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ON r-EQUITABLE COLORING OF COMPLETE MULTIPARTITE GRAPHS

Chih-Hung Yen*

Abstract. Let $r \geqslant 0$ and $k \geqslant 1$ be integers. We say that a graph G has an r-equitable k-coloring if there exists a proper k-coloring of G such that the sizes of any two color classes differ by at most r. The least k such that a graph G has an r-equitable k-coloring is denoted by $\chi_{r=}(G)$, and the least n such that a graph G has an r-equitable k-coloring for all $k \geqslant n$ is denoted by $\chi^*_{r=}(G)$. In this paper, we propose a necessary and sufficient condition for a complete multipartite graph G to have an r-equitable k-coloring, and also give exact values of $\chi_{r=}(G)$ and $\chi^*_{r=}(G)$.

1. INTRODUCTION

A graph $G = (V, E)$ is composed of a nonempty vertex set V and an edge set E. All graphs we consider in this paper are presumed to be undirected, finite, loopless, and without multiple edges. For a positive integer k, a (proper) *k-coloring* of a graph G is a mapping $f : V \to \{1, 2, \ldots, k\}$ such that adjacent vertices have different images. The images $1, 2, \ldots, k$ are called *colors* and the corresponding sets $\{u \in V : f(u) = 1\}, \{u \in V : f(u) = 2\}, \ldots, \{u \in V : f(u) = k\}$ are called *color classes*. Obviously, a color class is an independent set whose size may be equal to zero in G. And one color in a k-coloring of a graph G is said to be *missing* if its corresponding color class is an empty set of size zero. Moreover, a graph is *k-colorable* if it has a k-coloring. The *chromatic number* of a graph G, written $\chi(G)$, is the least k such that G is k-colorable.

A k-coloring of a graph G is said to be *equitable* provided that the sizes of any two color classes differ by at most one. A graph G is *equitably k-colorable* if G has an equitable k-coloring. The least k such that a graph G is equitably k-colorable is called the *equitable chromatic number* of G and denoted by $\chi_=(G)$. The notion of

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*Corresponding author.

equitable colorability was first introduced by Meyer [9] in 1973. His motivation came from the problem of assigning one of the six days of the work week to each garbage collection route. And so far, quite a few results on equitable coloring of graphs have been obtained in the literature, see $[1, 2, 4\text{-}10]$.

Recently, Hertz and Ries [3] generalized the notion of equitable colorability. They said that a k-coloring of a graph G is *r-equitable* for an integer $r \geqslant 0$ if the sizes of any two color classes differ by at most r. And a graph G is *r-equitably k-colorable* if there exists an r-equitable k-coloring of G. The least k such that a graph G is r-equitably k-colorable is called the *r-equitable chromatic number* of G and denoted by $\chi_{r=}(G)$. It is clear that an r-equitably k-colorable graph is certainly $(r+1)$-equitably k-colorable. Moreover, an equitably k-colorable graph is also 1-equitably k-colorable, and vice versa. In fact, such a generalization is quite natural since many k-colorable graphs do not have equitable k-colorings.

Unlike proper colorings of graphs, an equitably (or r-equitably) k-colorable graph may not be equitably (or r-equitably) $(k+1)$-colorable. For example, the graph in Figure 1, denoted by $K_{3,3}$, is equitably 2-colorable, yet it is not equitably 3-colorable. Hence, we also have an interest in finding the least n such that a graph G is equitably (or r-equitably) k-colorable for all $k \geqslant n$, called the *equitable* (or *r-equitable*) *chromatic threshold* of G and denoted by $\chi^*_{=}(G)$ (or $\chi^*_{r=}(G)$). Note that $\chi^*_{0=}(G)$ does not exist for any graph G. Because a graph G is not 0-equitably k-colorable for any $k \geqslant |V(G)|+1$.

