - s\gamma(tVy + (1-t)Sy) - (I - s\mu F)T_\lambda Q_r y\| \\
&\le s\gamma\|(tVx + (1-t)Sx) - (tVy + (1-t)Sy)\| \\
&\quad + \|(I - s\mu F)T_\lambda Q_r x - (I - s\mu F)T_\lambda Q_r y\| \\
&\le s\gamma\|tVx + (1-t)Sx - tVy - (1-t)Sy\| + (1 - s\tau)\|x - y\| \\
&\le s\gamma[t\|Vx - Vy\| + (1-t)\|Sx - Sy\|] + (1 - s\tau)\|x - y\| \\
&\le s\gamma[t\rho\|x - y\| + (1-t)\|x - y\|] + (1 - s\tau)\|x - y\| \\
&= s\gamma(1 - t(1-\rho))\|x - y\| + (1 - s\tau)\|x - y\| \\
&= \{1 - s[\tau - \gamma(1 - t(1-\rho))]\}\|x - y\| \\
&\le (1 - st\gamma(1-\rho))\|x - y\|
\end{aligned}$$

due to $0 < \gamma \le \tau$. Since $0 < \gamma \le \tau \le 1,\ 0 \le \rho < 1,\ 0 < s < \frac{1}{2}$ and $0 < t < 1$, we get $st\gamma(1-\rho) < \frac{1}{2}\gamma(1-\rho) \le \frac{1}{2}$. This implies that the contraction coefficient of $G_{s,t}$ is $1 - (1-\rho)\gamma st$. Hence, by the Banach contraction principle, $G_{s,t}$ has a unique fixed point which is denoted by $x_{s,t} \in H$, that is, $x_{s,t}$ is the unique solution in H of the fixed-point equation

$$x_{s,t} = s\gamma(tVx_{s,t} + (1-t)Sx_{s,t}) + (I - s\mu F)T_\lambda Q_r x_{s,t}. \tag{3.1}$$

Additionally, if we take $V = 0$, then (3.1) reduces to

$$x_{s,t} = \gamma s(1-t)Sx_{s,t} + (I - s\mu F)T_\lambda Q_r x_{s,t}. \tag{3.2}$$

In particular, whenever $\mu = 2,\ F = \frac{1}{2}I$ and $\gamma = \tau = 1$, the implicit schemes (3.1) and (3.2) reduce to the following implicit schemes, respectively:

$$x_{s,t} = s(tVx_{s,t} + (1-t)Sx_{s,t}) + (1-s)T_\lambda Q_r x_{s,t}, \tag{3.3}$$

and

$$x_{s,t} = s(1-t)Sx_{s,t} + (1-s)T_\lambda Q_r x_{s,t}. \tag{3.4}$$

Below is the first result of this paper which displays the behavior of the net $\{x_{s,t}\}$ as $s \to 0$ and $t \to 0$ successively.

Theorem 3.1. *Let $\Theta : C \times C \to \mathrm{R}$ be a bifunction satisfying (A1)-(A5). Let $f : C \to \mathrm{R}$ be a convex function such that ∇f is an L-Lipschitzian mapping with $L > 0$. Let $S : H \to H$ be a nonexpansive mapping, $V : H \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1)$ and $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants κ and $\eta > 0$, respectively. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < \gamma \le \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose that the solution set Ξ of HVI (1.5) is nonempty and that either (B1) or (B2) holds. For each $(s, t) \in (0, \frac{1}{2}) \times (0, 1)$ (with $s = \frac{2-\lambda L}{4}$ for each $\lambda \in (0, \frac{2}{L})$), let $x_{s,t}$ be defined implicitly by (3.1). Then, for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ converges in norm, as $s \to 0$, to a point $x_t \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$. Moreover, as $t \to 0$, the net $\{x_t\}$ converges in norm to a unique solution $x^* \in \Xi$ of the THVI (1.4). Moreover, for each null sequence $\{s_n\}$ (with $s_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$), there exists another null sequence $\{t_n\}$ in $(0, 1)$ such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.*

In particular, if we take $V = 0$ and if $x_{s,t}$ is defined by the implicit scheme (3.2), then the iterated limit in the norm topology

$$s - \lim_{t \to 0} \lim_{s \to 0} x_{s,t}$$

exists and is a unique solution x^* of the variational inequality (VI), which consists in finding $x^* \in \Xi$ such that

$$\langle Fx^*, x - x^* \rangle \ge 0, \quad \forall x \in \Xi. \tag{3.5}$$

Furthermore, for each null sequence $\{s_n\}$ (with $s_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$), there exists another null sequence $\{t_n\}$ in $(0, 1)$, such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.

Proof. We first show that $\{x_{s,t}\}$ is bounded. Indeed, take any $p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$. Observe that for each $(s, t) \in (0, \frac{1}{2}) \times (0, 1)$ (with $s = \frac{2-\lambda L}{4}$ for each $\lambda \in (0, \frac{2}{L})$),

$$\begin{aligned} x_{s,t} - p &= s\gamma(tVx_{s,t} + (1-t)Sx_{s,t}) + (I - s\mu F)T_\lambda Q_r x_{s,t} - p \\ &= [(I - s\mu F)T_\lambda Q_r x_{s,t} - (I - s\mu F)T_\lambda Q_r p] + st\gamma(Vx_{s,t} - Vp) \\ &\quad + s(1-t)\gamma(Sx_{s,t} - Sp) + st(\gamma V - \mu F)p + s(1-t)(\gamma S - \mu F)p. \end{aligned}$$

Noticing $0 < \gamma \le \tau$ and utilizing Lemma 2.7, we have

$$\begin{aligned} &\|x_{s,t} - p\| \\ &= \|s\gamma(tVx_{s,t} + (1-t)Sx_{s,t}) + (I - s\mu F)T_\lambda Q_r x_{s,t} - p\| \\ &= \|[(I - s\mu F)T_\lambda Q_r x_{s,t} - (I - s\mu F)T_\lambda Q_r p] + st\gamma(Vx_{s,t} - Vp) \\ &\quad + s(1-t)\gamma(Sx_{s,t} - Sp) + st(\gamma V - \mu F)p + s(1-t)(\gamma S - \mu F)p\| \end{aligned}$$

$$
\begin{aligned}
&\le \|(I-s\mu F)T_\lambda Q_r x_{s,t}-(I-s\mu F)T_\lambda Q_r p\|+st\gamma\|Vx_{s,t}-Vp\| \\
&\quad +s(1-t)\gamma\|Sx_{s,t}-Sp\|+st\|(\gamma V-\mu F)p\|+s(1-t)\|(\gamma S-\mu F)p\| \\
&\le (1-s\tau)\|x_{s,t}-p\|+st\gamma\rho\|x_{s,t}-p\| \\
&\quad +s(1-t)\gamma\|x_{s,t}-p\|+st\|(\gamma V-\mu F)p\|+s(1-t)\|(\gamma S-\mu F)p\| \\
&\le [1-s\tau+st\gamma\rho+s(1-t)\gamma]\|x_{s,t}-p\| \\
&\quad +(st+s(1-t))\max\{\|(\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\} \\
&\le (1-st\gamma(1-\rho))\|x_{s,t}-p\|+s\max\{\|(\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}.
\end{aligned}
$$

This implies that

$$\|x_{s,t}-p\|\le \frac{1}{t\gamma(1-\rho)}\max\{\|(\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}.$$

It follows that for each fixed $t\in(0,1)$, $\{x_{s,t}\}$ is bounded and so are the nets $\{Q_r x_{s,t}\}$, $\{T_\lambda Q_r x_{s,t}\}, \{Vx_{s,t}\}, \{Sx_{s,t}\}$ and $\{FT_\lambda Q_r x_{s,t}\}$. We note that

$$
\begin{aligned}
&\|Q_r x_{s,t}-T_\lambda Q_r x_{s,t}\| \\
&\le \|Q_r x_{s,t}-x_{s,t}\|+\|x_{s,t}-T_\lambda Q_r x_{s,t}\| \\
&= \|Q_r x_{s,t}-x_{s,t}\|+s\|\gamma(tVx_{s,t} \\
&\quad +(1-t)Sx_{s,t})-\mu FT_\lambda Q_r x_{s,t}\| \\
&= \|Q_r x_{s,t}-x_{s,t}\|+s\|t(\gamma Vx_{s,t}-\mu FT_\lambda Q_r x_{s,t}) \\
&\quad +(1-t)(\gamma Sx_{s,t}-\mu FT_\lambda Q_r x_{s,t})\| \\
&\le \|Q_r x_{s,t}-x_{s,t}\|+s\max\{\|\gamma Vx_{s,t}-\mu FT_\lambda Q_r x_{s,t}\|, \\
&\quad \|\gamma Sx_{s,t}-\mu FT_\lambda Q_r x_{s,t}\|\}.
\end{aligned}
\tag{3.6}
$$

