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INFINITELY MANY HOMOCLINIC ORBITS OF SECOND-ORDER p-LAPLACIAN SYSTEMS

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Abstract. In this paper, we give several new sufficient conditions for the existence of infinitely many homoclinic orbits of the second-order ordinary p-Laplacian system

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0,$$

where $p > 1,\ t \in \mathbb{R},\ u \in \mathbb{R}^N$, $a \in C(\mathbb{R}, \mathbb{R})$ and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ are no periodic in t, which greatly improve the known results due to Rabinowitz and Willem.

1. Introduction

Consider the second-order ordinary p-Laplacian system

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0, \tag{1.1}$$

where $p > 1,\ t \in \mathbb{R},\ u \in \mathbb{R}^N$, $a : \mathbb{R} \to \mathbb{R}$ and $W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. As usual, we say that a solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t) \to 0$ as $t \to \pm\infty$. In addition, if $u(t) \not\equiv 0$ then $u(t)$ is called a nontrivial homoclinic solution.

It is well-known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably product chaotic.

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Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic orbits of (1.1) emanating from 0.

When $p = 2$, system (1.1) reduces second-order Hamiltonian system

$$\ddot{u}(t) - a(t)u(t) + \nabla W(t, u(t)) = 0. \tag{1.2}$$

In recent years, the existence and multiplicity of homoclinic orbits for Hamiltonian systems have been investigated in many papers via variational methods and many results were obtained based on various hypotheses on the potential functions, see, e.g., [1, 3-10, 12, 13, 19-23, 25-27, 29-32].

In the last decade there has been an increasing interest in the study of ordinary differential systems driven by the p-Laplacian (or the generalization of Laplacian), see [14-18, 28] and the references cited therein. In most of these papers, the well-known global Ambrosetti-Rabinowitz superquadratic condition was commonly assumed:

(AR) there exists $\mu > p$ such that

$$0 < \mu W(t, x) \leq (\nabla W(t, x), x), \quad \forall\ (t, x) \in \mathbb{R} \times (\mathbb{R}^{\mathcal{N}} \setminus \{0\}),$$

where and in the sequel, $(\cdot, \cdot)$ denotes the standard inner product in $\mathbb{R}^{\mathcal{N}}$ and $|\cdot|$ is the induced norm.

In the present paper, we are interested in the existence of infinitely many homoclinic solutions for system (1.1), where $a(t)$ and $W(t, x)$ are no periodic in t. Under some weaker assumptions on $W(t, x)$ than (AR), we establish some existence criteria to guarantee that system (1.1) has infinitely many homoclinic solutions by using the Symmetric Mountain Pass Theorem.

Our main results are the following theorems.

Theorem 1.1. *Assume that a and W satisfy the following assumptions:*

(A) $a \in C(\mathbb{R}, (0, \infty))$ *and* $a(t) \to +\infty$ *as* $|t| \to \infty$*;*

(W1) $W(t, x) = W_1(t, x) - W_2(t, x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $W_2(t, 0) \equiv 0$, *and there are constants* $\mu > p$ *and* $\varrho \in [p, \mu)$ *such that*

$$0 < \mu W_1(t, x) \leq (\nabla W_1(t, x), x), \quad \forall\ (t, x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\},$$

and

$$W_2(t, x) \geq 0, \quad (\nabla W_2(t, x), x) \leq \varrho W_2(t, x), \quad \forall\ (t, x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W2) *There is a* $R > 0$ *such that*

$$\frac{1}{a(t)} |\nabla W(t, x)| = o(|x|^{p-1}) \quad \textit{as} \quad x \to 0$$

uniformly in $t \in (-\infty, -R] \cup [R, +\infty)$.

(W3) $W(t, -x) = W(t, x), \quad \forall\ (t, x) \in \mathbb{R} \times \mathbb{R}^N$.

Then there exists an unbounded sequence of homoclinic solutions for system (*1.1*).