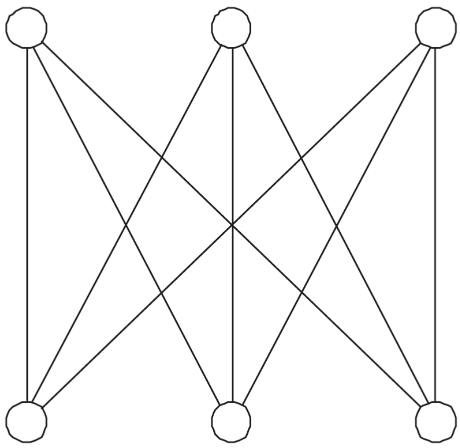

Fig. 1. The graph $K_{3,3}$.

In this paper, we pay attention to r-equitable coloring of a particular class of graphs, called *complete multipartite graphs*. We first give a brief review for equitable coloring on complete multipartite graphs related to our results in this paper. Then, for any $r \geqslant 0$, we propose a necessary and sufficient condition for a complete multipartite graph G to have an r-equitable k-coloring, and also give exact values of $\chi_{r=}(G)$ and $\chi^*_{r=}(G)$.

2. Known Results

Recall that a graph G is *t-partite* if its vertex set can be partitioned into t independent sets $V_1, V_2, \ldots, V_t$, and *complete t-partite*, denoted by $K_{n_1,n_2,\ldots,n_t}$, if every vertex in V_i is adjacent to every vertex in V_j whenever $i \neq j$ and $|V_i| = n_i \geqslant 1$ for all $1 \leqslant i \leqslant t$. $V_1, V_2, \ldots, V_t$ are called *partite sets* of G. By convention it is always assumed that $t \geqslant 2$ and $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$. And a graph is said to be *complete multipartite* if it is complete t-partite for some t. Furthermore, a complete t-partite graph $K_{n_1,n_2,\ldots,n_t}$ satisfies $n_1 = n_2 = \cdots = n_t = n$ is also denoted by $K_{t(n)}$.

Let $\lceil x \rceil$ and $\lfloor x \rfloor$ denote, respectively, the smallest integer not less than x and the largest integer not greater than x. Also, let $\mathbb{N}$ denote the set of all positive integers. In 1994, Wu [10] proved the followings.

Theorem 1. *For any $K_{n_1,n_2,\ldots,n_t}$, let $p = n_1 + n_2 + \cdots + n_t$. Then $K_{n_1,n_2,\ldots,n_t}$ is equitably k-colorable if and only if either $k > p$ or $n_i \geqslant \lceil n_i/\lceil p/k \rceil \rceil \lfloor p/k \rfloor$ for all $1 \leqslant i \leqslant t$ and $\sum_{i=1}^{t} \lfloor n_i/\lfloor p/k \rfloor \rfloor \geqslant k \geqslant \sum_{i=1}^{t} \lceil n_i/\lceil p/k \rceil \rceil$ when $k \leqslant p$.*

Theorem 2. $\chi_{=}(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t} \lceil n_i/h \rceil$, *where* $h = \max\{m \in \mathbb{N} \colon n_i \geqslant \lceil n_i/m \rceil (m-1)$ *for all* $1 \leqslant i \leqslant t\}$.

Theorem 3. $\chi^*_{=}(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t} \lceil n_i/h \rceil$, *where* $h = \min\{m \in \mathbb{N} \colon$ *there exists some i such that $n_i < \lceil n_i/(m+1) \rceil m$ or there exist n_i and n_j, $i \neq j$, such that both of n_i and n_j are not divisible by m*$\}$.

Later, in 2001, Lam et al. [5] also showed the following result which is equivalent to Theorem 2.

Theorem 4. $\chi_{=}(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t} \lceil n_i/(h+1) \rceil$, *where* $h = \max\{m \in \mathbb{N} \colon n_i \pmod{m} < \lceil n_i/m \rceil$ *for all* $1 \leqslant i \leqslant t\}$.

Recently, in 2010, Lin and Chang [6] showed the following results for $K_{t(n)}$.