Utilizing Lemma 2.2, we obtain

$$
\begin{aligned}
\|Q_r x_{s,t}-p\|^2 &= \|Q_r x_{s,t}-Q_r p\|^2 \\
&\le \langle x_{s,t}-p, Q_r x_{s,t}-Q_r p\rangle \\
&= \frac{1}{2}(\|x_{s,t}-p\|^2+\|Q_r x_{s,t}-p\|^2-\|x_{s,t}-Q_r x_{s,t}\|^2),
\end{aligned}
$$

and so

$$\|Q_r x_{s,t}-p\|^2\le \|x_{s,t}-p\|^2-\|x_{s,t}-Q_r x_{s,t}\|^2. \tag{3.7}$$

Then, from Lemma 2.6 and (3.7), we have

$$
\begin{aligned}
&\|x_{s,t}-p\|^2 \\
&= \|[(I-s\mu F)T_\lambda Q_r x_{s,t}-(I-s\mu F)p]+st\gamma(Vx_{s,t}-Vp) \\
&\quad +s(1-t)\gamma(Sx_{s,t}-Sp)+st(\gamma V-\mu F)p+s(1-t)(\gamma S-\mu F)p\|^2 \\
&\le (1-s\tau)^2\|T_\lambda Q_r x_{s,t}-p\|^2+2st\gamma\langle Vx_{s,t}-Vp, x_{s,t}-p\rangle \\
&\quad +2s(1-t)\gamma\langle Sx_{s,t}-Sp, x_{s,t}-p\rangle+2\langle st(\gamma V-\mu F)p \\
&\quad +s(1-t)(\gamma S-\mu F)p, x_{s,t}-p\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-s\tau)^2\|T_\lambda Q_r x_{s,t}-p\|^2+2st\gamma\rho\|x_{s,t}-p\|^2+2s(1-t)\gamma\|x_{s,t}-p\|^2\\
&\quad+2s\|t(\gamma V-\mu F)p+(1-t)(\gamma S-\mu F)p\|\|x_{s,t}-p\|\\
&\leq (1-s\tau)^2\|Q_r x_{s,t}-p\|^2+2st\gamma\rho\|x_{s,t}-p\|^2+2s(1-t)\gamma\|x_{s,t}-p\|^2\\
&\quad+2s\|t(\gamma V-\mu F)p+(1-t)(\gamma S-\mu F)p\|\|x_{s,t}-p\|\\
&\leq (1-s\tau)^2(\|x_{s,t}-p\|^2-\|x_{s,t}-Q_r x_{s,t}\|^2)+2st\gamma\rho\|x_{s,t}-p\|^2\\
&\quad+2s(1-t)\gamma\|x_{s,t}-p\|^2+2s\max\{\|\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{s,t}-p\|\\
&=[1-2s\tau+s^2\tau^2+2st\gamma\rho+2s(1-t)\gamma]\|x_{s,t}-p\|^2-(1-s\tau)^2\|x_{s,t}-Q_r x_{s,t}\|^2\\
&\quad+2s\max\{\|\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{s,t}-p\|\\
&\leq [1-2st\gamma(1-\rho)+s^2\tau^2]\|x_{s,t}-p\|^2-(1-s\tau)^2\|x_{s,t}-Q_r x_{s,t}\|^2\\
&\quad+2s\max\{\|\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{s,t}-p\|\\
&\leq \|x_{s,t}-p\|^2+s^2\tau^2\|x_{s,t}-p\|^2-(1-s\tau)^2\|x_{s,t}-Q_r x_{s,t}\|^2\\
&\quad+2s\max\{\|\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{s,t}-p\|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&(1-s\tau)^2\|x_{s,t}-Q_r x_{s,t}\|^2\\
&\leq s^2\tau^2\|x_{s,t}-p\|^2+2s\max\{\|\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{s,t}-p\|.
\end{aligned}
$$

This together with (3.6), implies that

$$
\lim_{s\to 0}\|x_{s,t}-Q_r x_{s,t}\|=0 \quad\text{and}\quad \lim_{s\to 0}\|Q_r x_{s,t}-T_\lambda Q_r x_{s,t}\|=0. \tag{3.8}
$$

Now, observe that

$$
\begin{aligned}
\|x_{s,t}-p\|^2 &= \langle s\gamma(tVx_{s,t}+(1-t)Sx_{s,t})+(I-s\mu F)T_\lambda Q_r x_{s,t}-p,x_{s,t}-p\rangle\\
&=\langle (I-s\mu F)T_\lambda Q_r x_{s,t}-(I-s\mu F)T_\lambda Q_r p,x_{s,t}-p\rangle\\
&\quad+st\gamma\langle Vx_{s,t}-Vp,x_{s,t}-p\rangle+s(1-t)\gamma\langle Sx_{s,t}-Sp,x_{s,t}-p\rangle\\
&\quad+st\langle(\gamma V-\mu F)p,x_{s,t}-p\rangle+s(1-t)\langle(\gamma S-\mu F)p,x_{s,t}-p\rangle\\
&\leq [1-s\tau+st\gamma\rho+s(1-t)\gamma]\|x_{s,t}-p\|^2\\
&\quad+st\langle(\gamma V-\mu F)p,x_{s,t}-p\rangle+s(1-t)\langle(\gamma S-\mu F)p,x_{s,t}-p\rangle\\
&\leq (1-st\gamma(1-\rho))\|x_{s,t}-p\|^2+st\langle(\gamma V-\mu F)p,x_{s,t}-p\rangle\\
&\quad+s(1-t)\langle(\gamma S-\mu F)p,x_{s,t}-p\rangle.
\end{aligned}
$$

It turns out that

$$
\begin{aligned}
&\|x_{s,t}-p\|^2\\
&\leq \frac{1}{t\gamma(1-\rho)}\langle(t\gamma V+(1-t)\gamma S-\mu F)p,x_{s,t}-p\rangle,\quad \forall p\in \mathrm{MEP}(\Theta,\varphi)\cap\Gamma.
\end{aligned}
\tag{3.9}
$$

Assume $\{s_n\} \subset (0, \frac{1}{2})$ is such that $s_n \to 0$ ($\Leftrightarrow$ $\lambda_n \to \frac{2}{L}$). From (3.9), we obtain immediately that

$$\|x_{s_n,t} - p\|^2 \tag{3.10}$$
$$\leq \frac{1}{t\gamma(1-\rho)}\langle(t\gamma V+(1-t)\gamma S-\mu F)p, x_{s_n,t}-p\rangle, \ \forall p\in \mathrm{MEP}(\Theta,\varphi)\cap \Gamma.$$

Since $\{x_{s_n,t}\}$ is bounded, without loss of generality, we may assume that $\{x_{s_n,t}\}$ converges weakly to a point $x_t \in H$. From (3.8), we get $\|x_{s_n,t} - Q_r x_{s_n,t}\| = 0$ and $\|Q_r x_{s_n,t} - T_{\lambda_n} Q_r x_{s_n,t}\| \to 0$. So, we get $Q_r x_{s_n,t} \rightharpoonup x_t$. Utilizing Lemmas 2.2 and 2.4 we know that $x_t \in \mathrm{Fix}(Q_r) = \mathrm{MEP}(\Theta, \varphi)$. Now, observe that

$$\begin{aligned}
&\|P_C(I - \lambda_n \nabla f)Q_r x_{s_n,t} - Q_r x_{s_n,t}\| \\
&= \|s_n Q_r x_{s_n,t} + (1 - s_n)T_{\lambda_n} Q_r x_{s_n,t} - Q_r x_{s_n,t}\| \\
&= (1 - s_n)\|T_{\lambda_n} Q_r x_{s_n,t} - Q_r x_{s_n,t}\| \\
&\leq \|T_{\lambda_n} Q_r x_{s_n,t} - Q_r x_{s_n,t}\|,
\end{aligned}$$

where $s_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$. Hence, we have

$$\begin{aligned}
&\|P_C(I - \frac{2}{L}\nabla f)Q_r x_{s_n,t} - Q_r x_{s_n,t}\| \\
&\leq \|P_C(I - \frac{2}{L}\nabla f)Q_r x_{s_n,t} - P_C(I - \lambda_n \nabla f)Q_r x_{s_n,t}\| \\
&\quad +\|P_C(I - \lambda_n \nabla f)Q_r x_{s_n,t} - Q_r x_{s_n,t}\| \\
&\leq \|(I - \frac{2}{L}\nabla f)Q_r x_{s_n,t} - (I - \lambda_n \nabla f)Q_r x_{s_n,t}\| \\
&\quad +\|P_C(I - \lambda_n \nabla f)Q_r x_{s_n,t} - Q_r x_{s_n,t}\| \\
&\leq \|(I - \frac{2}{L}\nabla f)Q_r x_{s_n,t} - (I - \lambda_n \nabla f)Q_r x_{s_n,t}\| \\
&\quad +\|P_C(I - \lambda_n \nabla f)Q_r x_{s_n,t} - Q_r x_{s_n,t}\| \\
&\leq (\frac{2}{L} - \lambda_n)\|\nabla f(Q_r x_{s_n,t})\| + \|T_{\lambda_n} Q_r x_{s_n,t} - Q_r x_{s_n,t}\|.
\end{aligned}$$

From the boundedness of $\{Q_r x_{s_n,t}\}$, $s_n \to 0$ ($\Leftrightarrow$ $\lambda_n \to \frac{2}{L}$) and $\|Q_r x_{s_n,t} - T_{\lambda_n} Q_r x_{s_n,t}\| \to 0$, we conclude that

$$\lim_{n\to\infty} \|P_C(I - \frac{2}{L}\nabla f)Q_r x_{s_n,t} - Q_r x_{s_n,t}\| = 0.$$

Utilizing Lemma 2.4 we deduce from $Q_r x_{s_n,t} \rightharpoonup x_t$ that

$$x_t = P_C(I - \frac{2}{L}\nabla f)x_t.$$

This means that $x_t \in \Gamma$. Therefore, $x_t \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$. We can then substitute x_t for p in (3.10) to derive

$$\|x_{s_n,t} - x_t\|^2 \le \frac{1}{t\gamma(1-\rho)} \langle (t\gamma V + (1-t)\gamma S - \mu F)x_t, x_{s_n,t} - x_t \rangle.$$

Consequently, the weak convergence of $\{x_{s_n,t}\}$ to x_t actually implies that $x_{s_n,t} \to x_t$ strongly. This has proved the relative norm-compactness of the net $\{x_{s,t}\}$ as $s \to 0$.