Theorem 1.2. *Assume that* a *and* W *satisfy* (*A*), (*W3*) *and the following assumptions:*

(W1′) $W(t, x) = W_1(t, x) - W_2(t, x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $W_2(t, 0) \equiv 0$, *and there are constants* $\mu > p$ *and* $\varrho \in [p, \mu)$ *such that*

$$0 < \mu W_1(t, x) \le (\nabla W_1(t, x), x), \quad \forall\ (t, x) \in \mathbb{R} \times (\mathbb{R}^N \setminus \{0\}),$$

and

$$(\nabla W_2(t, x), x) \le \varrho W_2(t, x), \quad \forall\ (t, x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W2′) $\frac{1}{a(t)}|\nabla W(t, x)| = o(|x|^{p-1})$ *as* $x \to 0$ *uniformly with respect to* $t \in \mathbb{R}$.

Then there exists an unbounded sequence of homoclinic solutions for system (*1.1*).

Theorem 1.3. *Assume that* a *and* W *satisfy (A), (W2′) and (W3) and the following assumptions:*

(W4) *For any* $r > 0$, *there exist* $\alpha,\ \beta > 0$ *and* $\nu < p$ *such that*

$$0 \le \left(p + \frac{1}{\alpha + \beta|x|^\nu}\right) W(t, x) \le (\nabla W(t, x), x), \quad \forall\ (t, x) \in \mathbb{R} \times \{x \in \mathbb{R}^N : |x| \ge r\};$$

(W5) *For any* $\gamma > 0$ *and* $\varepsilon > 0$

$$\lim_{s \to +\infty} s^{-p} \int_{t-\varepsilon}^{t+\varepsilon} \min_{|x| \ge 1} W(\tau, sx)d\tau = +\infty$$

uniformly with respect to $t \in [-\gamma, \gamma]$.

Then there exists an unbounded sequence of homoclinic solutions for system (1.1).

Remark 1.4. If assumption (AR) holds, then (W4) also holds by choosing $\alpha > 1/(\mu - p),\ \beta > 0$ and $\nu \in (0, p)$. In addition, by (AR), we have

$$W(t, sx) \ge s^\mu W(t, x) \quad \text{for} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad s \ge 1.$$

It follows that for any $\gamma > 0$ and $\varepsilon > 0$

$$\begin{aligned} s^{-p} \int_{t-\varepsilon}^{t+\varepsilon} \min_{|x| \ge 1} W(\tau, sx)d\tau &\ge s^{\mu-p} \int_{t-\varepsilon}^{t+\varepsilon} \min_{|x| \ge 1} W(\tau, x)d\tau \\ &\ge 2\varepsilon s^{\mu-p} \min_{-\gamma-\varepsilon \le \tau \le \gamma+\varepsilon, |x| \ge 1} W(\tau, x) \\ &\to +\infty, \quad s \to +\infty \end{aligned}$$

uniformly with respect to $t \in [-\gamma, \gamma]$. This shows that (AR) implies (W5).

The rest of the this paper is organized as follows: In Section 2, we introduce some notations and preliminary results, and establish an interesting imbedding inequality from $W^{1,p}(\mathbb{R}, \mathbb{R}^N)$ into $L^\infty(\mathbb{R}, \mathbb{R}^N)$, moreover, the constant in the imbedding inequality is the best possible. In Section 3, we complete the proofs of Theorems 1.1-1.3. In Section 4, we give some examples to to illustrate our results.

Throughout this paper, we let $q \in (0, \infty)$ such that $1/p + 1/q = 1$.

2. Preliminaries

Let

$$E = \left\{ u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} [|\dot{u}(t)|^p + a(t)|u(t)|^p]\, dt < +\infty \right\}$$

and for $u \in E$, let

$$\|u\| = \left\{ \int_{\mathbb{R}} [|\dot{u}(t)|^p + a(t)|u(t)|^p]\, dt \right\}^{1/p}.$$

Then E is a uniform convex Banach space with this norm, see [11].

