TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 4, pp. 1413-1423, August 2013
DOI: 10.11650/tjm.17.2013.2814
This paper is available online at http://journal.taiwanmathsoc.org.tw

FIXED POINTS OF MEROMORPHIC SOLUTIONS FOR DIFFERENCE RICCATI EQUATION

Ye-Yang Jiang and Zong-Xuan Chen*

Abstract. In this paper, we investigate fixed points of meromorphic functions $f(z)$ for difference Riccati equations, and obtain some estimates of exponents of convergence of fixed points of $f(z)$ and shifts $f(z+n)$, differences $\triangle f(z) = f(z+1) - f(z)$ and divided differences $\frac{\triangle f(z)}{f(z)}$.

1. Introduction

In this paper, we assume that the reader is familiar with the standard notations and basic results of Nevanlinna's value distribution theory (see [11, 15]). In addition, we use the notions $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the exponents of convergence of zeros and poles of $f(z)$, respectively. We also use the notion $\tau(f)$ to denote the exponent of convergence of fixed points of f that is defined as

$$\tau(f) = \overline{\lim_{r\to\infty}} \frac{\log N(r, \frac{1}{f-z})}{\log r}.$$

Early results for difference equations were motivated by the work of Kimura [14] on the iteration of analytic functions. Bank and Kaufman [2], Shimomura [17] and Yanagihara [18] studied complex non-linear difference equations from the viewpoint of Nevanlinna theory and obtained a series of original results on the existence of meromorphic solutions of complex difference equations. As the difference analogues of Nevanlinna's theory are being investigated, many results on the complex difference

Received February 1, 2013, accepted February 4, 2013.
Communicated by Alexander Vasiliev.
2010 *Mathematics Subject Classification*: 30D35, 39B12.
Key words and phrases: Riccati equation, Meromorphic function, Difference, Fixed points.
This work is supported by the National Natural Science Foundation of China (No. 11171119) and the Tianyuan Fund for Mathematics (No.11226090).
*Corresponding author.

equations are got rapidly. Many papers [1, 3, 5, 7, 13, 16] mainly deal with the growth of meromorphic solutions of difference equations.

In [9], Halburd and Korhonen use value distribution theory to single out the difference Painlevé II equation from a large class of difference equations of the form

$$y(z+1)+y(z-1)=\frac{c_2y^2+c_1y+c_0}{y^2-p^2},$$

where $c_j's$, $p(\not\equiv 0)$ are rational functions. In their proof, Halburd and Korhonen are concerned with the Riccati difference equations of the form

$$w(z+1)=\frac{A+\delta w(z)}{\delta-w(z)},$$

where A is a polynomial, $\delta=\pm 1$ (see [10, p. 197]).

In [13], Ishizaki illustrates that the difference Riccati equation

$$f(z+1)=\frac{A(z)+f(z)}{1-f(z)}$$

and the second linear difference equation

$$y(z+2)-2y(z+1)+(A(z)+1)y(z)=0$$

are closed related by the passage

$$f(z)=-\frac{\triangle y(z)}{y(z)}.$$

From the above, we see that the difference Riccati equation is an important class of difference equation, it will play an important role in research of difference equations.

Recently, many papers [3, 4, 6, 13] deal with complex difference Riccati equations. In [4], Chen and Shon investigated the existence and forms of rational solutions, and the Borel exceptional value, zeros, poles and fixed points of transcendental solutions, and they proved the following theorem.

Theorem A. *Let $\delta=\pm 1$ be a constant and $A(z)=\frac{m(z)}{n(z)}$ be an irreducible nonconstant rational function, where $m(z)$ and $n(z)$ are polynomials with* $\deg m(z)=m$ *and* $\deg n(z)=n$.