Now, we return to (3.10) and take the limit as $n \to \infty$ to get

$$\|x_t - p\|^2 \le \frac{1}{t\gamma(1-\rho)} \langle (t\gamma V + (1-t)\gamma S - \mu F)p, x_t - p \rangle, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma.$$

In particular, x_t solves the HVI of finding $x_t \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ such that

$$\langle (t\gamma V + (1-t)\gamma S - \mu F)p, x_t - p \rangle \ge 0, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma,$$

that is,

$$\langle (\mu F - t\gamma V - (1-t)\gamma S)p, p - x_t \rangle \ge 0, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma.$$

Let us show that the mapping $\mu F - t\gamma V - (1-t)\gamma S$ is monotone. Indeed, we observe that for each $x, y \in H$,

$$\begin{aligned}
&\langle (\mu F - t\gamma V - (1-t)\gamma S)x - (\mu F - t\gamma V - (1-t)\gamma S)y, x - y \rangle \\
&= \mu \langle Fx - Fy, x - y \rangle - t\gamma \langle Vx - Vy, x - y \rangle - (1-t)\gamma \langle Sx - Sy, x - y \rangle \\
&\ge \mu\eta \|x - y\|^2 - t\gamma\rho \|x - y\|^2 - (1-t)\gamma \|x - y\|^2 \\
&= [(\mu\eta - \gamma) + t\gamma(1-\rho)] \|x - y\|^2.
\end{aligned}$$

Noticing the inequality $\mu\eta \ge \tau$ (the argument can be seen in the sequel), we conclude from $0 < \gamma \le \tau$ and $\rho \in [0, 1)$ that

$$(\mu\eta - \gamma) + t\gamma(1-\rho) \ge (\mu\eta - \tau) + t\gamma(1-\rho) > 0.$$

This shows that the mapping $(\mu F - t\gamma V - (1-t)\gamma S)$ is strongly monotone, and hence, monotone. It is easy to see that the mapping $(\mu F - t\gamma V - (1-t)\gamma S)$ is Lipschitz continuous. It is well known that the set $\mathrm{MEP}(\Theta, \varphi) \cap \Gamma \ne \emptyset$ is closed and convex. Then, by applying the well-known Minty lemma (see [22]) for the operator $(\mu F - t\gamma V - (1-t)\gamma S)$ and the set $\mathrm{MEP}(\Theta, \varphi) \cap \Gamma$, we conclude that x_t solves the Minty variational inequality of finding $x_t \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ such that

$$\langle (t\gamma V + (1-t)\gamma S - \mu F)x_t, x_t - p \rangle \ge 0, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma. \tag{3.11}$$

Note that (3.11) is equivalent to the fact that

$$x_t = P_{\mathrm{MEP}(\Theta, \varphi) \cap \Gamma}(I - \mu F + t\gamma V + (1-t)\gamma S)x_t.$$

That is, x_t is a unique fixed point in $\mathrm{MEP}(\Theta,\varphi)\cap\Gamma$ of the contraction

$$P_{\mathrm{MEP}(\Theta,\varphi)\cap\Gamma}(I-\mu F+t\gamma V+(1-t)\gamma S).$$

Obviously, this is sufficient to conclude that the entire net $\{x_{s,t}\}$ converges in norm to x_t as $s\to 0$.

Next, we show that as $t\to 0$, the net $\{x_t\}$ converges strongly to x^* which is a unique solution of the HVI (1.5).

In (3.11), we take any $y\in\Xi$ to derive

$$\langle (t\gamma V+(1-t)\gamma S-\mu F)x_t, x_t-y\rangle \geq 0. \tag{3.12}$$

Note that $0<\gamma\leq\tau$ and

$$\begin{aligned}\mu\eta\geq\tau &\Leftrightarrow \mu\eta\geq 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}\\ &\Leftrightarrow \sqrt{1-\mu(2\eta-\mu\kappa^2)}\geq 1-\mu\eta\\ &\Leftrightarrow 1-2\mu\eta+\mu^2\kappa^2\geq 1-2\mu\eta+\mu^2\eta^2\\ &\Leftrightarrow \kappa^2\geq\eta^2\\ &\Leftrightarrow \kappa\geq\eta.\end{aligned}$$

It is clear that

$$\langle(\mu F-\gamma S)x-(\mu F-\gamma S)y, x-y\rangle\geq(\mu\eta-\gamma)\|x-y\|^2,\quad \forall x,y\in H.$$

Hence, it follows from $0<\gamma\leq\tau\leq\mu\eta$ that $\mu F-\gamma S$ is monotone. Thus, we have

$$\langle\gamma Sx_t-\mu Fx_t, x_t-y\rangle\leq\langle\gamma Sy-\mu Fy, x_t-y\rangle\leq 0. \tag{3.13}$$

It follows from (3.11) and (3.12) that

$$\langle(\gamma V-\mu F)x_t, x_t-y\rangle\geq 0,\quad \forall y\in\Xi. \tag{3.14}$$

Hence,

$$\begin{aligned}\mu\eta\|x_t-y\|^2 &\leq \mu\langle Fx_t-Fy, x_t-y\rangle\\ &\leq\langle\mu Fy-\gamma Vx_t, y-x_t\rangle\\ &=\langle(\mu F-\gamma V)y, y-x_t\rangle+\gamma\langle Vy-Vx_t, y-x_t\rangle\\ &\leq\langle(\mu F-\gamma V)y, y-x_t\rangle+\gamma\rho\|y-x_t\|^2.\end{aligned}$$

Therefore,

$$\|x_t-y\|^2\leq\frac{1}{\mu\eta-\gamma\rho}\langle(\mu F-\gamma V)y, y-x_t\rangle,\quad\forall y\in\Xi. \tag{3.15}$$

In particular,

$$\|x_t - y\| \leq \frac{1}{\mu\eta - \gamma\rho}\|\mu F - \gamma V)y\|, \quad \forall t \in (0,1),$$

which implies that $\{x_t\}$ is bounded.

Next, let us show that $\omega_w(x_t) \subset \Xi$; namely, if $\{t_n\}$ is a null sequence in $(0,1)$ such that $x_{t_n} \rightharpoonup x'$, then $x' \in \Xi$. To see this, we use (3.11) to get

$$\langle(\mu F - \gamma S)x_t, p - x_t\rangle \geq \frac{t}{1-t}\langle(\mu F - \gamma V)x_t, p - x_t\rangle, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma.$$

However, since $\mu F - \gamma S$ is monotone,

$$\langle(\mu F - \gamma S)p, p - x_t\rangle \geq \langle(\mu F - \gamma S)x_t, p - x_t\rangle.$$

Combining the last two relations yields

$$\langle(\mu F - \gamma S)p, p - x_t\rangle \geq \frac{t}{1-t}\langle(\mu F - \gamma V)x_t, p - x_t\rangle, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma. \tag{3.16}$$

Letting $t = t_n \to 0$ as $n \to \infty$ in (3.16), we get

$$\langle(\mu F - \gamma S)p, p - x'\rangle \geq 0, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma. \tag{3.17}$$

Since $\mu F - \gamma S$ is monotone and Lipschitz continuous, and $\mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ is nonempty, closed and convex, by applying Minty lemma [22] on the set $\mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ and on the operator $\mu F - \gamma S$, the inequality (3.17) is equivalent to

$$\langle(\mu F - \gamma S)x', p - x'\rangle \geq 0, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma.$$

Namely, x' is a solution of the HVI (1.5); hence $x' \in \Xi$.

We further prove that $x' = x^*$, a unique solution of the THVI (1.4). As a matter of fact, it follows from (3.15) that for $x' \in \Xi$

$$\|x_{t_n} - x'\|^2 \leq \frac{1}{\mu\eta - \gamma\rho}\langle(\gamma V - \mu F)x', x_{t_n} - x'\rangle.$$

Therefore, the weak convergence to x' of $\{x_{t_n}\}$ implies that $x_{t_n} \to x'$ in norm. Now, we can let $t = t_n \to 0$ in (3.14) to get

$$\langle(\gamma V - \mu F)x', x' - y\rangle \geq 0, \quad \forall y \in \Xi.$$

It turns out that $x' \in \Xi$ solves the THVI (1.4). By uniqueness, we have $x' = x^*$. This is sufficient to guarantee that $x_t \to x^*$ in norm, as $t \to 0$.

Finally, put $V = 0$ and let $\{x_{s,t}\}$ be defined by the implicit scheme (3.2). Then the THVI (1.4) reduces to (3.5). Moreover, the iterated limit in the norm topology

$$s - \lim_{t\to 0}\lim_{s\to 0} x_{s,t}$$

exists and is the unique solution $x^* \in \Xi$ of VI (3.5). In addition, it is easy to see that for each null sequence $\{s_n\}$ (with $s_n = \frac{2-\lambda L}{4}$ for each $\lambda_n \in (0, \frac{2}{L})$), there exists another null sequence $\{t_n\}$ in $(0,1)$, such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$. This completes the proof. ■

In the above Theorem 3.1, put $\mu = 2,\ F = \frac{1}{2}I$ and $\gamma = \tau = 1$. Then the HVI (1.5) reduces to the HVI of finding $x^* \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ such that

$$\langle (I-S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma, \tag{3.18}$$

In this case, THVI (1.4) reduces to the VI of finding $x^* \in \Xi$ such that

$$\langle (I-V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi. \tag{3.19}$$

In terms of Theorem 3.1, for each fixed $t \in (0,1)$, the net $\{x_{s,t}\}$ converges in norm, as $s \to 0$ ($\Leftrightarrow\ \lambda \to \frac{2}{L}$), to a point $x_t \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$. Moreover, as $t \to 0$, the net $\{x_t\}$ converges in norm to the unique solution $x^* \in \Xi$ of VI (3.19). Hence, for each null sequence $\{s_n\}$ (with $s_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$), there exists another null sequence $\{t_n\}$ in $(0,1)$, such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.

Additionally, if we take $V = 0$, then VI (3.19) reduces to the following VI:

$$\text{find } x^* \in \Xi \text{ such that } \langle x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi,$$

which is equivalent to

$$x^* = P_\Xi(0).$$

Note that

$$x^* = P_\Xi(0) \ \Leftrightarrow\ \|0 - x^*\| \leq \|0 - y\| \ (\forall y \in \Xi) \ \Leftrightarrow\ \|x^*\| = \min_{y \in \Xi} \|y\|.$$

Thus, by Theorem 3.1, the iterated limit in the norm topology

$$s - \lim_{t \to 0} \lim_{s \to 0} x_{s,t}$$

exists and is the minimum-norm solution x^* of VI (3.18). Moreover, for each null sequence $\{s_n\}$ (with $s_n = \frac{2-\lambda_n L}{4} \in (0, \frac{1}{2})$ for each $\lambda_n \in (0, \frac{2}{L})$), there exists another null sequence $\{t_n\}$ in $(0,1)$, such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$. Therefore, we obtain the following conclusion.

