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A LEWENT TYPE DETERMINANTAL INEQUALITY

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Abstract. We prove a Lewent type determinantal inequality: Let A_i, $i = 1, \dots, n$, be (strictly) contractive trace class operators over a separable Hilbert space. Then

$$\left|\det\left(\frac{I+\sum_{i=1}^{n}\lambda_i A_i}{I-\sum_{i=1}^{n}\lambda_i A_i}\right)\right| \le \prod_{i=1}^{n}\det\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i},$$

where $\sum_{i=1}^{n}\lambda_i = 1$, $\lambda_i \ge 0$, $i = 1, \dots, n$, are (scalar) weights and $|A| = (A^*A)^{1/2}$.

1. Main Results

All the operators considered in this paper are trace class operators over a separable Hilbert space except I, the identity operator. The (Fredholm) determinant of $I + A$ is thus well defined and the trace functional of A is denoted by $\operatorname{tr} A$. A positive operator A (written $A \ge 0$) has a unique positive square root B with $B^2 = A$; we write $B = A^{1/2}$. For two self-adjoint operators A, B, we say $A \ge B$ whenever $A - B \ge 0$. For any operator A, one defines its absolute value $|A| = (A^*A)^{1/2}$; A is (strictly) contractive if $\|A\| < 1$, such an A is called a contraction.

We start with an elegant result as part of the motivation for this article.

Theorem 1. *Let A, B be trace class operators. Then*

$$|\det(I + A + B)| \le \det(I + |A|)\det(I + |B|). \tag{1.1}$$

This interesting inequality was discovered first by Grothendieck (unpublished; see [3]) and reproved many times; see, for example, [9, 10, 7]. A quick application of Bernoulli's inequality gives, for contractions A, B and $\lambda \in [0, 1]$,

$$\begin{aligned} \det(I+|A|)^{\lambda}\det(I+|B|)^{1-\lambda} &\le \det(I+\lambda|A|)\det(I+(1-\lambda)|B|);\\ \det(I-|A|)^{\lambda}\det(I-|B|)^{1-\lambda} &\le \det(I-\lambda|A|)\det(I-(1-\lambda)|B|). \end{aligned} \tag{1.2}$$

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If A, B are positive contractions, then using a result of Ky Fan [4, p. 467] gives

$$\begin{aligned} \det(I+A)^{\lambda}\det(I+B)^{1-\lambda} &\leq \det(I+\lambda A+(1-\lambda)B); \\ \det(I-A)^{\lambda}\det(I-B)^{1-\lambda} &\leq \det(I-\lambda A-(1-\lambda)B). \end{aligned} \tag{1.3}$$

In view of (1.1), (1.3) is stronger than (1.2) in the positive case.

The following numerical inequality is due to Lewent [6]; see also [5]:

$$\frac{1+\sum_{i=1}^{n}\lambda_i x_i}{1-\sum_{i=1}^{n}\lambda_i x_i} \leq \prod_{i=1}^{n}\left(\frac{1+x_i}{1-x_i}\right)^{\lambda_i}, \tag{1.4}$$

where $x_i \in [0,1)$ and $\sum_{i=1}^{n}\lambda_i = 1$, $\lambda_i \geq 0$, $i = 1, \ldots, n$, are (scalar) weights.

This paper proves an analogue of (1.4) for determinant functional.

Theorem 2. *Let A_i, $i = 1, \ldots, n$, be contractive trace class operators. Then*

$$\left|\det\left(\frac{I+\sum_{i=1}^{n}\lambda_i A_i}{I-\sum_{i=1}^{n}\lambda_i A_i}\right)\right| \leq \prod_{i=1}^{n}\det\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}, \tag{1.5}$$

where $\sum_{i=1}^{n}\lambda_i = 1$, $\lambda_i \geq 0$, $i = 1, \ldots, n$.

Here $\frac{I+A}{I-A}$ is understood as $(I+A)(I-A)^{-1}$, or equivalently, $(I-A)^{-1}(I+A)$ provided the inverse exists.

Remark 3. When A_i $(i = 1, \ldots, n)$ in the previous theorem are moreover self-adjoint, a much more general result has been stated in [2, Theorem 3.3].

Theorem 2 is essentially equivalent to the following, though the latter looks stronger in the form.

Theorem 4. *Let A_i, $i = 1, \ldots, n$, be contractive trace class operators. Then*

$$\det\left(\frac{I+\left|\sum_{i=1}^{n}\lambda_i A_i\right|}{I-\left|\sum_{i=1}^{n}\lambda_i A_i\right|}\right) \leq \prod_{i=1}^{n}\det\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}, \tag{1.6}$$

where $\sum_{i=1}^{n}\lambda_i = 1$, $\lambda_i \geq 0$, $i = 1, \ldots, n$.