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{p}\|u\|^p - \int_{\mathbb{R}} W(t, u(t))dt. \tag{2.1}$$

If (A), (W1) and (W2) or (W1′) and (W2′) hold, then $I \in C^1(E, \mathbb{R})$ and one can easily check that

(2.2)
$$\langle I'(u), v\rangle = \int_{\mathbb{R}} \left[|\dot{u}(t)|^{p-2}(\dot{u}(t), \dot{v}(t)) + a(t)|u(t)|^{p-2}(u(t), v(t)) - (\nabla W(t, u(t)), v(t))\right] dt.$$

Furthermore, the critical points of I in E are classical solutions of (1.1) with $u(\pm\infty) = 0$.

We will obtain the critical points of I by using the Symmetric Mountain Pass Theorem. Since the minimax characterisation provides the critical value it is important for what follows. Therefore, we state this theorem precisely.

Lemma 2.1. ([24]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ with I even. Suppose that I satisfies (PS)-condition and the following conditions:*

- (i) $I(0) = 0$;
- (ii) *there exist constants $\rho, \alpha > 0$ such that $I|_{tialB_\rho(0)} \geq \alpha$;*

(*iii*) *for each finite dimensional subspace* $E' \subset E$*, there is* $r = r(E') > 0$ *such that* $I(u) \leq 0$ *for* $u \in E' \setminus B_r(0)$*, where* $B_r(0)$ *is an open ball in* E *of radius* r *centered at* 0*.*

Then I *possesses an unbounded sequence of critical values.*

Remark 2.2. As shown in [2], a deformation lemma can be proved with condition (C) replacing the usual (PS)-condition, and it turns out that Lemmas 2.1 holds true under condition (C). We say I satisfies condition (C), i.e., for every sequence $\{u_k\} \subset E$, $\{u_k\}$ has a convergent subsequence if $I(u_k)$ is bounded and $(1 + \|u_k\|)\|I'(u_k)\| \to 0$ as $k \to \infty$.

Lemma 2.3. *For* $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$

$$\|u\|_{L^\infty(\mathbb{R})} \leq \left(\frac{p-1}{2^q}\right)^{1/pq} \left[\int_{\mathbb{R}} (|\dot{u}(s)|^p + |u(s)|^p)\, ds\right]^{1/p}; \tag{2.3}$$

and for $u \in E$

$$\|u\|_{L^\infty(\mathbb{R})} \leq \left(\frac{p-1}{2^q a_*}\right)^{1/pq} \|u\|, \tag{2.4}$$

$$|u(t)| \leq (p-1)^{1/pq} \left\{\int_t^\infty [a(s)]^{-1/q} \left[|\dot{u}(s)|^p + a(s)|u(s)|^p\right] ds\right\}^{1/p}, \quad t \in \mathbb{R}, \tag{2.5}$$

and

$$|u(t)| \leq (p-1)^{1/pq} \left\{\int_{-\infty}^t [a(s)]^{-1/q} \left[|\dot{u}(s)|^p + a(s)|u(s)|^p\right] ds\right\}^{1/p}, \quad t \in \mathbb{R}, \tag{2.6}$$

where $a_* = \min\{a(t) : t \in \mathbb{R}\}$.

Proof. Since $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}} (|\dot{u}(t)|^p + |u(t)|^p)\, dt < \infty,$$

and so

$$\lim_{r\to\infty} \int_{|t|\geq r} (|\dot{u}(t)|^p + |u(t)|^p)\, dt = 0.$$

It is not difficulty to show that $\lim_{|t|\to\infty} |u(t)| = 0$, see, e.g. [(2.10), 28]. Hence, if $u \in W^{1,p}(\mathbb{R}, \mathbb{R}^N)$, then there exists $t^* \in (-\infty, \infty)$ such that

$$|u(t^*)| = \max_{t\in\mathbb{R}} |u(t)| = \|u\|_{L^\infty(\mathbb{R})}. \tag{2.7}$$