Theorem 5. *For any $k \geqslant t$, $K_{t(n)}$ is equitably k-colorable if and only if $\lceil n/\lfloor k/t \rfloor \rceil - \lfloor n/\lceil k/t \rceil \rfloor \leqslant 1$.*

Theorem 6. $\chi^*_{=}(K_{t(n)}) = t\lceil n/h \rceil$, *where h is the least positive integer such that n is not divisible by h.*

3. Our Results

In what follows, let I_n denote the graph consisting of n isolated vertices, where $n \geqslant 1$.

Lemma 7. *For any $r \geqslant 0$, I_n has an r-equitable k-coloring if and only if there exists an integer $m \geqslant 0$ such that $(m+r)k \geqslant n \geqslant mk$.*

Proof. ($\Rightarrow$) Suppose that I_n has an r-equitable k-coloring. Then there exists a k-coloring of I_n such that each of the k color classes is of size $m, m+1, \ldots,$ or $m+r$ for some integer $m \geqslant 0$. Hence, we have $(m+r)k \geqslant n \geqslant mk$.

($\Leftarrow$) Firstly, since $n = \lceil n/k \rceil + \lceil (n-1)/k \rceil + \cdots + \lceil (n-(k-1))/k \rceil$, we partition the vertex set of I_n into k independent sets $V_1, V_2, \ldots, V_k$ of sizes $\lceil n/k \rceil$, $\lceil (n-1)/k \rceil$, $\ldots$, $\lceil (n-(k-1))/k \rceil$, respectively. Next, since there exists an integer $m \geqslant 0$ such that $(m+r)k \geqslant n \geqslant mk$, we have $m+r \geqslant n/k \geqslant m$. It implies that $m+r \geqslant \lceil n/k \rceil \geqslant \lfloor n/k \rfloor \geqslant m$ because $m+r$ and m are integers. Then I_n has a k-coloring such that each of the k color classes is of size $m, m+1, \ldots,$ or $m+r$ by letting each of $V_1, V_2, \ldots, V_k$ be a color class and $m+r \geqslant \lceil n/k \rceil \geqslant \lceil (n-1)/k \rceil \geqslant \cdots \geqslant \lceil (n-(k-1))/k \rceil = \lfloor n/k \rfloor \geqslant m$. Hence, I_n has an r-equitable k-coloring. ∎

Lemma 8. *For any $r \geqslant 1$, $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring such that at least one color is missing if and only if $k \geqslant (\sum_{i=1}^{t} \lceil n_i/r \rceil) + 1$.*

Proof. ($\Rightarrow$) Clearly, if $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring such that at least one color is missing, then there exists an r-equitable $(k-1)$-coloring of $K_{n_1,n_2,\ldots,n_t}$ such that each of the $k-1$ color classes is of size $0, 1, \ldots,$ or r. Hence, it implies that we can certainly find positive integers $k_1, k_2, \ldots, k_t$ such that $k-1 = \sum_{i=1}^{t} k_i$ and I_{n_i} has a k_i-coloring in which each of the k_i color classes is of size $0, 1, \ldots,$ or r for all $1 \leqslant i \leqslant t$. Then we have $rk_i \geqslant n_i$ for all $1 \leqslant i \leqslant t$. Since $r \geqslant 1$ and $k_1, k_2, \ldots, k_t$ are positive integers, $k_i \geqslant \lceil n_i/r \rceil$ for all $1 \leqslant i \leqslant t$. Therefore, $k-1 = \sum_{i=1}^{t} k_i \geqslant \sum_{i=1}^{t} \lceil n_i/r \rceil$ and thereby $k \geqslant (\sum_{i=1}^{t} \lceil n_i/r \rceil) + 1$.