If $f(z)$ is a transcendental finite order meromorphic solution of the difference Riccati equation

$$f(z+1)=\frac{A(z)+\delta f(z)}{\delta-f(z)}, \tag{1.1}$$

then

(i) *if* $\sigma(f) > 0$, *then* $f(z)$ *has at most one Borel exceptional value;*

(ii) $\lambda(\frac{1}{f}) = \lambda(f) = \sigma(f)$;

(iii) *if* $A(z) \not\equiv -z^2 - z + 1$, *then the exponent of convergence of fixed points of* f *satisfies* $\tau(f) = \sigma(f)$.

In this paper, we continue to investigate the difference Riccati equation (1.1), and obtain some estimates of fixed points of difference and shift of meromorphic solutions of (1.1). We prove the following theorem.

Theorem 1.1. *Let* $\delta = \pm 1$ *be a constant and* $A(z)$ *be a nonconstant rational function, not be a polynomial with* $\deg A = 2$. *Set* $\triangle f(z) = f(z+1) - f(z)$. *Then every finite order transcendental meromorphic solution of the difference Riccati equation* (1.1) *satisfies:*

(i) $\tau(f(z+n)) = \sigma(f(z))$ $(n = 0, 1, 2, \cdots)$;

(ii) *if there is a rational function* $m(z)$ *satisfying*

$$m^2(z) = \left(\frac{z}{1+z}\right)^2 - \frac{4A(z)}{1+z},$$

then $\tau\left(\frac{\triangle f(z)}{f(z)}\right) = \sigma(f(z))$;

(iii) *if there is a rational function* $n(z)$ *satisfying* $n^2(z) = z^2 - 4A(z)$, *then* $\tau(\triangle f(z)) = \sigma(f(z))$.

Remark 1.2 Generally, for a meromorphic function $f(z)$, $\tau(f(z+n)) \neq \tau(f(z))$. For example, the function $f(z) = e^{z^2} + z$ satisfies

$$\tau(f(z)) = 0, \quad \text{but} \quad \tau(f(z+n)) = 2 \ (n = 1, 2, \cdots).$$

2. Lemmas for Proof of Theorem 1.1

Firstly we need the following lemma for the proof of Theorem 1.1.

Lemma 2.1. (See [8, 16]). *Let* $w(z)$ *is a nonconstant finite order transcendental meromorphic solution of the difference equation of*

$$P(z, w) = 0$$

where $P(z, w)$ *is a difference polynomial in* $w(z)$. *If* $P(z, \alpha) \not\equiv 0$ *for a meromorphic function* $\alpha(z)$ *satisfying* $T(r, \alpha) = S(r, w)$, *then*

$$m\left(r, \frac{1}{w - \alpha}\right) = S(r, w)$$

holds for all r *outside of a possible exceptional set with finite logarithmic measure.*

Lemma 2.2. *Let $A(z)$ be a nonconstant rational function, and $f(z)$ be a nonconstant meromorphic function. Then*

$$y_1(z) = (1+z)f(z) + A(z) - z \quad \textit{and} \quad y_2(z) = 1 - f(z)$$

have at most finitely many common zeros.

Proof Suppose that z_0 is a common zero of $y_1(z)$ and $y_2(z)$. Then $y_1(z_0) = 1 - f(z_0) = 0$. Thus, $f(z_0) = 1$. Substituting $f(z_0) = 1$ into $y_1(z)$, we obtain

$$y_1(z_0) = (1+z_0) + A(z_0) - z_0 = A(z_0) + 1 = 0.$$

Since $A(z)$ is a nonconstant rational function, $A(z) + 1$ has only finitely many zeros. Thus, $y_1(z)$ and $y_2(z)$ have at most finitely many common zeros.

Lemma 2.3. (See [7]). *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f), \sigma < +\infty$, and let η be a fixed non zero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Using the same discussion as Lemma 2.2, we have the following two lemmas.

Lemma 2.4. *Let $A(z)$ be a nonconstant rational function, and $f(z)$ be a nonconstant meromorphic function. Then*

$$y_1(z) = (1+z)f^2(z) - zf(z) + A(z) \quad \textit{and} \quad y_2(z) = (1 - f(z))f(z)$$

have at most finitely many common zeros.