2. The Proof

The proof of Theorem 2 relies on a few auxiliary results.

Proposition 5. *Let A, B be positive trace class operators with $A \geq B$ and C be any self-adjoint trace class operator. Then* $\operatorname{tr}(AC)^2 \geq \operatorname{tr}(BC)^2$.

Proof. Note that if X, Y, Z are positive trace operators with $X \geq Y$, then $Z^{1/2}XZ^{1/2} \geq Z^{1/2}YZ^{1/2}$ and so $\operatorname{tr} XZ = \operatorname{tr} Z^{1/2}XZ^{1/2} \geq \operatorname{tr} Z^{1/2}YZ^{1/2} = \operatorname{tr} YZ$, i.e., $\operatorname{tr} XZ \geq \operatorname{tr} YZ$. Thus

$$\operatorname{tr}(AC)^2 = \operatorname{tr} A(CAC) \geq \operatorname{tr} B(CAC) = \operatorname{tr} A(CBC) \geq \operatorname{tr} B(CBC) = \operatorname{tr}(BC)^2. \blacksquare$$

The next lemma is a special case of Theorem 2. As in Remark 3, Lemma 6 below is also an implication of [2, Theorem 3.3], nevertheless, we provide a self-contained proof here.

Lemma 6. *Let A_i, $i = 1, \ldots, n$, be positive contractions and trace class. Then*

$$\det \left(\frac{I + \sum_{i=1}^{n} \lambda_i A_i}{I - \sum_{i=1}^{n} \lambda_i A_i} \right) \leq \prod_{i=1}^{n} \det \left(\frac{I + A_i}{I - A_i} \right)^{\lambda_i}, \tag{2.1}$$

where $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \geq 0$, $i = 1, \ldots, n$.

Proof. It suffices to show the case where A_i, $i = 1, \ldots, n$, are positive (semi) definite matrices, the general case is via limiting arguments. The alleged inequality is equivalent to showing that $f(A) = \ln \det(I + A) - \ln \det(I - A)$ is convex over the set of positive (semi)definite contractions.

The following claim is readily verified.

Claim. A real-valued function g is convex over the convex set S if and only if $\forall x, y \in S$ ($x \neq y$),

$$\phi(t) \equiv g(y + t\frac{x - y}{\|x - y\|})$$

is convex over $t \in [0, \|x - y\|]$.

Suppose $U = U(t), V = V(t)$ are matrices of the same size, we have the known formulae

$$\frac{\partial(UV)}{\partial t} = U\frac{\partial V}{\partial t} + \frac{\partial U}{\partial t}V;$$

and if U is invertible, then

$$\frac{\partial \ln\det U}{\partial t} = \operatorname{tr}\frac{\partial U}{\partial t}U^{-1},$$
$$\frac{\partial U^{-1}}{\partial t} = -U^{-1}\frac{\partial U}{\partial t}U^{-1}.$$

Now let X, Y $(X \neq Y)$ be positive definite contractions, denoted by $M = (X - Y)/\|X-Y\|$. Then $Y + tM = \frac{t}{\|X-Y\|}X + (1 - \frac{t}{\|X-Y\|})Y$ is again a positive definite contraction for $t \in [0, \|X-Y\|]$. Define

$$\phi(t) = f(Y+tM) = \ln\det(I+Y+tM) - \ln\det(I-(Y+tM)),$$

$t \in [0, \|X-Y\|]$. Then

$$\phi'(t) = \operatorname{tr}[(I+Y+tM)^{-1}M] + \operatorname{tr}[(I-(Y+tM))^{-1}M]$$

and

$$\phi''(t) = -\operatorname{tr}[(I+Y+tM)^{-1}M]^2 + \operatorname{tr}[(I-(Y+tM))^{-1}M]^2.$$

As $(I-(Y+tM))^{-1} \geq (I+Y+tM)^{-1}$, by Proposition 5, we have $\phi''(t) \geq 0$. Then the Claim tells us $f(A)$ is convex over the set of positive (semi)definite contractions. ■

Let Φ be a map between C^*-algebras. We say that Φ is 2-positive if whenever the 2×2 operator matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ then so is $\begin{bmatrix} \Phi(A) & \Phi(B) \\ \Phi(B^*) & \Phi(C) \end{bmatrix} \geq 0$. It is clear that any Liebian function [11, p. 70] is 2-positive.

The following identity carries the name of L.-K. Hua (see, for example, [8]),

$$(I-B^*B)-(I-B^*A)(I-A^*A)^{-1}(I-A^*B) = -(A-B)^*(I-AA^*)^{-1}(A-B),$$

where A, B are contractions.