($\Leftarrow$) If $k \geqslant (\sum_{i=1}^{t} \lceil n_i/r \rceil) + 1$, then $k-1 \geqslant \sum_{i=1}^{t} \lceil n_i/r \rceil$. Hence, we can certainly find positive integers $k_1, k_2, \ldots, k_t$ such that $k-1 = \sum_{i=1}^{t} k_i$ and $k_i \geqslant \lceil n_i/r \rceil$ for all $1 \leqslant i \leqslant t$. So, $k_i \geqslant n_i/r$ and $rk_i \geqslant n_i \geqslant 1 > 0 = 0 \cdot k_i$ for all $1 \leqslant i \leqslant t$. Then I_{n_i} has a k_i-coloring such that each of the k_i color classes is of size $0, 1, \ldots,$ or r for all $1 \leqslant i \leqslant t$ by the proof of Lemma 7. Therefore, there exists an r-equitable $(k-1)$-coloring of $K_{n_1,n_2,\ldots,n_t}$ such that each of the $k-1$ color class is of size $0, 1, \ldots,$ or r by $k-1 = \sum_{i=1}^{t} k_i$. It implies that $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring such that at least one color is missing. ∎

Note that $K_{n_1,n_2,\ldots,n_t}$ has no 0-equitable k-coloring such that at least one color is missing; otherwise, the order of $K_{n_1,n_2,\ldots,n_t}$ is equal to zero.

Lemma 9. *For any $r \geqslant 0$, $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring such that no color is missing if and only if there exists a positive integer m such that $\lfloor n_i/m \rfloor \geqslant \lceil n_i/(m+r) \rceil$ for all $1 \leqslant i \leqslant t$ and $\sum_{i=1}^{t} \lfloor n_i/m \rfloor \geqslant k \geqslant \sum_{i=1}^{t} \lceil n_i/(m+r) \rceil$.*

Proof. ($\Rightarrow$) It is obvious that if $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring such that no color is missing, then we can certainly find positive integers $k_1, k_2, \ldots, k_t$, and m such that $k = \sum_{i=1}^{t} k_i$ and I_{n_i} has a k_i-coloring in which each of the k_i color classes is of size $m, m+1, \ldots,$ or $m+r$ for all $1 \leqslant i \leqslant t$. Hence, we have $(m+r)k_i \geqslant n_i \geqslant mk_i$ for all $1 \leqslant i \leqslant t$. Since $k_1, k_2, \ldots, k_t$, and m are positive integers, it implies that $n_i/m \geqslant k_i \geqslant n_i/(m+r)$ and thereby $\lfloor n_i/m \rfloor \geqslant k_i \geqslant \lceil n_i/(m+r) \rceil$ for all $1 \leqslant i \leqslant t$. Therefore, $\lfloor n_i/m \rfloor \geqslant \lceil n_i/(m+r) \rceil$ for all $1 \leqslant i \leqslant t$ and $\sum_{i=1}^{t} \lfloor n_i/m \rfloor \geqslant \sum_{i=1}^{t} k_i = k \geqslant \sum_{i=1}^{t} \lceil n_i/(m+r) \rceil$.

($\Leftarrow$) If there exists a positive integer m such that $\lfloor n_i/m \rfloor \geqslant \lceil n_i/(m+r) \rceil$ for all $1 \leqslant i \leqslant t$ and $\sum_{i=1}^{t} \lfloor n_i/m \rfloor \geqslant k \geqslant \sum_{i=1}^{t} \lceil n_i/(m+r) \rceil$, then we can certainly find positive integers $k_1, k_2, \ldots, k_t$ such that $k = \sum_{i=1}^{t} k_i$ and $\lfloor n_i/m \rfloor \geqslant k_i \geqslant \lceil n_i/(m+r) \rceil$ for all $1 \leqslant i \leqslant t$. Hence, $n_i/m \geqslant k_i \geqslant n_i/(m+r)$ and thereby $(m+r)k_i \geqslant n_i \geqslant mk_i$ for all $1 \leqslant i \leqslant t$. Then I_{n_i} has a k_i-coloring in which each of the k_i color classes is of size $m, m+1, \ldots,$ or $m+r$ for all $1 \leqslant i \leqslant t$ by the proof of Lemma 7. Therefore, $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring such that no color is missing by $k = \sum_{i=1}^{t} k_i$ and $m \geqslant 1$. ∎

By the conclusions of Lemmas 8 and 9, we can conclude the necessary and sufficient condition for a complete t-partite graph $K_{n_1,n_2,\ldots,n_t}$ to have an r-equitable k-coloring.