Choose two sequences $\{t_k\}$ and $\{t_{-k}\}$ such that

$$\cdots < t_{-3} < t_{-2} < t_{-1} < t^* < t_1 < t_2 < t_3 < \cdots,$$
$$\lim_{k\to\infty} t_k = +\infty, \qquad \lim_{k\to\infty} t_{-k} = -\infty,$$

and

$$\lim_{k\to\infty} |u(t_k)| = \lim_{k\to\infty} |u(t_{-k})| = 0.$$

Observe that

$$|u(t^*)|^p = |u(t_k)|^p - p\int_{t^*}^{t_k} |u(s)|^{p-2}(u(s), \dot{u}(s))ds, \tag{2.8}$$

and

$$|u(t^*)|^p = |u(t_{-k})|^p + p\int_{t_{-k}}^{t^*} |u(s)|^{p-2}(u(s), \dot{u}(s))ds. \tag{2.9}$$

From (2.8), (2.9) and Young's inequality, we have

$$\begin{aligned} |u(t^*)|^p &= \frac{1}{2}\left(|u(t_k)|^p + |u(t_{-k})|^p\right) - \frac{p}{2}\int_{t^*}^{t_k} |u(s)|^{p-2}(u(s), \dot{u}(s))ds \\ &\quad + \frac{p}{2}\int_{t_{-k}}^{t^*} |u(s)|^{p-2}(u(s), \dot{u}(s))ds \\ &\le \frac{1}{2}\left(|u(t_k)|^p + |u(t_{-k})|^p\right) + \frac{p}{2}\int_{t_{-k}}^{t_k} |u(s)|^{p-1}|\dot{u}(s)|ds \\ &\le \frac{1}{2}\left(|u(t_k)|^p + |u(t_{-k})|^p\right) + \frac{(p-1)^{1/q}}{2}\int_{t_{-k}}^{t_k} \left(|\dot{u}(s)|^p + |u(s)|^p\right)ds, \ k \in \mathbb{N}. \end{aligned}$$

Let $k \to \infty$ in the above, we obtain

$$|u(t^*)|^p \le \frac{(p-1)^{1/q}}{2}\int_{-\infty}^{\infty} \left(|\dot{u}(s)|^p + |u(s)|^p\right)ds,$$

which, together with (2.7), implies that (2.3) holds.

For $u \in E$, we have by (2.8), (2.9) and Young's inequality,

$$\begin{aligned} &|u(t^*)|^p \\ &\le \frac{1}{2}\left(|u(t_k)|^p + |u(t_{-k})|^p\right) + \frac{p}{2}\int_{t_{-k}}^{t_k} |u(s)|^{p-1}|\dot{u}(s)|ds \\ &\le \frac{1}{2}\left(|u(t_k)|^p + |u(t_{-k})|^p\right) + \frac{(p-1)^{1/q}}{2}\int_{t_{-k}}^{t_k} [a(s)]^{-1/q}\left(|\dot{u}(s)|^p + a(s)|u(s)|^p\right)ds \\ &\le \frac{1}{2}\left(|u(t_k)|^p + |u(t_{-k})|^p\right) + \left(\frac{p-1}{2^q a_*}\right)^{1/q}\int_{t_{-k}}^{t_k} \left(|\dot{u}(s)|^p + a(s)|u(s)|^p\right)ds, \quad k \in \mathbb{N}. \end{aligned}$$

Let $k \to \infty$ in the above, we obtain

$$|u(t^*)|^p \leq \left(\frac{p-1}{2^q a_*}\right)^{1/q} \int_{-\infty}^{\infty} [|\dot{u}(s)|^p + a(s)|u(s)|^p]\, ds,$$

which, together with (2.7), implies that (2.4) holds.