Corollary 3.1. *Let $\Theta : C \times C \to \mathrm{R}$ be a bifunction satisfying (A1)-(A5). Let $f : C \to \mathrm{R}$ be a convex function such that ∇f is an L-Lipschitzian mapping with $L > 0$. Let $S : H \to H$ be a nonexpansive mapping and $V : H \to H$ be a ρ-contraction with coefficient $\rho \in [0,1)$. Suppose that the solution set Ξ of HVI (3.18)* *is nonempty and that either* $(B1)$ *or* $(B2)$ *holds. For each* $(s,t) \in (0,\frac{1}{2}) \times (0,1)$ *(with* $s = \frac{2-\lambda L}{4}$ *for each* $\lambda \in (0,\frac{2}{L})$*), let* $x_{s,t}$ *be defined implicitly by* (3.3)*. Then, for each fixed* $t \in (0,1)$*, the net* $\{x_{s,t}\}$ *converges in norm, as* $s \to 0$*, to a point* $x_t \in \mathrm{MEP}(\Theta,\varphi) \cap \Gamma$*. Moreover, as* $t \to 0$*, the net* $\{x_t\}$ *converges in norm to a unique solution* $x^* \in \Xi$ *of the VI* (3.19)*. Moreover, for each null sequence* $\{s_n\}$ *(with* $s_n = \frac{2-\lambda_n L}{4} \in (0,\frac{1}{2})$ *for each* $\lambda_n \in (0,\frac{2}{L})$*), there exists another null sequence* $\{t_n\}$ *in* $(0,1)$ *such that the sequence* $x_{s_n,t_n} \to x^*$ *in norm as* $n \to \infty$*.*

In particular, if we take $V = 0$ and if $x_{s,t}$ is defined by the implicit scheme (3.4), then the iterated limit in the norm topology

$$s - \lim_{t\to 0}\lim_{s\to 0} x_{s,t}$$

exists and is the minimum-norm solution x^* of the HVI (3.18). Furthermore, for each null sequence $\{s_n\}$ (with $s_n = \frac{2-\lambda_n L}{4} \in (0,\frac{1}{2})$ for each $\lambda_n \in (0,\frac{2}{L})$), there exists another null sequence $\{t_n\}$ in $(0,1)$ such that the sequence $x_{s_n,t_n} \to x^*$ in norm as $n \to \infty$.

Next, we introduce an explicit scheme for finding a unique solution of the THVI (1.4). This scheme is indeed obtained by discretizing the implicit scheme as investigated in the above. Recall that $P_C(I-\lambda_n\nabla f)$ is $\frac{2+\lambda_n L}{4}$-averaged for each $\lambda_n \in (0,\frac{2}{L})$. Therefore, we can write

$$P_C(I-\lambda_n\nabla f) = \frac{2-\lambda_n L}{4} I + \frac{2+\lambda_n L}{4} T_{\lambda_n} = s_n I + (1-s_n)T_{\lambda_n},$$

where T_{λ_n} is nonexpansive and $s_n := s_n(\lambda_n) = \frac{2-\lambda_n L}{4} \in (0,\frac{1}{2})$ for each $\lambda_n \in (0,\frac{2}{L})$. It is easy to see that

$$\lambda_n \to \frac{2}{L} \;\Leftrightarrow\; s_n \to 0.$$

Now, starting with an arbitrary initial guess $x_0 \in H$, we define a sequence $\{x_n\}$ iteratively by

$$x_{n+1} = s_n\gamma(t_nVx_n + (1-t_n)Sx_n) + (I - s_n\mu F)T_{\lambda_n}Q_{r_n}x_n, \quad \forall n \ge 0, \tag{3.20}$$

where the functions Θ,φ, the mappings S, V, F and the parameters μ,γ are the same as stated in the above, $\{r_n\}$ is a sequence in $(0,\infty)$ and $\{t_n\}$ is a sequence $(0,1)$. Additionally, if we take $V = 0$, then (3.20) reduces to the following iterative scheme:

$$x_{n+1} = s_n(1-t_n)\gamma Sx_n + (I - s_n\mu F)T_{\lambda_n}Q_{r_n}x_n, \quad \forall n \ge 0. \tag{3.21}$$

In particular, whenever $\mu = 2,\ F = \frac{1}{2}I$ and $\gamma = \tau = 1$, the explicit schemes (3.20) and (3.21) reduce to the following explicit schemes, respectively,

$$x_{n+1} = s_n(t_nVx_n + (1-t_n)Sx_n) + (1-s_n)T_{\lambda_n}Q_{r_n}x_n, \quad \forall n \ge 0, \tag{3.22}$$

and

$$x_{n+1} = s_n(1-t_n)Sx_n + (1-s_n)T_{\lambda_n}Q_{r_n}x_n, \quad \forall n \geq 0. \tag{3.23}$$

Comparing with the convergence of the implicit scheme (3.1), the convergence of the explicit scheme (3.20) seems much more subtle.

Theorem 3.2. *Let $\Theta : C \times C \to \mathrm{R}$ be a bifunction satisfying (A1)-(A5). Let $f : C \to \mathrm{R}$ be a convex function such that ∇f is an L-Lipschitzian mapping with $L > 0$. Let $S : H \to H$ be a nonexpansive mapping, $V : H \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1)$ and $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants κ and $\eta > 0$, respectively. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < \gamma \leq \tau$, where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)}$. Assume that the solution set Ξ of HVI (1.5) is nonempty, that either (B1) or (B2) holds and that the following conditions hold:*

(i) $\lim_{n\to\infty} t_n = 0$ *and* $\lim_{n\to\infty} s_n = 0$ *(*$\Leftrightarrow \lambda_n \to \frac{2}{L}$*);*

(ii) $\lim_{n\to\infty} \frac{r_n - r_{n-1}}{s_n^2 t_n} = 0$, $\lim_{n\to\infty} \frac{s_n t_n - s_{n-1}t_{n-1}}{s_n^2 t_n} = 0$ *and* $\lim_{n\to\infty} \frac{s_n - s_{n-1}}{s_n^2 s_{n-1} t_n} = 0$*;*

(iii) $\sum_{n=0}^{\infty} s_n t_n = \infty$ *and* $\liminf_{n\to\infty} r_n > 0$*;*

(iv) *there is a constant* $\bar{k} > 0$ *satisfying* $\|x - P_C(I - \frac{2}{L}\nabla f)x\| \geq \bar{k}[d(x, \mathrm{MEP}(\Theta, \varphi) \cap \Gamma)]$ *for each* $x \in C$*, where* $d(x, \mathrm{MEP}(\Theta, \varphi) \cap \Gamma) = \inf_{y \in \mathrm{MEP}(\Theta,\varphi)\cap\Gamma} \|x - y\|$*;*

(v) $\lim_{n\to\infty} \frac{s_n^{1/2}}{t_n} = 0$.

We have

(a) *If $\{x_n\}$ is the sequence generated by the scheme (3.20) and is bounded, then $\{x_n\}$ converges in norm to the point $x^* \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ which is a unique solution of the THVI (1.4).*

(b) *If $\{x_n\}$ is the sequence generated by the scheme (3.21) and is bounded, then $\{x_n\}$ converges in norm to a unique solution x^* of the VI of finding $x^* \in \Xi$ such that*

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi. \tag{3.24}$$

Proof. We treat only case (a); that is, the sequence $\{x_n\}$ is generated by the scheme (3.20). We divide the proof into several steps.

Step 1. $\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{s_n} = 0$.

Indeed, from (3.20), we observe that

$$\begin{aligned}
&x_{n+1} - x_n \\
&= s_n t_n \gamma(Vx_n - Vx_{n-1}) + s_n(1-t_n)\gamma(Sx_n - Sx_{n-1}) \\
&\quad +[(I - s_n\mu F)T_{\lambda_n}Q_{r_n}x_n - (I - s_n\mu F)T_{\lambda_{n-1}}Q_{r_{n-1}}x_{n-1}] \\
&\quad +(s_n t_n - s_{n-1}t_{n-1})\gamma[Vx_{n-1} - Sx_{n-1}] \\
&\quad +(s_n - s_{n-1})(\gamma Sx_{n-1} - \mu FT_{\lambda_{n-1}}Q_{r_{n-1}}x_{n-1}).
\end{aligned}$$

Thus, by Lemma 2.7 we obtain that

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\le s_n t_n \gamma\rho\|x_n-x_{n-1}\| + s_n(1-t_n)\gamma\|x_n-x_{n-1}\| \\
&\quad +(1-s_n\tau)\|T_{\lambda_n}Q_{r_n}x_n - T_{\lambda_{n-1}}Q_{r_{n-1}}x_{n-1}\| \\
&\quad +|s_n t_n - s_{n-1}t_{n-1}|\gamma\|Vx_{n-1}-Sx_{n-1}\| \\
&\quad +|s_n - s_{n-1}|\|\gamma Sx_{n-1} - \mu F T_{\lambda_{n-1}}Q_{r_{n-1}}x_{n-1}\| \\
&\le s_n t_n \gamma\rho\|x_n-x_{n-1}\| + s_n(1-t_n)\gamma\|x_n-x_{n-1}\| \\
&\quad +(1-s_n\tau)\|T_{\lambda_n}Q_{r_n}x_n - T_{\lambda_{n-1}}Q_{r_{n-1}}x_{n-1}\| \\
&\quad +M_1(|s_n t_n - s_{n-1}t_{n-1}| + |s_n - s_{n-1}|),
\end{aligned}
\tag{3.25}
$$

where $\gamma\|Vx_n - Sx_n\| + \|\gamma Sx_n - \mu F T_{\lambda_n}Q_{r_n}x_n\| \le M_1,\ \forall n \ge 0$ for some $M_1 > 0$.