Theorem 10. *For any $r \geqslant 0$, $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring if and only if at least one of the following statements holds.*

1. *$r \geqslant 1$ and $k \geqslant (\sum_{i=1}^{t} \lceil n_i/r \rceil) + 1$.*
2. *There exists a positive integer m such that $\lfloor n_i/m \rfloor \geqslant \lceil n_i/(m+r) \rceil$ for all $1 \leqslant i \leqslant t$ and $\sum_{i=1}^{t} \lfloor n_i/m \rfloor \geqslant k \geqslant \sum_{i=1}^{t} \lceil n_i/(m+r) \rceil$.*

For example, $K_{3,5,7}$ has a 2-equitable k-coloring such that at least one color is missing if and only if $k \geqslant 10$. Moreover, if we choose $m = 1, 2, 3$, then we get that $K_{3,5,7}$ has a 2-equitable k-coloring such that no color is missing if and only if $15 \geqslant k \geqslant 4$. Hence, $K_{3,5,7}$ has a 2-equitable k-coloring if and only if $k \geqslant 4$.

Theorem 11. *For any $r \geqslant 0$ and $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$, let $\theta = \max\{m \in \mathbb{N} \colon \lfloor n_i/m \rfloor \geqslant \lceil n_i/(m+r) \rceil$ for all $1 \leqslant i \leqslant t\}$. Then $\chi_{r=}(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t} \lceil n_i/(\theta+r) \rceil$.*

Proof. Firstly, since $\lfloor n_i/1 \rfloor = n_i \geqslant \lceil n_i/(1+r) \rceil$ for all $1 \leqslant i \leqslant t$, we have that θ exists with $\theta \geqslant 1$. Secondly, if $m \geq n_1+1$, then $\lfloor n_1/m \rfloor = 0 < 1 = \lceil n_1/(m+r) \rceil$. Hence, $\theta \leqslant n_1$. Finally, if $k = \sum_{i=1}^{t} \lceil n_i/(\theta+r) \rceil$, then $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring by the choice of θ and Theorem 10. Now, let $k < \sum_{i=1}^{t} \lceil n_i/(\theta+r) \rceil$, and suppose that $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring. By $k < \sum_{i=1}^{t} \lceil n_i/r \rceil$ if $r \geqslant 1$ and Theorem 10, we know that there exists a positive integer m such that $\lfloor n_i/m \rfloor \geqslant \lceil n_i/(m+r) \rceil$ for all $1 \leqslant i \leqslant t$ and $\sum_{i=1}^{t} \lfloor n_i/m \rfloor \geqslant k \geqslant \sum_{i=1}^{t} \lceil n_i/(m+r) \rceil$. Then $m \leqslant \theta$ by the choice of θ, and thereby $k \geqslant \sum_{i=1}^{t} \lceil n_i/(m+r) \rceil \geqslant \sum_{i=1}^{t} \lceil n_i/(\theta+r) \rceil$. It is a contradiction. Thus $K_{n_1,n_2,\ldots,n_t}$ has no r-equitable k-coloring when $k <$

$\sum_{i=1}^{t}\lceil n_i/(\theta+r)\rceil$. Therefore, we can conclude that $\chi_{r=}(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t}\lceil n_i/(\theta+r)\rceil$. ∎

Theorem 12. *For any $r \geqslant 1$ and $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t$, let $m_1, m_2, \ldots, m_x$ be all positive integers such that $m_1 < m_2 < \cdots < m_x$ and $\lfloor n_i/m_j\rfloor \geqslant \lceil n_i/(m_j+r)\rceil$ for all $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant x$. Also, let $M = \{m_1, m_2, \ldots, m_x\}$ and $\theta = \min\{m_j \in M\colon \sum_{i=1}^{t}\lceil n_i/(m_j+r)\rceil > (\sum_{i=1}^{t}\lfloor n_i/m_{j+1}\rfloor) + 1$ or $m_j = m_x\}$. Then $\chi^*_{r=}(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t}\lceil n_i/(\theta+r)\rceil$.*