For any $t \in \mathbb{R}$, choose $k \in \mathbb{N}$ such that $t_{-k} < t < t_k$. Then we have

$$|u(t)|^p = |u(t_k)|^p - p \int_t^{t_k} |u(s)|^{p-2}(u(s), \dot{u}(s))ds, \tag{2.10}$$

and

$$|u(t)|^p = |u(t_{-k})|^p + 2 \int_{t_{-k}}^{t} |u(s)|^{p-2}(u(s), \dot{u}(s))ds. \tag{2.11}$$

By (2.10) and Young's inequality, we have

$$\begin{aligned} |u(t)|^p &\leq |u(t_k)|^p + p \int_t^{t_k} |u(s)|^{p-1}|\dot{u}(s)|ds \\ &\leq |u(t_k)|^p + (p-1)^{1/q} \int_t^{t_k} [a(s)]^{-1/q} (|\dot{u}(s)|^p + a(s)|u(s)|^p)\, ds, \quad k \in \mathbb{N}. \end{aligned}$$

Let $k \to \infty$ in the above, we obtain

$$|u(t)|^p \leq (p-1)^{1/q} \int_t^{\infty} [a(s)]^{-1/q} [|\dot{u}(s)|^p + a(s)|u(s)|^p]\, ds,$$

which implies that (2.5) holds.

Similarly, (2.6) can be proved by using (2.11) insteads of (2.10). The proof is complete.

Remark 2.4. The constant $\left(\frac{p-1}{2^q}\right)^{1/pq}$ in (2.3) is the best possible. For example, let

$$u(t) = \left(e^{-|t|/(p-1)^{1/p}}, 0, \cdots, 0\right)^\top \in \mathbb{R}^N.$$

Then

$$\|u\|_{L^\infty(\mathbb{R})} = |u(0)| = 1,$$

and

$$\int_{\mathbb{R}} [|\dot{u}(s)|^p + |u(s)|^p]\, ds = \frac{2p}{p-1} \int_0^{\infty} e^{-pt/(p-1)^{1/p}} dt = \frac{2}{(p-1)^{1/q}}.$$

This shows that the constant $\left(\frac{p-1}{2^q}\right)^{1/pq}$ *in (2.3) is the best possible.*

Lemma 2.5. *Assume that* $(W1)$ *or* $(W1')$ *hold. Then for every* $(t,x)\in\mathbb{R}\times\mathbb{R}^N$*,*
(i) $s^{-\mu}W_1(t,sx)$ *is nondecreasing on* $(0,+\infty)$*;*
(ii) $s^{-\varrho}W_2(t,sx)$ *is nonincreasing on* $(0,+\infty)$.

The proof of Lemma 2.5 is routine and so we omit it.

3. Proofs of Theorems

Proof of Theorem 1.1. It is clear that $I(0)=0$ and (W3) implies that I is even. We first show that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}}\subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k)\to 0$ as $k\to+\infty$. Then there exists a constant $c>0$ such that

$$|I(u_k)|\le c,\quad \|I'(u_k)\|_{E^*}\le \mu c\quad \text{for } k\in\mathbb{N}. \tag{3.1}$$

From (2.1), (2.2), (3.1) and (W1), we obtain

$$\begin{aligned}
&pc+pc\|u_k\|\\
\ge\ & pI(u_k)-\frac{p}{\mu}\langle I'(u_k),u_k\rangle\\
=\ & \frac{\mu-p}{\mu}\|u_k\|^p+p\int_{\mathbb{R}}\left[W_2(t,u_k(t))-\frac{1}{\mu}(\nabla W_2(t,u_k(t)),u_k(t))\right]dt\\
&-p\int_{\mathbb{R}}\left[W_1(t,u_k(t))-\frac{1}{\mu}(\nabla W_1(t,u_k(t)),u_k(t))\right]dt\\
\ge\ & \frac{\mu-p}{\mu}\|u_k\|^p,\quad k\in\mathbb{N}.
\end{aligned}$$

It follows that there exists a constant $A>0$ such that

$$\|u_k\|\le A\quad \text{for } k\in\mathbb{N}. \tag{3.2}$$

So passing to a subsequence if necessary, it can be assumed that $u_k\rightharpoonup u_0$ in E. For any given number $\varepsilon>0$, by (W2), we can choose $\xi>0$ such that

$$|\nabla W(t,x)|\le \varepsilon a(t)|x|^{p-1}\quad \text{for } |t|\ge R,\ \text{ and } |x|\le\xi. \tag{3.3}$$