An application of a result of Schur [4, p. 472] reveals that

$$\begin{bmatrix} (I-A^*A)^{-1} & (I-B^*A)^{-1} \\ (I-A^*B)^{-1} & (I-B^*B)^{-1} \end{bmatrix} \geq 0.$$

Exchanging the role of A, B and their adjoints gives

$$\begin{bmatrix} (I-AA^*)^{-1} & (I-BA^*)^{-1} \\ (I-AB^*)^{-1} & (I-BB^*)^{-1} \end{bmatrix} \geq 0. \tag{2.2}$$

Proposition 7. $\Phi(t) = (1-t)^{-1}$ *is* 2*-positive over the contractions.*

Proof. If A, B are contractions, so are A^*B, B^*A. With (2.2), we have

$$\begin{aligned}&\begin{bmatrix}(I-A^*A)^{-1} & (I-A^*B)^{-1}\\(I-B^*A)^{-1} & (I-B^*B)^{-1}\end{bmatrix}\\=\;&\begin{bmatrix}I+A^*(I-AA^*)^{-1}A & I+A^*(I-BA^*)^{-1}B\\I+B^*(I-AB^*)^{-1}A & I+B^*(I-BB^*)^{-1}B\end{bmatrix}\\=\;&\begin{bmatrix}I & I\\I & I\end{bmatrix}+(A\oplus B)^*\begin{bmatrix}(I-AA^*)^{-1} & (I-BA^*)^{-1}\\(I-AB^*)^{-1} & (I-BB^*)^{-1}\end{bmatrix}(A\oplus B)\geq 0.\end{aligned}$$

Here we use $A\oplus B$ to denote the 2×2 operator matrix $\begin{bmatrix}A & 0\\0 & B\end{bmatrix}$. As we can identify any positive 2×2 operator matrix with the form $\begin{bmatrix}X^*X & X^*Y\\Y^*X & Y^*Y\end{bmatrix}$, the conclusion follows. ■

Remark 8. The above argument can also be found in [1, Theorem 1.1].

Proposition 9. *$\Phi(t)=(1+t)(1-t)^{-1}$ is 2-positive over the contractions.*

Proof. From the proof of previous proposition, we have $(1-t)^{-1}-1$ is also 2-positive over the contractions. The proof is complete by noting $\Phi(t)=2(1-t)^{-1}-1$. ■

Lemma 10. *Let A_i, $i=1,\ldots,n$, be contractive trace class operators. Then for any Liebian function f,*

$$\left|f\left(\frac{I+\sum_{i=1}^n\lambda_iA_i}{I-\sum_{i=1}^n\lambda_iA_i}\right)\right|^2\leq f\left(\frac{I+\sum_{i=1}^n\lambda_i|A_i|}{I-\sum_{i=1}^n\lambda_i|A_i|}\right)f\left(\frac{I+\sum_{i=1}^n\lambda_i|A_i^*|}{I-\sum_{i=1}^n\lambda_i|A_i^*|}\right),\tag{2.3}$$

where $\sum_{i=1}^n\lambda_i=1$, $\lambda_i\geq 0$, $i=1,\ldots,n$.

Proof. An application of the polar decomposition reveals $\begin{bmatrix}|A_i^*| & A_i\\A_i^* & |A_i|\end{bmatrix}\geq 0$ for any i, so

$$\begin{bmatrix}\sum_{i=1}^n\lambda_i|A_i^*| & \sum_{i=1}^n\lambda_iA_i\\\sum_{i=1}^n\lambda_iA_i^* & \sum_{i=1}^n\lambda_i|A_i|\end{bmatrix}=\sum_{i=1}^n\lambda_i\begin{bmatrix}|A_i^*| & A_i\\A_i^* & |A_i|\end{bmatrix}\geq 0.$$

The conclusion follows from Proposition 9. ■

Proof of Theorem 2. Determinant functional is a Liebian function, so by Lemma 10, we have

$$\begin{aligned}\left|\det\left(\frac{I+\sum_{i=1}^n \lambda_i A_i}{I-\sum_{i=1}^n \lambda_i A_i}\right)\right|^2 &\le \det\left(\frac{I+\sum_{i=1}^n \lambda_i |A_i|}{I-\sum_{i=1}^n \lambda_i |A_i|}\right)\det\left(\frac{I+\sum_{i=1}^n \lambda_i |A_i^*|}{I-\sum_{i=1}^n \lambda_i |A_i^*|}\right)\\ &\le \prod_{i=1}^n \det\left(\frac{I+|A_i|}{I-|A_i|}\right)^{\lambda_i}\prod_{i=1}^n \det\left(\frac{I+|A_i^*|}{I-|A_i^*|}\right)^{\lambda_i}\\ &= \prod_{i=1}^n \det\left(\frac{I+|A_i|}{I-|A_i|}\right)^{2\lambda_i},\end{aligned}$$

in which the second inequality is by Lemma 6 and the third equality is by the fact $\det(I+|A|)=\det(I+|A^*|)$. ■

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