Proof. Firstly, since $\lfloor n_i/1\rfloor = n_i \geqslant \lceil n_i/(1+r)\rceil$ for all $1 \leqslant i \leqslant t$, we have $m_1 = 1$, and thereby M is a nonempty set and θ exists with $\theta \geqslant 1$. Secondly, if $k \geqslant (\sum_{i=1}^{t}\lceil n_i/r\rceil) + 1$, then $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring by Theorem 10. Finally, let k satisfy $\sum_{i=1}^{t}\lceil n_i/r\rceil \geqslant k \geqslant \sum_{i=1}^{t}\lceil n_i/(\theta+r)\rceil$, and also let $m_\ell = \max\{m_j \in M\colon \sum_{i=1}^{t}\lfloor n_i/m_j\rfloor \geqslant k\}$. Since $\sum_{i=1}^{t}\lfloor n_i/m_1\rfloor = \sum_{i=1}^{t} n_i \geqslant \sum_{i=1}^{t}\lceil n_i/r\rceil \geqslant k$, we know that m_ℓ exists with $m_\ell \geqslant 1$. Also, $m_\ell \leqslant \theta$ by the choice of θ. Then we want to show that $\sum_{i=1}^{t}\lfloor n_i/m_\ell\rfloor \geqslant k \geqslant \sum_{i=1}^{t}\lceil n_i/(m_\ell+r)\rceil$, and thus $K_{n_1,n_2,\ldots,n_t}$ has an r-equitable k-coloring by Theorem 10. Suppose that $k < \sum_{i=1}^{t}\lceil n_i/(m_\ell+r)\rceil$. Then $m_\ell < m_{\ell+1} \leqslant \theta$ by $k \geqslant \sum_{i=1}^{t}\lceil n_i/(\theta+r)\rceil$, and $k > \sum_{i=1}^{t}\lfloor n_i/m_{\ell+1}\rfloor$ by the choice of m_ℓ. Hence, we have $\sum_{i=1}^{t}\lceil n_i/(m_\ell+r)\rceil > (\sum_{i=1}^{t}\lfloor n_i/m_{\ell+1}\rfloor) + 1$. It is a contradiction by $m_\ell < \theta$ and the choice of θ. Now, let $k = (\sum_{i=1}^{t}\lceil n_i/(\theta+r)\rceil) - 1$. Then $k < \sum_{i=1}^{t}\lceil n_i/r\rceil$ by $\theta \geqslant 1$. Also, $k < \sum_{i=1}^{t}\lceil n_i/(m_j+r)\rceil$ for each $m_j \leqslant \theta$. Moreover, by the choice of θ, $k > \sum_{i=1}^{t}\lfloor n_i/m_j\rfloor$ for each $m_j > \theta$. So, there exists no $m_j \in M$ such that $\sum_{i=1}^{t}\lfloor n_i/m_j\rfloor \geqslant k \geqslant \sum_{i=1}^{t}\lceil n_i/(m_j+r)\rceil$. Then $K_{n_1,n_2,\ldots,n_t}$ has no r-equitable k-coloring by Theorem 10. Thus we can conclude that $\chi^*_{r=}(K_{n_1,n_2,\ldots,n_t}) = \sum_{i=1}^{t}\lceil n_i/(\theta+r)\rceil$. ∎

In fact, it is not difficult to observe that if a graph G has an r-equitable k-coloring such that at least one color is missing, then there must exist a positive integer $k' < k$ such that G has an r-equitable k'-coloring in which no color is missing. Hence, the r-equitable chromatic number $\chi_{r=}(G)$ of a graph G is actually equal to the least k such that G has an r-equitable k-coloring in which no color is missing. Similarly, the r-equitable chromatic threshold $\chi^*_{r=}(G)$ of a graph G is actually equal to the least n such that G has an r-equitable k-coloring for all $k > n$ and G has an r-equitable n-coloring in which no color is missing. Finally, according to the above theorems, we have the following corollaries.