Let $p \in \mathrm{MEP}(\Theta,\varphi) \cap \Gamma$. Since ∇f is $\frac{1}{L}$-ism, $P_C(I-\lambda_n\nabla f)$ is nonexpansive. So, it follows that

$$
\begin{aligned}
&\|P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1}\| \\
&\le \|P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1} - p\| + \|p\| \\
&= \|P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1} - P_C(I-\lambda_n\nabla f)p\| + \|p\| \\
&\le \|Q_{r_{n-1}}x_{n-1} - p\| + \|p\| \\
&\le \|x_{n-1} - p\| + \|p\|.
\end{aligned}
$$

This together with the boundedness of $\{x_n\}$ implies that $\{P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1}\}$ is bounded. Also, observe that

$$
\begin{aligned}
&\|T_{\lambda_n}Q_{r_{n-1}}x_{n-1} - T_{\lambda_{n-1}}Q_{r_{n-1}}x_{n-1}\| \\
&= \|\tfrac{4P_C(I-\lambda_n\nabla f)-(2-\lambda_n L)I}{2+\lambda_n L}Q_{r_{n-1}}x_{n-1} - \tfrac{4P_C(I-\lambda_{n-1}\nabla f)-(2-\lambda_{n-1}L)I}{2+\lambda_{n-1}L}Q_{r_{n-1}}x_{n-1}\| \\
&\le \|\tfrac{4P_C(I-\lambda_n\nabla f)}{2+\lambda_n L}Q_{r_{n-1}}x_{n-1} - \tfrac{4P_C(I-\lambda_{n-1}\nabla f)}{2+\lambda_{n-1}L}Q_{r_{n-1}}x_{n-1}\| \\
&\quad +\|\tfrac{2-\lambda_{n-1}L}{2+\lambda_{n-1}L}Q_{r_{n-1}}x_{n-1} - \tfrac{2-\lambda_n L}{2+\lambda_n L}Q_{r_{n-1}}x_{n-1}\| \\
&= \|\tfrac{4(2+\lambda_{n-1}L)P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1} - 4(2+\lambda_n L)P_C(I-\lambda_{n-1}\nabla f)Q_{r_{n-1}}x_{n-1}}{(2+\lambda_n L)(2+\lambda_{n-1}L)}\| \\
&\quad +\tfrac{4L|\lambda_n-\lambda_{n-1}|}{(2+\lambda_n L)(2+\lambda_{n-1}L)}\|Q_{r_{n-1}}x_{n-1}\| \\
&\le \|\tfrac{4L(\lambda_{n-1}-\lambda_n)P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1} + 4(2+\lambda_n L)(P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1} - P_C(I-\lambda_{n-1}\nabla f)Q_{r_{n-1}}x_{n-1})}{(2+\lambda_n L)(2+\lambda_{n-1}L)}\| \\
&\quad +\tfrac{4L|\lambda_n-\lambda_{n-1}|}{(2+\lambda_n L)(2+\lambda_{n-1}L)}\|Q_{r_{n-1}}x_{n-1}\| \\
&\le \tfrac{4L|\lambda_{n-1}-\lambda_n|\|P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1}\|}{(2+\lambda_n L)(2+\lambda_{n-1}L)} + \tfrac{4(2+\lambda_n L)\|P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1} - P_C(I-\lambda_{n-1}\nabla f)Q_{r_{n-1}}x_{n-1}\|}{(2+\lambda_n L)(2+\lambda_{n-1}L)} \\
&\quad +\tfrac{4L|\lambda_n-\lambda_{n-1}|}{(2+\lambda_n L)(2+\lambda_{n-1}L)}\|Q_{r_{n-1}}x_{n-1}\| \\
&\le |\lambda_n-\lambda_{n-1}|[L\|P_C(I-\lambda_n\nabla f)Q_{r_{n-1}}x_{n-1}\| + 4\|\nabla f(Q_{r_{n-1}}x_{n-1})\| + L\|Q_{r_{n-1}}x_{n-1}\|] \\
&\le M_2|\lambda_n-\lambda_{n-1}|,
\end{aligned}
$$

where $L\|P_C(I-\lambda_{n+1}\nabla f)Q_{r_n}x_n\| + 4\|\nabla f(Q_{r_n}x_n)\| + L\|Q_{r_n}x_n\| \le M_2,\ \forall n \ge 0$ for some $M_2 > 0$. This immediately implies that

$$
\begin{aligned}
&\|T_{\lambda_n}Q_{r_n}x_n - T_{\lambda_{n-1}}Q_{r_{n-1}}x_{n-1}\| \\
&\le \|T_{\lambda_n}Q_{r_n}x_n - T_{\lambda_n}Q_{r_{n-1}}x_{n-1}\| + \|T_{\lambda_n}Q_{r_{n-1}}x_{n-1} - T_{\lambda_{n-1}}Q_{r_{n-1}}x_{n-1}\| \\
&\le \|Q_{r_n}x_n - Q_{r_{n-1}}x_{n-1}\| + M_2|\lambda_n - \lambda_{n-1}| \\
&= \|Q_{r_n}x_n - Q_{r_{n-1}}x_{n-1}\| + \tfrac{4M_2}{L}|s_n - s_{n-1}|.
\end{aligned} \tag{3.26}
$$

Furthermore, for simplicity, we write $u_n = Q_{r_n}x_n$ for all $n \ge 0$. Then we have

$$
\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \tag{3.27}
$$

and

$$
\Theta(u_{n-1}, y) + \varphi(y) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C. \tag{3.28}
$$

Putting $y = u_{n-1}$ in (3.27) and $y = u_n$ in (3.28), respectively, we obtain that

$$
\Theta(u_n, u_{n-1}) + \varphi(u_{n-1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n-1} - u_n, u_n - x_n\rangle \ge 0,
$$

and

$$
\Theta(u_{n-1}, u_n) + \varphi(u_n) - \varphi(u_{n-1}) + \frac{1}{r_{n-1}}\langle u_n - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0.
$$

By (A2), we have

$$
\langle u_n - u_{n-1}, \frac{u_{n-1} - x_{n-1}}{r_{n-1}} - \frac{u_n - x_n}{r_n}\rangle \ge 0,
$$

and hence

$$
\langle u_n - u_{n-1}, u_{n-1} - u_n + u_n - x_{n-1} - \frac{r_{n-1}}{r_n}(u_n - x_n)\rangle \ge 0.
$$

Since $\liminf_{n\to\infty} r_n > 0$, we may assume, without loss of generality, that $r_n \ge c, \ \forall n \ge 0$ for some $c > 0$. Thus we have

$$
\begin{aligned}
\|u_n - u_{n-1}\|^2 &\le \langle u_n - u_{n-1}, x_n - x_{n-1} + (1 - \frac{r_{n-1}}{r_n})(u_n - x_n)\rangle \\
&\le \|u_n - u_{n-1}\|\{\|x_n - x_{n-1}\| + |1 - \frac{r_{n-1}}{r_n}|\|u_n - x_n\|\},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|u_n - u_{n-1}\| &\le \|x_n - x_{n-1}\| + \frac{1}{r_n}|r_n - r_{n-1}|\|u_n - x_n\| \\
&\le \|x_n - x_{n-1}\| + \frac{M_3}{c}|r_n - r_{n-1}|.
\end{aligned} \tag{3.29}
$$

where $\|u_n - x_n\| \leq M_3,\ \forall n \geq 0$ for some $M_3 > 0$. Therefore, from (3.25), (3.26) and (3.29) it follows that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq s_n t_n \gamma\rho\|x_n - x_{n-1}\| + s_n(1-t_n)\gamma\|x_n - x_{n-1}\| + (1-s_n\tau)[\|Q_{r_n}x_n - Q_{r_{n-1}}x_{n-1}\| \\
&\quad + \tfrac{4M_2}{L}|s_n - s_{n-1}|] + M_1(|s_n t_n - s_{n-1}t_{n-1}| + |s_n - s_{n-1}|) \\
&\leq s_n t_n \gamma\rho\|x_n - x_{n-1}\| + s_n(1-t_n)\gamma\|x_n - x_{n-1}\| + (1-s_n\tau)[\|x_n - x_{n-1}\| \\
&\quad + \tfrac{M_3}{c}|r_n - r_{n-1}| + \tfrac{4M_2}{L}|s_n - s_{n-1}|] + M_1(|s_n t_n - s_{n-1}t_{n-1}| + |s_n - s_{n-1}|) \\
&= [s_n t_n \gamma\rho + s_n(1-t_n)\gamma + (1-s_n\tau)]\|x_n - x_{n-1}\| \\
&\quad + (1-s_n\tau)[\tfrac{M_3}{c}|r_n - r_{n-1}| + \tfrac{4M_2}{L}|s_n - s_{n-1}|] + M_1(|s_n t_n - s_{n-1}t_{n-1}| + |s_n - s_{n-1}|) \\
&\leq (1 - s_n t_n \gamma(1-\rho))\|x_n - x_{n-1}\| + \tfrac{M_3}{c}|r_n - r_{n-1}| + \tfrac{4M_2}{L}|s_n - s_{n-1}| \\
&\quad + M_1(|s_n t_n - s_{n-1}t_{n-1}| + |s_n - s_{n-1}|) \\
&\leq (1 - s_n t_n \gamma(1-\rho))\|x_n - x_{n-1}\| + \tfrac{M_3}{c}|r_n - r_{n-1}| + (M_1 + \tfrac{4M_2}{L})|s_n - s_{n-1}| \\
&\quad + M_1|s_n t_n - s_{n-1}t_{n-1}|,
\end{aligned}$$

which hence leads to

$$\begin{aligned}
&\tfrac{\|x_{n+1} - x_n\|}{s_n} \\
&\leq (1 - s_n t_n \gamma(1-\rho))\tfrac{\|x_n - x_{n-1}\|}{s_n} + \tfrac{M_3}{c}\tfrac{|r_n - r_{n-1}|}{s_n} \\
&\quad + (M_1 + \tfrac{4M_2}{L})\tfrac{|s_n - s_{n-1}|}{s_n} + M_1\tfrac{|s_n t_n - s_{n-1}t_{n-1}|}{s_n} \\
&= (1 - s_n t_n \gamma(1-\rho))\tfrac{\|x_n - x_{n-1}\|}{s_{n-1}} + (1 - s_n t_n \gamma(1-\rho))(\tfrac{\|x_n - x_{n-1}\|}{s_n} - \tfrac{\|x_n - x_{n-1}\|}{s_{n-1}}) \\
&\quad + \tfrac{M_3}{c}\tfrac{|r_n - r_{n-1}|}{s_n} + (M_1 + \tfrac{4M_2}{L})\tfrac{|s_n - s_{n-1}|}{s_n} + M_1\tfrac{|s_n t_n - s_{n-1}t_{n-1}|}{s_n} \\
&\leq (1 - s_n t_n \gamma(1-\rho))\tfrac{\|x_n - x_{n-1}\|}{s_{n-1}} + s_n t_n\|x_n - x_{n-1}\|\tfrac{1}{s_n t_n}|\tfrac{1}{s_n} - \tfrac{1}{s_{n-1}}| \\
&\quad + s_n t_n[\tfrac{M_3}{c}\tfrac{|r_n - r_{n-1}|}{s_n^2 t_n} + (M_1 + \tfrac{4M_2}{L})\tfrac{|s_n - s_{n-1}|}{s_n^2 t_n} + M_1\tfrac{|s_n t_n - s_{n-1}t_{n-1}|}{s_n^2 t_n}].
\end{aligned}$$

By conditions (ii) and (iii), we can apply Lemma 2.3 to the last inequality to conclude

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{s_n} = 0.$$

Step 2. $\lim_{n\to\infty}\|x_n - u_n\| = 0,\ \lim_{n\to\infty}\|u_n - T_{\lambda_n}u_n\| = 0$ and $\lim_{n\to\infty}\|u_n - P_C(I - \frac{2}{L}\nabla f)u_n\| = 0$, where $u_n := Q_{r_n}x_n$ for all $n \geq 0$.