Since $a(t)\to+\infty$ as $t\to\pm\infty$, we can also choose $T>R$ such that

$$a(t)\ge \frac{(p-1)A^{pq}}{\xi^{pq}},\quad |t|\ge T. \tag{3.4}$$

By (2.5), (3.2) and (3.4), we have

$$\begin{aligned}
|u_k(t)|^p &\le (p-1)^{1/q}\int_t^\infty [a(s)]^{-1/q}\left[|\dot u_k(s)|^p+a(s)|u_k(s)|^p\right]ds\\
&\le \frac{\xi^p}{A^p}\int_t^\infty \left[|\dot u_k(s)|^p+a(s)|u_k(s)|^p\right]ds\\
&\le \frac{\xi^p}{A^p}\|u_k\|^p\le \xi^p\quad \text{for } t\ge T,\ \ k\in\mathbb{N}.
\end{aligned} \tag{3.5}$$

Similarly, we have

$$|u_k(t)|^p \le \xi^p \quad \text{for } t \le -T, \quad k \in \mathbb{N}. \tag{3.6}$$

Since $u_k \rightharpoonup u_0$ in E, it is easy to verify that $u_k(t)$ converses to $u_0(t)$ pointwise for all $t \in \mathbb{R}$. Hence, we have by (3.5) and (3.6)

$$|u_0(t)| \le \xi \quad \text{for } t \in (-\infty, -T] \cup [T, +\infty). \tag{3.7}$$

Since $a(t) \ge a_* > 0$ on $[-T, T] = J$, the operator defined by $S : E \to W^{1,p}(J) : u \to u|_J$ is a linear continuous map. So $u_k \rightharpoonup u_0$ in $W^{1,p}(J)$. Sobolev's theorem (see e.g. [19]) implies that $u_k \to u_0$ uniformly on J, so there is $k_0 \in \mathbb{N}$ such that

$$\int_{-T}^{T} |\nabla W(t, u_k(t)) - \nabla W(t, u_0(t))||u_k(t) - u_0(t)|dt < \varepsilon \quad \text{for } k \ge k_0. \tag{3.8}$$

On the other hand, it follows from (3.2), (3.3), (3.5), (3.6) and (3.7) that

$$\begin{aligned} &\int_{\mathbb{R}\setminus[-T,T]} |\nabla W(t, u_k(t)) - \nabla W(t, u_0(t))||u_k(t) - u_0(t)|dt \\ &\le \int_{\mathbb{R}\setminus[-T,T]} (|\nabla W(t, u_k(t))| + |\nabla W(t, u_0(t))|)(|u_k(t)| + |u_0(t)|)dt \\ &\le \varepsilon \int_{\mathbb{R}\setminus[-T,T]} a(t)(|u_k(t)|^{p-1} + |u_0(t)|^{p-1})(|u_k(t)| + |u_0(t)|)dt \\ &\le 2\varepsilon \int_{\mathbb{R}\setminus[-T,T]} a(t)(|u_k(t)|^p + |u_0(t)|^p)dt \\ &\le 2\varepsilon(\|u_k\|^p + \|u_0\|^p) \\ &\le 2\varepsilon(A^p + \|u_0\|^p), \quad k \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Combining (3.8) with (3.9) we get

$$\int_{\mathbb{R}} |\nabla W(t, u_k(t)) - \nabla W(t, u_0(t))|\,|u_k(t) - u_0(t)|dt \to 0 \quad \text{as } k \to \infty. \tag{3.10}$$