Corollary 13. *For any $r \geqslant 0$ and $k \geqslant t$, $K_{t(n)}$ has an r-equitable k-coloring if and only if $\lceil n/\lfloor k/t\rfloor\rceil - \lfloor n/\lceil k/t\rceil\rfloor \leqslant r$.*

Proof. (⇒) Suppose that $K_{t(n)}$ has an r-equitable k-coloring. Then, either $r \geqslant 1$ and $k \geqslant t\lceil n/r\rceil + 1$ or there exists a positive integer m such that $t\lfloor n/m\rfloor \geqslant k \geqslant t\lceil n/(m+r)\rceil$ by Theorem 10. If $r \geqslant 1$ and $k \geqslant t\lceil n/r\rceil + 1$, then $\lfloor k/t\rfloor \geqslant$

$\lceil n/r \rceil \geqslant n/r$. Hence, we have $n/\lfloor k/t \rfloor \leqslant n/(n/r) = r$. Since r is an integer, it implies that $\lceil n/\lfloor k/t \rfloor \rceil \leqslant r$ and thereby $\lceil n/\lfloor k/t \rfloor \rceil - \lfloor n/\lceil k/t \rceil \rfloor \leqslant r$. If there exists a positive integer m such that $t\lfloor n/m \rfloor \geqslant k \geqslant t\lceil n/(m+r) \rceil$, then $n/m \geqslant \lfloor n/m \rfloor \geqslant \lceil k/t \rceil \geqslant k/t \geqslant \lfloor k/t \rfloor \geqslant \lceil n/(m+r) \rceil \geqslant n/(m+r)$. Hence, we have $m \leqslant \lfloor n/\lceil k/t \rceil \rfloor \leqslant \lceil n/\lfloor k/t \rfloor \rceil \leqslant m+r$ because m and $m+r$ are positive integers. It implies that $\lceil n/\lfloor k/t \rfloor \rceil - \lfloor n/\lceil k/t \rceil \rfloor \leqslant r$.

($\Leftarrow$) Let $m = \lfloor n/\lceil k/t \rceil \rfloor$. Firstly, if $m = 0$, then $\lceil k/t \rceil \geqslant n+1$ and $\lfloor k/t \rfloor \geqslant n$. It implies that $r \geqslant \lceil n/\lfloor k/t \rfloor \rceil - \lfloor n/\lceil k/t \rceil \rfloor = 1 - 0 = 1$ and $k/t > n$. Hence, $k \geqslant tn+1 = t\lceil n/1 \rceil + 1 \geqslant t\lceil n/r \rceil + 1$. Therefore, $K_{t(n)}$ has an r-equitable k-coloring by Theorem 10. Next, if $m \geqslant 1$, by $\lceil n/\lfloor k/t \rfloor \rceil - \lfloor n/\lceil k/t \rceil \rfloor \leqslant r$, then we have $m = \lfloor n/\lceil k/t \rceil \rfloor \leqslant n/\lceil k/t \rceil \leqslant n/(k/t) \leqslant n/\lfloor k/t \rfloor \leqslant \lceil n/\lfloor k/t \rfloor \rceil \leqslant m+r$. Hence, $n/m \geqslant \lfloor n/m \rfloor \geqslant \lceil k/t \rceil \geqslant k/t \geqslant \lfloor k/t \rfloor \geqslant \lceil n/(m+r) \rceil \geqslant n/(m+r)$. It implies that $\lfloor n/m \rfloor \geqslant \lceil n/(m+r) \rceil$ and $t\lfloor n/m \rfloor \geqslant k \geqslant t\lceil n/(m+r) \rceil$. Therefore, $K_{t(n)}$ has an r-equitable k-coloring by Theorem 10. ■

Corollary 14. *For any* $r \geqslant 0$, $\chi_{r=}(K_{t(n)}) = t = \chi_{=}(K_{t(n)})$.