Lemma 2.5. *Let $A(z)$ be a nonconstant rational function, and $f(z)$ be a nonconstant meromorphic function. Then*

$$y_1(z) = f^2(z) + zf(z) + A(z) - z \quad \textit{and} \quad y_2(z) = 1 - f(z)$$

have at most finitely many common zeros.

3. Proof of Theorem 1.1

Suppose that $\delta = 1$. We only prove the case $\delta = 1$. We can use the same method to prove the case $\delta = -1$.

(i) We prove that $\tau(f(z+n)) = \sigma(f(z))$ $(n = 0, 1, 2, \cdots)$. Firstly, if $n = 0$, by theorem A(iii), the conclusion holds.

Now suppose that $n = 1$. By (1.1), we obtain

$$\begin{aligned} f(z+1) - z &= \frac{A(z) + f(z)}{1 - f(z)} - z = \frac{(1+z)f(z) + A(z) - z}{1 - f(z)} \\ &= \frac{(1+z)\left(f(z) + \frac{A(z)-z}{1+z}\right)}{1 - f(z)}. \end{aligned} \tag{3.1}$$

Since $A(z)$ is a nonconstant rational function, by (3.1), we know that $f(z)+\frac{A(z)-z}{1+z}$ and $1-f(z)$ have the same poles, except possibly finitely many. By Lemma 2.2, we see that $(1+z)f(z)+A(z)-z$ and $1-f(z)$ have at most finitely many common zeros. Hence, by (3.1), we have that

$$\tau(f(z+1))=\lambda(f(z+1)-z)=\lambda\left(f(z)+\frac{A(z)-z}{1+z}\right). \tag{3.2}$$

Suppose that $\lambda\left(f(z)+\frac{A(z)-z}{1+z}\right)<\sigma(f(z))$. Thus, $f(z)+\frac{A(z)-z}{1+z}$ can be rewritten as form

$$f(z)+\frac{A(z)-z}{1+z}=z^s\frac{P_0(z)}{Q_0(z)}e^{h(z)}=\frac{P(z)}{Q(z)}, \tag{3.3}$$

where $h(z)$ is a polynomial with $\deg h(z)\le\sigma(f(z)), P_0(z)$ and $Q_0(z)$ are canonical products ($P_0(z)$ may be a polynomial) formed by nonzero zeros and poles of $f(z)+\dfrac{A(z)-z}{1+z}$, respectively, s is an integer, if $s\ge 0$, then $P(z)=z^sP_0(z), Q(z)=Q_0(z)e^{-h(z)}$; if $s<0$, then $P(z)=P_0(z), Q(z)=z^{-s}Q_0(z)e^{-h(z)}$. Combining Theorem A with property of canonical product, we have

$$\begin{cases}\lambda(P(z))=\sigma(P(z))=\lambda\left(f(z)+\frac{A(z)-z}{1+z}\right)<\sigma(f(z)),\\ \lambda(Q(z))=\sigma(Q(z))=\sigma(f(z))\end{cases} \tag{3.4}$$

by (3.3), we obtain

$$f(z)=\frac{z-A(z)}{1+z}+P(z)y(z),\ f(z+1)=\frac{z+1-A(z+1)}{2+z}+P(z+1)y(z+1), \tag{3.5}$$

where $y(z)=\frac{1}{Q(z)}$. Thus, by (3.4), we have

$$\sigma(y(z))=\sigma(Q(z))=\sigma(f(z)),\ \sigma(P(z+1))=\sigma(P(z))<\sigma(f(z)). \tag{3.6}$$