It follows from (2.2) and the Hölder's inequality that

$$\begin{aligned} &\langle I'(u_k) - I'(u_0), u_k - u_0\rangle \\ &= \int_{\mathbb{R}} |\dot{u}_k(t)|^{p-2}(\dot{u}_k(t), \dot{u}_k(t) - \dot{u}_0(t))dt \\ &\quad + \int_{\mathbb{R}} a(t)|u_k(t)|^{p-2}(u_k(t), u_k(t) - u_0(t))dt \\ &\quad - \int_{\mathbb{R}} |\dot{u}_0(t)|^{p-2}(\dot{u}_0(t), \dot{u}_k(t) - \dot{u}_0(t))dt \end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}} a(t)|u_0(t)|^{p-2}(u_0(t), u_k(t)-u_0(t))dt \\
&-\int_{\mathbb{R}} (\nabla W(t,u_k(t)) - \nabla W(t,u_0(t)), u_k(t)-u_0(t))dt \\
=\ & \|u_k\|^p + \|u_0\|^p - \int_{\mathbb{R}} |\dot{u}_k(t)|^{p-2}(\dot{u}_k(t), \dot{u}_0(t))dt \\
&-\int_{\mathbb{R}} a(t)|u_k(t)|^{p-2}(u_k(t), u_0(t))dt \\
&-\int_{\mathbb{R}} |\dot{u}_0(t)|^{p-2}(\dot{u}_0(t), \dot{u}_k(t))dt - \int_{\mathbb{R}} a(t)|u_0(t)|^{p-2}(u_0(t), u_k(t))dt \\
&-\int_{\mathbb{R}} (\nabla W(t,u_k(t)) - \nabla W(t,u_0(t)), u_k(t)-u_0(t))dt \\
\geq\ & \|u_k\|^p + \|u_0\|^p - \int_{\mathbb{R}} |\dot{u}_k(t)|^{p-1}|\dot{u}_0(t)|dt - \int_{\mathbb{R}} a(t)|u_k(t)|^{p-1}|u_0(t)|dt \\
&-\int_{\mathbb{R}} |\dot{u}_0(t)|^{p-1}|\dot{u}_k(t)|dt - \int_{\mathbb{R}} a(t)|u_0(t)|^{p-1}|u_k(t)|dt \\
&-\int_{\mathbb{R}} (\nabla W(t,u_k(t)) - \nabla W(t,u_0(t)), u_k(t)-u_0(t))dt \\
\geq\ & \|u_k\|^p + \|u_0\|^p - \left(\int_{\mathbb{R}} |\dot{u}_0(t)|^p dt\right)^{1/p} \left(\int_{\mathbb{R}} |\dot{u}_k(t)|^p dt\right)^{1/q} \\
&-\left(\int_{\mathbb{R}} a(t)|u_0(t)|^p dt\right)^{1/p} \left(\int_{\mathbb{R}} a(t)|u_k(t)|^p dt\right)^{1/q} \\
&-\left(\int_{\mathbb{R}} |\dot{u}_k(t)|^p dt\right)^{1/p} \left(\int_{\mathbb{R}} |\dot{u}_0(t)|^p dt\right)^{1/q} \\
&-\left(\int_{\mathbb{R}} a(t)|u_k(t)|^p dt\right)^{1/p} \left(\int_{\mathbb{R}} a(t)|u_0(t)|^p dt\right)^{1/q} \\
&-\int_{\mathbb{R}} (\nabla W(t,u_k(t)) - \nabla W(t,u_0(t)), u_k(t)-u_0(t))dt \\
\geq\ & \|u_k\|^p + \|u_0\|^p - \left(\int_{\mathbb{R}} \left[|\dot{u}_0(t)|^p + a(t)|u_0(t)|^p\right] dt\right)^{1/p} \\
&\left(\int_{\mathbb{R}} \left[|\dot{u}_k(t)|^p + a(t)|u_k(t)|^p dt\right] dt\right)^{1/q} - \left(\int_{\mathbb{R}} \left[|\dot{u}_k(t)|^p + a(t)|u_k(t)|^p\right] dt\right)^{1/p} \\
&\left(\int_{\mathbb{R}} \left[|\dot{u}_0(t)|^p + a(t)|u_0(t)|^p\right] dt\right)^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u_0(t)), u_k(t) - u_0(t))dt \\
&= \|u_k\|^p + \|u_0\|^p - \|u_0\|\|u_k\|^{p-1} - \|u_k\|\|u_0\|^{p-1} \\
&\quad - \int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u_0(t)), u_k(t) - u_0(t))dt \\
&= \left(\|u_k\|^{p-1} - \|u_0\|^{p-1}\right)(\|u_k\| - \|u_0\|) \\
&\quad - \int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u_0(t)), u_k(t) - u_0(t))dt.
\end{aligned}
$$