Corollary 15. *For any* $r \geqslant 1$ *and* $n \geqslant 1$, *let* θ *be the least positive integer such that* $\lfloor n/(\theta+1) \rfloor < \lceil n/(\theta+r) \rceil$. *Then* $\chi^*_{r=}(K_{t(n)}) = t\lceil n/(\theta+r) \rceil$.

Proof. Firstly, since $\lfloor n/(n+1) \rfloor = 0 < 1 = \lceil n/(n+r) \rceil$, we know that θ exists with $\theta \leqslant n$. Also, $\lfloor n/m \rfloor \geqslant \lfloor n/(m+1) \rfloor \geqslant \lceil n/(m+r) \rceil$ for each $m \in \{1, 2, \dots, \theta-1\}$ by the choice of θ. Moreover, if $m = \theta - 1$, then $\lfloor n/(\theta-1+1) \rfloor = \lfloor n/\theta \rfloor \geqslant \lceil n/(\theta-1+r) \rceil \geqslant \lceil n/(\theta+r) \rceil$. Hence, we have that $\lfloor n/m \rfloor \geqslant \lceil n/(m+r) \rceil$ for each $m \in \{1, 2, \dots, \theta\}$. Next, let $m_1, m_2, \dots, m_x$ be all positive integers such that $m_1 < m_2 < \cdots < m_x$ and $\lfloor n/m_j \rfloor \geqslant \lceil n/(m_j+r) \rceil$ for all $1 \leqslant j \leqslant x$. Also, let $M = \{m_1, m_2, \dots, m_x\}$. Then $m_1 = 1, m_2 = 2, \dots, m_\theta = \theta$, and thereby $(\sum_{i=1}^t \lfloor n/m_{j+1} \rfloor) + 1 > \sum_{i=1}^t \lfloor n/m_{j+1} \rfloor = t\lfloor n/m_{j+1} \rfloor \geqslant t\lceil n/(m_j+r) \rceil = \sum_{i=1}^t \lceil n/(m_j+r) \rceil$ for each $m_j \in \{m_1, m_2, \dots, m_{\theta-1}\}$ by $\lfloor n/(m+1) \rfloor \geqslant \lceil n/(m+r) \rceil$ for each $m \in \{1, 2, \dots, \theta-1\}$. Furthermore, since $\lfloor n/(\theta+1) \rfloor < \lceil n/(\theta+r) \rceil$, it implies that $t\lfloor n/(\theta+1) \rfloor + t \leqslant t\lceil n/(\theta+r) \rceil$. Therefore, if $m_{\theta+1}$ exists, then $m_{\theta+1} \geqslant \theta+1$ and $(\sum_{i=1}^t \lfloor n/m_{\theta+1} \rfloor) + 1 = t\lfloor n/m_{\theta+1} \rfloor + 1 < t\lfloor n/(\theta+1) \rfloor + t \leqslant t\lceil n/(\theta+r) \rceil = \sum_{i=1}^t \lceil n/(m_\theta+r) \rceil$ for all $t \geqslant 2$. Thus we can conclude that $m_\theta = \min\{m_j \in M \colon \sum_{i=1}^t \lceil n/(m_j+r) \rceil > (\sum_{i=1}^t \lfloor n/m_{j+1} \rfloor) + 1 \text{ or } m_j = m_x\}$. Then $\chi^*_{r=}(K_{t(n)}) = t\lceil n/(\theta+r) \rceil$ by Theorem 12 and $m_\theta = \theta$. ■

4. Some Concluding Remarks

The motivation for writing this paper was reading a paper titled "on r-equitable colorings of trees and forests" uploaded to the personal home page of Alain Heartz, see [3]. Although the notion of r-equitable colorability is a quite natural generalization, it seems to be proposed without precedent. Hence, we believe that such a paper might open the door for more interesting problems on equitable coloring of graphs in the future, and perhaps, for more valuable research. In this paper, we do some things on this side and view them as the beginning.

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Chih-Hung Yen
Department of Applied Mathematics
National Chiayi University
Chiayi 60004, Taiwan
E-mail: chyen@mail.ncyu.edu.tw