Substituting (3.5) into (1.1), we obtain

$$\begin{aligned}K_1(z,y):=&\left(P(z+1)y(z+1)+\frac{z+1-A(z+1)}{2+z}\right)\\ &\cdot\left(1-\frac{z-A(z)}{1+z}-P(z)y(z)\right)-A(z)-P(z)y(z)-\frac{z-A(z)}{1+z}\\ =&\,0\end{aligned}$$

and

$$\begin{aligned}
K_1(z,0) :&= \frac{z+1-A(z+1)}{2+z}\cdot\frac{1+A(z)}{1+z} - A(z) - \frac{z-A(z)}{1+z}\\
&= \frac{(z+1-A(z+1)-z(1+z))\,(1+A(z))}{(2+z)(1+z)}\\
&= \frac{\left(1-z^2-A(z+1)\right)(1+A(z))}{(2+z)(1+z)}.
\end{aligned} \tag{3.7}$$

Since $A(z)$ is neither a constant nor a polynomial with $\deg A = 2$, we see that $1 - z^2 - A(z+1) \not\equiv 0$ and $1 + A(z) \not\equiv 0$, so that

$$K_1(z,0) \not\equiv 0. \tag{3.8}$$

Thus, by (3.4), (3.8) and Lemma 2.1, we obtain for any given $\varepsilon\,(0 < \varepsilon < \sigma(f(z)) - \sigma(P(z)))$

$$N\left(r, \frac{1}{y(z)}\right) = T(r, y(z)) + S(r, y(z)) + O\left(r^{\sigma(P(z))+\varepsilon}\right) \tag{3.9}$$

holds for all r outside of a possible exceptional set with finite logarithmic measure. On the other hand, by $y(z) = \dfrac{1}{Q(z)}$ and the fact that $Q(z)$ is an entire function, we see that

$$N\left(r, \frac{1}{y(z)}\right) = 0,$$

which contradicts (3.9). Hence, $\lambda\left(f(z) + \dfrac{A(z)-z}{1+z}\right) = \sigma(f(z))$. By (3.2), we have

$$\tau(f(z+1)) = \sigma(f(z)).$$

Now suppose that $n = 2$. By (1.1), we obtain

$$g(z+1) = \frac{A(z+1)+g(z)}{1-g(z)}, \tag{3.10}$$

where $g(z) = f(z+1)$. By Lemma 2.3, we know that $\sigma(g(z)) = \sigma(f(z))$. By the assumption, we know that $A(z+1)$ is a nonconstant rational function and is not a polynomial with $\deg A = 2$. Thus for (3.10), applying the conclusion of $n = 1$ above, we have

$$\tau(f(z+2)) = \tau(g(z+1)) = \sigma(g(z)) = \sigma(f(z)). \tag{3.11}$$

Continuing to use the same method as above, we can obtain

$$\tau(f(z+n)) = \sigma(f(z))\,(n = 1, 2, \cdots).$$

(ii) Suppose that there is a rational function $m(z)$ satisfying

$$m^2(z) = (\frac{z}{1+z})^2 - \frac{4A(z)}{1+z}. \tag{3.12}$$

Now we prove that $\tau\left(\frac{\triangle f(z)}{f(z)}\right) = \sigma(f(z))$. By (1.1), we have

$$\begin{aligned} \frac{\triangle f(z)}{f(z)} - z &= \frac{f(z+1)-f(z)}{f(z)} - z = \frac{(1+z)f^2(z) - zf(z) + A(z)}{(1-f(z))f(z)} \\ &= \frac{(1+z)\left(f(z) - \frac{\frac{z}{1+z}+m(z)}{2}\right)\left(f(z) - \frac{\frac{z}{1+z}-m(z)}{2}\right)}{(1-f(z))f(z)}. \end{aligned} \tag{3.13}$$

Since $\frac{z}{1+z}$ and $m(z)$ are rational functions, we see that $(1+z)f^2(z) - zf(z) + A(z)$ and $(1-f(z))f(z)$ have the same poles, except possibly finitely many. By Lemma 2.4 and (3.13), in order to prove $\tau\left(\frac{\triangle f(z)}{f(z)}\right) = \sigma(f(z))$, we only need to prove that

$$\lambda\left(f(z) - \frac{1}{2}\left(\frac{z}{1+z} - m(z)\right)\right) = \sigma(f(z)) \tag{3.14}$$

or

$$\lambda\left(f(z) - \frac{1}{2}\left(\frac{z}{1+z} + m(z)\right)\right) = \sigma(f(z)). \tag{3.15}$$