Indeed, since

$$x_{n+1} - T_{\lambda_n}u_n = s_n[\gamma(t_n V x_n + (1-t_n)Sx_n) - \mu F T_{\lambda_n}u_n],$$

we have

$$\begin{aligned} &\|x_n - T_{\lambda_n} u_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_{\lambda_n} u_n\| \\ &\le \|x_n - x_{n+1}\| + s_n[t_n\|\gamma V x_n - \mu F T_{\lambda_n} u_n\| + (1-t_n)\|\gamma S x_n - \mu F T_{\lambda_n} u_n\|] \\ &\le \|x_n - x_{n+1}\| + s_n \max\{\|\gamma V x_n - \mu F T_{\lambda_n} u_n\|, \|\gamma S x_n - \mu F T_{\lambda_n} u_n\|\}. \end{aligned} \tag{3.30}$$

Therefore, from $s_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ it follows that

$$\lim_{n\to\infty} \|x_n - T_{\lambda_n} u_n\| = 0.$$

Noticing the firmly nonexpansivity of Q_{r_n}, for each $p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ we have

$$\begin{aligned} \|u_n - p\|^2 &= \|Q_{r_n} x_n - Q_{r_n} p\|^2 \\ &\le \langle x_n - p, u_n - p\rangle \\ &= \frac{1}{2}(\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2), \end{aligned}$$

and hence

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

Utilizing Lemmas 2.6 and 2.7 we know that

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &= \|s_n\gamma(t_n V x_n + (1-t_n) S x_n) + (I - s_n\mu F) T_{\lambda_n} u_n - p\|^2 \\ &= \|[(I - s_n\mu F) T_{\lambda_n} u_n - (I - s_n\mu F) T_{\lambda_n} p] + s_n t_n \gamma(V x_n - V p) \\ &\quad + s_n(1-t_n)\gamma(S x_n - S p) + s_n t_n(\gamma V - \mu F)p + s_n(1-t_n)(\gamma S - \mu F)p\|^2 \\ &\le \|[(I - s_n\mu F) T_{\lambda_n} u_n - (I - s_n\mu F) T_{\lambda_n} p] + s_n t_n \gamma(V x_n - V p) \\ &\quad + s_n(1-t_n)\gamma(S x_n - S p)\|^2 \\ &\quad + 2s_n[t_n\langle(\gamma V - \mu F)p, x_{n+1} - p\rangle + (1-t_n)\langle(\gamma S - \mu F)p, x_{n+1} - p\rangle] \\ &\le [(1 - s_n\tau)\|u_n - p\| + s_n t_n \gamma\rho\|x_n - p\| + s_n(1-t_n)\gamma\|x_n - p\|]^2 \\ &\quad + 2s_n \max\{\|(\gamma V - \mu F)p\|, \|(\gamma S - \mu F)p\|\}\|x_{n+1} - p\| \\ &= [(1 - s_n\tau)\|u_n - p\| + s_n\gamma(1 - (1-\rho)t_n)\|x_n - p\|]^2 \\ &\quad + 2s_n \max\{\|(\gamma V - \mu F)p\|, \|(\gamma S - \mu F)p\|\}\|x_{n+1} - p\| \\ &\le [(1 - s_n\tau)\|u_n - p\| + s_n\gamma\|x_n - p\|]^2 \\ &\quad + 2s_n \max\{\|\gamma V - \mu F)p\|, \|\gamma S - \mu F)p\|\}\|x_{n+1} - p\| \\ &\le (1 - s_n\tau)\|u_n - p\|^2 + s_n\tfrac{\gamma^2}{\tau}\|x_n - p\|^2 \\ &\quad + 2s_n \max\{\|(\gamma V - \mu F)p\|, \|(\gamma S - \mu F)p\|\}\|x_{n+1} - p\| \\ &\le (1 - s_n\tau)(\|x_n - p\|^2 - \|x_n - u_n\|^2) + s_n\tfrac{\gamma^2}{\tau}\|x_n - p\|^2 \\ &\quad + 2s_n \max\{\|(\gamma V - \mu F)p\|, \|(\gamma S - \mu F)p\|\}\|x_{n+1} - p\| \\ &= (1 - s_n\tfrac{\tau^2 - \gamma^2}{\tau})\|x_n - p\|^2 - (1 - s_n\tau)\|x_n - u_n\|^2 \end{aligned}$$

$$\begin{aligned}&+2s_n\max\{\|(\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{n+1}-p\|\\ \le\ &\|x_n-p\|^2-(1-s_n\tau)\|x_n-u_n\|^2\\ &+2s_n\max\{\|(\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{n+1}-p\|,\end{aligned}$$

which hence implies that

$$\begin{aligned}&(1-s_n\tau)\|x_n-u_n\|^2\\ \le\ &\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &+2s_n\max\{\|(\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{n+1}-p\|\\ \le\ &(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\ &+2s_n\max\{\|(\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{n+1}-p\|.\end{aligned}\tag{3.31}$$

Since $s_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$, we get

$$\lim_{n\to\infty}\|x_n-u_n\|=0.$$

Thus, from $\|x_n-T_{\lambda_n}u_n\|\to 0$, we also have

$$\|u_n-T_{\lambda_n}u_n\|\le\|u_n-x_n\|+\|x_n-T_{\lambda_n}u_n\|\to 0,\quad \text{as } n\to\infty.$$

That is,

$$\lim_{n\to\infty}\|u_n-T_{\lambda_n}u_n\|=0.$$

Furthermore, utilizing the arguments similar to those in the proof of Theorem 3.1, we can derive

$$\lim_{n\to\infty}\|u_n-P_C(I-\frac{2}{L}\nabla f)u_n\|=0.$$

Step 3. $\omega_w(x_n)\subset\Xi\subset\mathrm{MEP}(\Theta,\varphi)\cap\Gamma$.

Indeed, since H is reflexive and $\{x_n\}$ is bounded, there exists a weakly convergent subsequence of $\{x_n\}$ and hence $\omega_w(x_n)$ is nonempty. Let $q\in\omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i}\rightharpoonup q$. Hence it is clear from $\|x_n-u_n\|\to 0$ that $u_{n_i}\rightharpoonup q$. Utilizing Lemma 2.4, from $\|u_n-P_C(I-\frac{2}{L}\nabla f)u_n\|\to 0$ we obtain $q=P_C(I-\frac{2}{L}\nabla f)q$. This means that $q\in\Gamma$.

Now, let us show that $q\in\mathrm{MEP}(\Theta,\varphi)$. Since $u_n=Q_{r_n}x_n$, for each $y\in C$ we have

$$\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0.$$

It follows from (A2) that

$$\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge\Theta(y,u_n).$$

Replacing n by n_i, we have

$$\varphi(y) - \varphi(u_{n_i}) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq \Theta(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup q$, it follows from (A4) that

$$0 \geq -\varphi(y) + \varphi(q) + \Theta(y, q), \quad \forall y \in C.$$

Put $z_t = ty + (1-t)q$ for all $t \in (0, 1]$ and $y \in C$. Then we have $z_t \in C$ and

$$-\varphi(z_t) + \varphi(q) + \Theta(z_t, q) \leq 0,$$

which together with (A1) and (A4), implies that

$$\begin{aligned}
0 &= \Theta(z_t, z_t) + \varphi(z_t) - \varphi(z_t) \\
&\leq t\Theta(z_t, y) + (1-t)\Theta(z_t, q) + t\varphi(y) + (1-t)\varphi(q) - \varphi(z_t) \\
&\leq t(\Theta(z_t, y) + \varphi(y) - \varphi(z_t)) + (1-t)(\Theta(z_t, q) + \varphi(q) - \varphi(z_t)) \\
&\leq t(\Theta(z_t, y) + \varphi(y) - \varphi(z_t)),
\end{aligned}$$

and hence

$$0 \leq \Theta(z_t, y) + \varphi(y) - \varphi(z_t).$$

From (A3), we conclude that as $t \to 0$,

$$0 \leq \Theta(q, y) + \varphi(y) - \varphi(q), \quad \forall y \in C.$$

This leads to $q \in \mathrm{MEP}(\Theta, \varphi)$. So, we get $q \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$. Therefore, $\omega_w(x_n) \subset \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$.