Now we prove that (3.14) holds. Suppose that $\lambda\left(f(z) - \frac{1}{2}\left(\frac{z}{1+z} - m(z)\right)\right) < \sigma(f(z))$. Using a similar method as in the proof of (i), we see that $f(z) - \frac{1}{2}(\frac{z}{1+z} + m(z))$ can be rewritten as form

$$f(z) = \frac{\frac{z}{1+z} - m(z)}{2} + P_2(z)y_2(z), \tag{3.16}$$

where $y_2(z) = \frac{1}{H_2(z)}$, $P_2(z)$ and $H_2(z)$ are nonzero entire functions, such that

$$\lambda(P_2(z)) = \sigma(P_2(z)) < \sigma(f(z)) \text{ and } \lambda(H_2(z)) = \sigma(H_2(z)) = \sigma(f(z))$$

Substituting (3.16) into (1.1), we have

$$\begin{aligned} K_2(z, y_2) :=& \left(\frac{z+1}{2z+4} - \frac{1}{2}m(z+1) + P_2(z+1)y_2(z+1)\right) \\ &\cdot \left(\frac{z+2}{2z+2} - \frac{1}{2}m(z) + P_2(z)y_2(z)\right) - A(z) - \frac{z}{2z+2} + \frac{1}{2}m(z) - P_2(z)y_2(z) \\ =& 0 \end{aligned}$$

and

$$
\begin{aligned}
&K_2(z,0):\\
&=\left(\frac{z+1}{2z+4}-\frac{1}{2}m(z+1)\right)\left(\frac{z+2}{2z+2}-\frac{1}{2}m(z)\right)-A(z)-\frac{z}{2z+2}+\frac{1}{2}m(z)\\
&=\frac{1-z}{4z+4}-\frac{z+2}{4z+4}m(z+1)+\frac{3z+5}{4z+8}m(z)-\frac{1}{4}m(z+1)m(z)-A(z)\\
&=\frac{1-z}{4z+4}-\frac{z+2}{4z+4}m(z+1)+\frac{3z+5}{4z+8}m(z)+(1+z)m^2(z)\\
&\quad-\frac{z^2}{z+1}-\frac{1}{4}m(z+1)m(z).
\end{aligned}
$$

We affirm that $K_2(z,0)\not\equiv 0$. We set $m(z)=\frac{S(z)}{T(z)}$, where $S(z)$ and $T(z)$ are mutually prime polynomials with $\deg S(z)=s$ and $\deg T(z)=t$. If $\deg m(z)=s-t>0$, we see that in the representation of $K_2(z,0)$, there exists only term being of the highest degree with $\deg(1+z)m^2(z)=1+2(s-t)$, then $K_2(z,0)\not\equiv 0$. If $\deg m(z)=s-t<0$, we have that $\dfrac{z^2}{z+1}$ is the unique highest degree term, then $K_2(z,0)\not\equiv 0$. If $\deg m(z)=s-t=0$, we obtain that $(1+z)m^2(z)$ and $\dfrac{z^2}{z+1}$ have the highest degree. If $(1+z)m^2(z)-\dfrac{z^2}{z+1}\equiv 0$, by (3.12), then $A(z)\equiv 0$, which contradicts the assumption. Hence, $K_2(z,0)\not\equiv 0$.

Using the same method as in the proof of (i), we see that (3.15) holds.

(iii) Suppose that there is a rational function $n(z)$ satisfying

$$n^2(z)=z^2+4z-4A(z), \tag{3.17}$$

then $\tau(\triangle f(z))=\sigma(f(z))$. By (1.1), we obtain

$$\triangle f(z)-z=f(z+1)-f(z)-z=\frac{f^2(z)+zf(z)+A(z)-z}{1-f(z)}. \tag{3.18}$$

By (3.17) and (3.18), we have

$$f(z+1)-f(z)-z=\frac{\left(f(z)+\dfrac{z-n(z)}{2}\right)\left(f(z)+\dfrac{z+n(z)}{2}\right)}{1-f(z)}. \tag{3.19}$$

Since $A(z)$ is a rational function, we know that poles of $1-f(z)$ must be poles of $f^2(z)+zf(z)+A(z)-z$. Thus, poles of $1-f(z)$ are not zeros of $f(z+1)-f(z)-z$. By Lemma 2.5, we see that the numerator and denominator of the right side of (3.18) have at most finite many common zeros. Thus, in order to prove $\tau(\triangle f(z)) = \sigma(f(z))$, by (3.19), we only need to prove that

$$\lambda\left(f(z) + \frac{z - n(z)}{2}\right) = \sigma(f(z)) \tag{3.20}$$

or

$$\lambda\left(f(z) + \frac{z + n(z)}{2}\right) = \sigma(f(z)). \tag{3.21}$$

Now we prove that (3.20) holds. Suppose that

$$\lambda\left(f(z) + \frac{z - n(z)}{2}\right) < \sigma(f(z)).$$

Using a similar method as in the proof of (i), we see that $f(z) + \frac{z - n(z)}{2}$ can be rewritten as form

$$f(z) = \frac{1}{2}n(z) - \frac{1}{2}z + P_3(z)y_3(z). \tag{3.22}$$

where $y_3(z) = \frac{1}{H_3(z)}$, $P_3(z)$ and $H_3(z)$ are non-zero entire functions such that

$$\lambda(P_3(z)) = \sigma(P_3(z)) < \sigma(f(z)) \quad \text{and} \quad \lambda(H_3(z)) = \sigma(H_3(z)) = \sigma(f(z))$$

Substituting (3.22) into (1.1), we have

$$\begin{aligned} K_3(z, y_3(z)) :=& \left[\frac{1}{2}n(z+1) - \frac{1}{2}(z+1) + P_3(z+1)y_3(z+1)\right] \cdot \\ & \left[1 - \frac{1}{2}n(z) + \frac{1}{2}z - P_3(z)y_3(z)\right] - A(z) - \frac{1}{2}n(z) + \frac{1}{2}z - P_3(z)y_3(z). \end{aligned}$$

By (3.17), we have

$$\begin{aligned} K_3(z, 0) :=& \left(\frac{1}{2}n(z+1) - \frac{1}{2}(z+1)\right)\left(1 - \frac{1}{2}n(z) + \frac{1}{2}z\right) - A(z) - \frac{1}{2}n(z) + \frac{1}{2}z \\ =& \frac{1}{4}(z+2)n(z+1) - \frac{1}{4}(z^2 + z + 2) - \frac{1}{4}n(z+1)n(z) \\ & + \frac{1}{4}(z-1)n(z) - A(z) \\ =& \frac{1}{4}(z+2)n(z+1) - \frac{1}{4}(z^2 + 3z + 2) - \frac{1}{4}n(z+1)n(z) \\ & + \frac{1}{4}(z-1)n(z) - \frac{1}{4}(z^2 + z) + \frac{1}{4}n^2(z) \\ =& \frac{1}{4}n(z)[n(z) - n(z+1)] + \frac{1}{4}z[n(z) + n(z+1)] \\ & + \frac{1}{4}[n(z+1) - n(z)] - \frac{1}{4}(2z^2 + 5z + 2). \end{aligned}$$