On the other hand, from (3.20) we observe that

$$\begin{aligned}
&x_n - x_{n+1} \\
&= s_n t_n(\mu F - \gamma V)x_n + s_n(1-t_n)(\mu F - \gamma S)x_n \\
&\quad + (1-s_n)(I - T_{\lambda_n}Q_{r_n})x_n + s_n[(I - \mu F)x_n - (I - \mu F)T_{\lambda_n}Q_{r_n}x_n].
\end{aligned} \tag{3.32}$$

Set

$$y_n = \frac{x_n - x_{n+1}}{s_n(1-t_n)}, \quad \forall n \geq 0.$$

It can be easily seen from (3.32) that

$$\begin{aligned}
y_n &= (\mu F - \gamma S)x_n + \frac{t_n}{1-t_n}(\mu F - \gamma V)x_n \\
&\quad + \frac{1-s_n}{s_n(1-t_n)}(I - T_{\lambda_n}Q_{r_n})x_n + \frac{1}{1-t_n}[(I - \mu F)x_n - (I - \mu F)T_{\lambda_n}Q_{r_n}x_n].
\end{aligned}$$

This yields that, for each $p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$,

$$\begin{aligned} & \langle y_n, x_n - p\rangle \\ & = \langle (\mu F - \gamma S)x_n, x_n - p\rangle + \tfrac{t_n}{1-t_n}\langle (\mu F - \gamma V)x_n, x_n - p\rangle \\ & \quad + \tfrac{1-s_n}{s_n(1-t_n)}\langle (I - T_{\lambda_n}Q_{r_n})x_n - (I - T_{\lambda_n}Q_{r_n})p, x_n - p\rangle \\ & \quad + \tfrac{1}{1-t_n}\langle (I - \mu F)x_n - (I - \mu F)T_{\lambda_n}Q_{r_n}x_n, x_n - p\rangle \\ & = \langle (\mu F - \gamma S)p, x_n - p\rangle + \langle (\mu F - \gamma S)x_n - (\mu F - \gamma S)p, x_n - p\rangle \\ & \quad + \tfrac{1-s_n}{s_n(1-t_n)}\langle (I - T_{\lambda_n}Q_{r_n})x_n - (I - T_{\lambda_n}Q_{r_n})p, x_n - p\rangle \\ & \quad + \tfrac{t_n}{1-t_n}\langle (\mu F - \gamma V)x_n, x_n - p\rangle \\ & \quad + \tfrac{1}{1-t_n}\langle (I - \mu F)x_n - (I - \mu F)T_{\lambda_n}Q_{r_n}x_n, x_n - p\rangle. \end{aligned} \tag{3.33}$$

In (3.33), the second and third terms are also nonnegative due to the monotonicity of $\mu F - \gamma S$ and $I - T_{\lambda_n}Q_{r_n}$. We, therefore, deduce from (3.33) that

$$\begin{aligned} \langle y_n, x_n - p\rangle & \geq \langle (\mu F - \gamma S)p, x_n - p\rangle + \tfrac{t_n}{1-t_n}\langle (\mu F - \gamma V)x_n, x_n - p\rangle \\ & \quad + \tfrac{1}{1-t_n}\langle (I - \mu F)x_n - (I - \mu F)T_{\lambda_n}Q_{r_n}x_n, x_n - p\rangle. \end{aligned} \tag{3.34}$$

Note that $\|x_n - T_{\lambda_n}Q_{r_n}x_n\| \to 0$ (due to Step 2) implies $\|(I - \mu F)x_n - (I - \mu F)T_{\lambda_n}Q_{r_n}x_n\| \to 0$. Also, since $\|y_n\| \to 0$ by (3.31), $t_n \to 0$ and $\{x_n\}$ is bounded by assumption which implies that $\{\mu F - \gamma V)x_n\}$ is bounded, we obtain from (3.34) that

$$\limsup_{n\to\infty}\langle (\mu F - \gamma S)p, x_n - p\rangle \leq 0, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma. \tag{3.35}$$

This suffices to guarantee that $q \in \Xi$. As a matter of fact, since $x_{n_i} \rightharpoonup q \in \omega_w(x_n)$, we deduce from (3.35) that

$$\langle (\mu F - \gamma S)p, q - p\rangle \leq \limsup_{n\to\infty}\langle (\mu F - \gamma S)p, x_n - p\rangle \leq 0, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma,$$

that is,

$$\langle (\mu F - \gamma S)p, p - q\rangle \geq 0, \quad \forall p \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma.$$

Since $\mu F - \gamma S$ is monotone and Lipschitz continuous, and $\mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ is nonempty, closed and convex, by the Minty Lemma [22] the last inequality is equivalent to the inequality (1.5). Thus, we derive $q \in \Xi$. This shows that $\omega_w(x_n) \subset \Xi$.

Step 4. $x_n \to x^* \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$, which is a unique solution of the THVI (1.4).

Indeed, since the mapping $\mu F - \gamma V$ is $(\mu\eta - \gamma\rho)$-strongly monotone and $\mu\kappa + \gamma\rho$-Lipschitz continuous, namely,

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma\rho)\|x - y\|^2, \quad \forall x, y \in H$$

and

$$\|(\mu F - \gamma V)x - (\mu F - \gamma V)y\| \leq (\mu\kappa + \gamma\rho)\|x - y\|, \quad \forall x, y \in H,$$

there exists a unique solution of the VI of finding $x^* \in \Xi$ such that

$$\langle(\mu F - \gamma V)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \Xi.$$

We take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)x^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle(\gamma V - \mu F)x^*, x_{n_j} - x^*\rangle.$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \tilde{x}$; then $\tilde{x} \in \Xi$ as we just proved. Since x^* is a solution of the THVI (1.4), we get

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)x^*, x_n - x^*\rangle = \langle(\gamma V - \mu F)x^*, \tilde{x} - x^*\rangle \leq 0. \tag{3.36}$$

From (3.20), it follows that (noticing $0 < \gamma \leq \tau$)

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \langle(I - s_n\mu F)T_{\lambda_n}u_n - (I - s_n\mu F)T_{\lambda_n}x^*, x_{n+1} - x^*\rangle \\
&\quad + s_n t_n\gamma\langle Vx_n - Vx^*, x_{n+1} - x^*\rangle \\
&\quad + s_n(1 - t_n)\gamma\langle Sx_n - Sx^*, x_{n+1} - x^*\rangle + s_n t_n\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + s_n(1 - t_n)\langle(\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\leq (1 - s_n\tau)\|u_n - x^*\|\|x_{n+1} - x^*\| + s_n t_n\gamma\rho\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + s_n(1 - t_n)\gamma\|x_n - x^*\|\|x_{n+1} - x^*\| + s_n t_n\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + s_n(1 - t_n)\langle(\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\leq [(1 - s_n\tau) + s_n t_n\gamma\rho + s_n(1 - t_n)\gamma]\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad + s_n t_n\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + s_n(1 - t_n)\langle(\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\leq [1 - s_n t_n\gamma(1 - \rho)]\|x_n - x^*\|\|x_{n+1} - x^*\| + s_n t_n\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + s_n(1 - t_n)\langle(\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\leq [1 - s_n t_n\gamma(1 - \rho)]\tfrac{1}{2}(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + s_n t_n\langle(\gamma V - \mu F)x^*, x_{n+1} - x^*\rangle \\
&\quad + s_n(1 - t_n)\langle(\gamma S - \mu F)x^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

It turns out that

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&\le \frac{1-s_nt_n\gamma(1-\rho)}{1+s_nt_n\gamma(1-\rho)}\|x_n-x^*\|^2+\frac{2}{1+s_nt_n\gamma(1-\rho)}[s_nt_n\langle(\gamma V-\mu F)x^*, x_{n+1}-x^*\rangle\\
&\quad +s_n(1-t_n)\langle(\gamma S-\mu F)x^*, x_{n+1}-x^*\rangle]\\
&\le [1-s_nt_n\gamma(1-\rho)]\|x_n-x^*\|^2+\frac{2}{1+s_nt_n\gamma(1-\rho)}[s_nt_n\langle(\gamma V-\mu F)x^*, x_{n+1}-x^*\rangle\\
&\quad +s_n(1-t_n)\langle(\gamma S-\mu F)x^*, x_{n+1}-x^*\rangle]\\
&= [1-s_nt_n\gamma(1-\rho)]\|x_n-x^*\|^2+\frac{2s_nt_n}{1+s_nt_n\gamma(1-\rho)}[\langle(\gamma V-\mu F)x^*, x_{n+1}-x^*\rangle\\
&\quad +\frac{(1-t_n)}{t_n}\langle(\gamma S-\mu F)x^*, x_{n+1}-x^*\rangle]\\
&= (1-\gamma_n)\|x_n-x^*\|^2+\delta_n,
\end{aligned}
\tag{3.37}
$$

where

$$\gamma_n = s_nt_n\gamma(1-\rho),$$

and

$$\delta_n = \frac{2s_nt_n}{1+s_nt_n\gamma(1-\rho)}[\langle(\gamma V-\mu F)x^*, x_{n+1}-x^*\rangle+\frac{(1-t_n)}{t_n}\langle(\gamma S-\mu F)x^*, x_{n+1}-x^*\rangle].$$

However, since $x^* \in \Xi$, by assumption (iv) we obtain that

$$
\begin{aligned}
&\langle(\gamma S-\mu F)x^*, x_{n+1}-x^*\rangle\\
&= \langle(\gamma S-\mu F)x^*, x_{n+1}-u_n\rangle + \langle(\gamma S-\mu F)x^*, u_n-P_{\mathrm{MEP}(\Theta,\varphi)\cap\Gamma}u_n\rangle\\
&\quad +\langle(\gamma S-\mu F)x^*, P_{\mathrm{MEP}(\Theta,\varphi)\cap\Gamma}u_n-x^*\rangle\\
&\le \langle(\gamma S-\mu F)x^*, x_{n+1}-u_n\rangle + \langle(\gamma S-\mu F)x^*, u_n-P_{\mathrm{MEP}(\Theta,\varphi)\cap\Gamma}u_n\rangle\\
&\le \|(\gamma S-\mu F)x^*\|\|x_{n+1}-u_n\|+\|(\gamma S-\mu F)x^*\|d(u_n, \mathrm{MEP}(\Theta,\varphi)\cap\Gamma)\\
&\le \|(\gamma S-\mu F)x^*\|\|x_{n+1}-u_n\|+\|(\gamma S-\mu F)x^*\|\tfrac{1}{k}\|u_n-P_C(I-\tfrac{2}{L}\nabla f)u_n\|\\
&\le \|(\gamma S-\mu F)x^*\|[\|x_{n+1}-u_n\| + \tfrac{1}{k}(\|u_n-P_C(I-\lambda_n\nabla f)u_n\|\\
&\quad +\|P_C(I-\lambda_n\nabla f)u_n-P_C(I-\tfrac{2}{L}\nabla f)u_n\|)]\\
&\le \|(\gamma S-\mu F)x^*\|[\|x_{n+1}-u_n\| + \tfrac{1}{k}(\|u_n-P_C(I-\lambda_n\nabla f)u_n\|\\
&\quad +\|(I-\lambda_n\nabla f)u_n-(I-\tfrac{2}{L}\nabla f)u_n\|)]\\
&= \|(\gamma S-\mu F)x^*\|[\|x_{n+1}-u_n\| + \tfrac{1}{k}(\|u_n-P_C(I-\lambda_n\nabla f)u_n\|\\
&\quad +(\tfrac{2}{L}-\lambda_n)\|\nabla f(u_n)\|)]\\
&= \|(\gamma S-\mu F)x^*\|[\|x_{n+1}-u_n\|\\
&\quad +\frac{1}{k}(\|u_n-P_C(I-\lambda_n\nabla f)u_n\| + \frac{4s_n}{L}\|\nabla f(u_n)\|)].
\end{aligned}
\tag{3.38}
$$