We affirm that $K_3(z,0) \not\equiv 0$. In fact, we set $n(z) = \dfrac{S_1(z)}{T_1(z)}$, where $S_1(z)$ and $T_1(z)$ are mutually prime polynomials with $\deg S_1(z) = s_1$ and $\deg T_1(z) = t_1$. If $s_1 - t_1 \leq 0$, we see that in the representation of $K_3(z,0)$, there exists only term being of the highest degree with $\deg \frac{1}{4}(2z^2+5z+2) = 2$, then $K_3(z,0) \not\equiv 0$. If $\deg n(z) = s_1 - t_1 > 0$, then $n(z)$ can be rewritten as $\dfrac{S_0(z)}{T_0(z)} + n_0(z)$, where $S_0(z)$, $T_0(z)$, $n_0(z)$ are polynomials with $\deg S_0(z) \leq \deg T_0(z)$ and $\deg n_0(z) = s_1 - t_1$. It is sufficient to consider $n_0(z)$ in place of $n_1(z)$. If $\deg n_0(z) = 1$ or 2, we may set $n_0(z) = az + b$ or $n_0(z) = az^2 + bz + c$. Substituting $n_0(z) = az + b$ or $n_0(z) = az^2 + bz + c$ into $K_3(z,0)$, and combining the coefficients of the polynomial $K_3(z,0)$, we have that $K_3(z,0) \not\equiv 0$. If $\deg n_0 \geq 3$, combining the degree of every terms in $K_3(z,0)$, we obtain that $\deg \dfrac{1}{4} n_0(z)[n_0(z) - n_0(z+1)]$ is the highest degree, then $K_3(z,0) \not\equiv 0$.

Using the same method as in the proof of (i), we know that (3.20) holds.

Thus, Theorem 1.1 is proved.

References

1. M. Ablowitz, R. G. Halburd and Herbst, On the extension of Painlevé property to difference equations, *Nonlinearty*, **13** (2000), 889-905.
2. S. B. Bank and R. P. Kaufman, An extension of Hölder's theoremconcerning the Gamma function, *Funkcial. Ekvac.*, **19** (1976), 53-63.
3. Z. X. Chen, On growth, zeros and poles of meromorphic functions of linear and nonlinear difference equations, *Sci. China, Ser. A*, **54** (2011), 2123-2133.
4. Z. X. Chen and K. H. Shon, Some results on Riccati equations, *Acta Mathematica Sinica*, **27** (2011), 1091-1100.
5. Z. X. Chen and K. H. Shon, On zeros and fixed points of difference of meromorphic functions, *J. Math. Anal. Appl.*, **344** (2008), 373-383.
6. Z. X. Chen, Z. Huang and R. Zhang, On difference equations relationg to Gamma function, *Acta Mathematica Sinica*, **31** (2011), 1281-1294.
7. Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, *Ramanujan J.*, **16** (2008), 105-129.
8. R. G. Halburd and R. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, *J. Math. Anal. Appl.*, **314** (2006), 477-487.
9. R. G. Halburd and R. Korhonen, Existence of finite-order meromorphic solutions as a detector of integrability in difference equations, *Physica D.*, **218** (2006), 191-203.
10. R. G. Halburd and R. Korhonen, Finite-order meromorphic solutions and the discrete Painlevé equations, *Proc. London Math. Soc.*, **94** (2007), 443-474.

11. W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

12. E. Hille, *Ordinary Differential Equations in the Complex Domain*, Dover Publications, Inc., Mineola, NY, 1997.

13. K. Ishizaki, On difference Riccati equations and second order linear difference equations, *Aequat. Math.*, **81** (2011), 185-198.

14. T. Kimura, On the iteration of analytic functions, *Funkcial. Ekvac.*, **14** (1971), 197-238.

15. I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, 1993.

16. I. Laine and C. C. Yang, Clunie theorems for difference and q-difference polymials, *J. Lond. Math. Soc.*, **76** (2007), 556-566.

17. S. Shimomura, Entire solutions of a polynomial difference equation, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **28** (1981), 253-266.

18. N. Yanagihara, Meromorphic solutions of some difference equations, *Funkcial. Ekvac.*, **23** (1980), 309-326.

Ye-Yang Jiang
School of Mathematics
South China Normal University
Guangzhou, Guangdong 510631
P. R. China
E-mail: jyyang1018@yahoo.com.cn

Zong-Xuan Chen
School of Mathematics
South China Normal University
Guangzhou, Guangdong 510631
P. R. China
E-mail: chzx@vip.sina.com