In addition, it is clear from (3.31) that

$$\begin{aligned}
&(1-s_n\tau)\frac{\|x_n-u_n\|^2}{t_n^2}\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\frac{\|x_n-x_{n+1}\|}{t_n^2}\\
&+2\frac{s_n}{t_n^2}\max\{\|(\gamma V-\mu F)p\|,\|(\gamma S-\mu F)p\|\}\|x_{n+1}-p\|.
\end{aligned}$$

Since $\frac{s_n}{t_n^2}\to 0$ and $\frac{\|x_n-x_{n+1}\|}{t_n^2}=\frac{\|x_n-x_{n+1}\|}{s_n}\cdot\frac{s_n}{t_n^2}\to 0$, we have

$$\lim_{n\to\infty}\frac{\|x_n-u_n\|}{t_n}=0.$$

Meantime, it is also clear from (3.30) that

$$\lim_{n\to\infty}\frac{\|x_n-T_{\lambda_n}u_n\|}{t_n}=0.$$

From (3.38) it follows that

$$\begin{aligned}
&\frac{1}{t_n}\langle(\gamma S-\mu F)x^*,x_{n+1}-x^*\rangle\\
&\le \|(\gamma S-\mu F)x^*\|[\tfrac{\|x_{n+1}-u_n\|}{t_n}+\tfrac{1}{k}(\tfrac{\|u_n-P_C(I-\lambda_n\nabla f)u_n\|}{t_n}+\tfrac{4s_n}{Lt_n}\|\nabla f(u_n)\|)]\\
&\le \|(\gamma S-\mu F)x^*\|[\tfrac{\|x_{n+1}-x_n\|+\|x_n-u_n\|}{t_n}+\tfrac{1}{k}(\tfrac{(1-s_n)\|u_n-T_{\lambda_n}u_n\|}{t_n}+\tfrac{4s_n}{Lt_n}\|\nabla f(u_n)\|)]\\
&\le \|(\gamma S-\mu F)x^*\|[\tfrac{\|x_{n+1}-x_n\|+\|x_n-u_n\|}{t_n}+\tfrac{1}{k}(\tfrac{\|u_n-x_n\|+\|x_n-T_{\lambda_n}u_n\|}{t_n}+\tfrac{4s_n}{Lt_n}\|\nabla f(u_n)\|)],
\end{aligned}$$

which leads to

$$\limsup_{n\to\infty}\frac{1}{t_n}\langle(\gamma S-\mu F)x^*,x_{n+1}-x^*\rangle\le 0. \tag{3.39}$$

Consequently, from (3.36), (3.39) and assumption (v) we infer that in inequality (3.37),

$$\limsup_{n\to\infty}\delta_n/\gamma_n\le 0\quad\text{and}\quad\sum_{n=0}^{\infty}\gamma_n=\infty.$$

Therefore, we can apply Lemma 2.3 to (3.37) to conclude that $x_n\to x^*$. The proof of part (a) is complete. It is easy to see that part (b) now becomes a straightforward consequence of part (a) since, if $V=0$, THVI (1.4) reduces to VI (3.24). This completes the proof. ■

In the above Theorem 3.2, put $\mu=2,\ F=\frac{1}{2}I$ and $\gamma=\tau=1$. Then the HVI (1.5) reduces to the HVI (3.18). In this case, the THVI (1.4) reduces to the VI (3.40). In terms of Theorem 3.2 (a), $\{x_n\}$ converges in norm to the point $x^*\in \mathrm{MEP}(\Theta,\varphi)\cap\Gamma$ which is a unique solution of VI (3.40).

Additionally, if we take $V=0$, then VI (3.24) reduces to the following VI:

$$\text{find } x^*\in\Xi \text{ such that } \langle x^*,x-x^*\rangle\ge 0,\quad\forall x\in\Xi,$$

which is equivalent to

$$x^* = P_{\Xi}(0).$$

Note that

$$x^* = P_{\Xi}(0) \;\Leftrightarrow\; \|0 - x^*\| \leq \|0 - y\| \;(\forall y \in \Xi) \;\Leftrightarrow\; \|x^*\| = \min_{y \in \Xi} \|y\|.$$

Thus, by Theorem 3.2 (b), $\{x_n\}$ converges in norm to the minimum-norm solution of the HVI (3.18). Therefore, we get the following conclusion.

Corollary 3.2. *Let $\Theta : C \times C \to \mathrm{R}$ be a bifunction satisfying (A1)-(A5). Let $f : C \to \mathrm{R}$ be a convex function such that ∇f is an L-Lipschitzian mapping with $L > 0$. Let $S : H \to H$ be a nonexpansive mapping and $V : H \to H$ be a ρ-contraction with coefficient $\rho \in [0, 1)$. Assume that the solution set Ξ of HVI (3.18) is nonempty, that either (B1) or (B2) holds and that the following conditions hold:*

(i) $\lim_{n\to\infty} t_n = 0$ *and* $\lim_{n\to\infty} s_n = 0$ *($\Leftrightarrow \lambda_n \to \frac{2}{L}$);*

(ii) $\lim_{n\to\infty} \frac{r_n - r_{n-1}}{s_n^2 t_n} = 0,\ \lim_{n\to\infty} \frac{s_n t_n - s_{n-1} t_{n-1}}{s_n^2 t_n} = 0$ *and* $\lim_{n\to\infty} \frac{s_n - s_{n-1}}{s_n^2 s_{n-1} t_n} = 0$*;*

(iii) $\sum_{n=0}^{\infty} s_n t_n = \infty$ *and* $\liminf_{n\to\infty} r_n > 0$*;*

(iv) *there is a constant $\bar{k} > 0$ satisfying $\|x - P_C(I - \frac{2}{L}\nabla f)x\| \geq \bar{k}[d(x, \mathrm{MEP}(\Theta, \varphi) \cap \Gamma)]$ for each $x \in C$, where $d(x, \mathrm{MEP}(\Theta, \varphi) \cap \Gamma) = \inf_{y \in \mathrm{MEP}(\Theta,\varphi)\cap\Gamma} \|x - y\|$;*

(v) $\lim_{n\to\infty} \frac{s_n^{1/2}}{t_n} = 0$. *We have*

(a) *If $\{x_n\}$ is the sequence generated by the scheme (3.22) and is bounded, then $\{x_n\}$ converges in norm to the point $x^* \in \mathrm{MEP}(\Theta, \varphi) \cap \Gamma$ which is a unique solution of the VI of finding $x^* \in \Xi$ such that*

$$\langle (I - V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Xi. \tag{3.40}$$

(b) *If $\{x_n\}$ is the sequence generated by the scheme (3.23) and is bounded, then $\{x_n\}$ converges in norm to a minimum-norm of the HVI (3.18).*

Remark 3.1. As an example, we consider the following sequences:

(i) $\{s_n\}$ and $\{t_n\}$ are essentially chosen the same as in [24], that is,

$$s_n = \frac{1}{(n+1)^s} \quad \text{and} \quad t_n = \frac{1}{(n+1)^t}.$$

(ii) $\{r_n\}$ is chosen as

$$r_n = \frac{1}{(n+1)^s} + \frac{1}{2}.$$

Conditions (i)-(iii) of Theorem 3.2 are satisfied provided $0 < s, t < 1$ and $2s + t \leq 1$. Also, condition (v) is satisfied provided $s/t > 2$.

4. Concluding Remarks

We consider a variational inequality with a variational inequality constraint over the intersection of the solution set of a mixed equilibrium problem (MEP) and the solution set of a minimization problem (MP) for a convex and continuously Fréchet differential functional, called a triple hierarchical variational inequality (THVI) over the common solution set of minimization and equilibrium problems. It is worth pointing out that the class of triple hierarchical variational inequalities over the fixed point set of a nonexpansive mapping has been introduced and studied in the setting of Hilbert spaces; see Ceng, Ansari and Yao [1]. In [1], the authors combined the regularization method, the hybrid steepest-descent method, and the projection method to propose an implicit scheme that generates a net in an implicit way, and derived its strong convergence to a unique solution of the THVI over the fixed point set of a nonexpansive mapping. Meantime, the authors also proposed an explicit scheme that generates a sequence via an iterative algorithm and proved that the sequence converges strongly to the unique solution of the THVI over the fixed point set of a nonexpansive mapping. In this paper, the THVI over the fixed point set of a nonexpansive mapping in [1] is extended to develop our triple hierarchical variational inequality (THVI) over the common solution set of minimization and equilibrium problems. Combining the hybrid steepest-descent method, the viscosity approximation method and the averaged mapping approach to the gradient-projection algorithm, we propose two iterative schemes: implicit and explicit ones, to compute the approximate solutions of the THVI over the common solution set of minimization and equilibrium problems. The convergence analysis of the proposed schemes is also studied. That is, it is proven not only that the net generated by the proposed implicit scheme converges strongly to the unique solution of the THVI over the common solution set of minimization and equilibrium problems but also that the sequence generated by the proposed explicit scheme converges strongly to the unique solution of the THVI over the common solution set of minimization and equilibrium problems. The argument techniques in our Theorems 3.1 and 3.2 are very different from the argument ones in [23, Theorems 3.1 and 3.2] because we use the properties of resolvent operator of Θ and φ and the averaged mapping approach to the gradient-projection algorithm.

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