Since $I'(u_k) \to 0$ as $k \to +\infty$ and $u_k \rightharpoonup u_0$ in E, it follows from (3.2) that

$$\langle I'(u_k) - I'(u_0), u_k - u_0\rangle \to 0 \quad \text{as} \quad k \to \infty,$$

which, together with (3.10) and (3.11), yields $\|u_k\| \to \|u\|$ as $k \to +\infty$. By the uniform convexity of E and the fact that $u_k \rightharpoonup u_0$ in E, it follows from the Kadec-Klee property [11] that $u_k \to u_0$ in E. Hence, I satisfies (PS)-condition.

We now show that there exist constants $\rho, \alpha > 0$ such that I satisfies assumption (ii) of Lemma 2.1 with these constants. By (W2), there exists $\eta \in (0, 1)$ such that

$$|\nabla W(t, x)| \le \frac{1}{2} a(t)|x|^{p-1} \quad \text{for} \ \ |t| \ge R, \ \ |x| \le \eta. \tag{3.12}$$

Since $W(t, 0) = 0$, it follows that

$$|W(t, x)| \le \frac{1}{2p} a(t)|x|^{p} \quad \text{for} \ \ |t| \ge R, \ \ |x| \le \eta. \tag{3.13}$$

Set

$$M = \sup\left\{ \frac{W_1(t, x)}{a(t)} \ \middle| \ t \in [-R, R], \ x \in \mathbb{R}^N, \ |x| = 1 \right\}. \tag{3.14}$$

Set $\delta = \min\{1/(2pM + 1)^{1/(\mu - p)}, \eta\}$. If $\|u\| = \left(\frac{2^q a_*}{p-1}\right)^{1/pq} \delta := \rho$, then by (2.4), $|u(t)| \le \delta \le \eta < 1$ for $t \in \mathbb{R}$. By (3.14) and Lemma 2.5 (i), we have

$$
\begin{aligned}
\int_{-R}^{R} W_1(t, u(t))dt &\le \int_{\{t \in [-R, R] \, : \, u(t) \neq 0\}} W_1\left(t, \frac{u(t)}{|u(t)|}\right) |u(t)|^{\mu} dt \\
&\le M \int_{-R}^{R} a(t)|u(t)|^{\mu} dt \\
&\le M\delta^{\mu - p} \int_{-R}^{R} a(t)|u(t)|^{p} dt \\
&\le \frac{1}{2p} \int_{-R}^{R} a(t)|u(t)|^{p} dt.
\end{aligned} \tag{3.15}
$$

Set

$$\alpha = \left(\frac{2^q a_*}{p-1}\right)^{1/q} \frac{\delta^p}{2p}.$$

Hence, from (2.1), (3.13), (3.15) and (W4), we have

$$\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p - \int_{\mathbb{R}} W(t,u(t))dt \\
&= \frac{1}{p}\|u\|^p - \int_{\mathbb{R}\setminus[-R,R]} W(t,u(t))dt - \int_{-R}^{R} W(t,u(t))dt \\
&\geq \frac{1}{p}\|u\|^p - \frac{1}{2p}\int_{\mathbb{R}\setminus[-R,R]} a(t)|u(t)|^p dt - \int_{-R}^{R} W_1(t,u(t))dt \\
&\geq \frac{1}{p}\|u\|^p - \frac{1}{2p}\int_{\mathbb{R}\setminus[-R,R]} a(t)|u(t)|^p dt - \frac{1}{2p}\int_{-R}^{R} a(t)|u(t)|^p dt \\
&= \frac{1}{p}\int_{\mathbb{R}} |\dot{u}(t)|^p dt + \frac{1}{2p}\int_{\mathbb{R}} a(t)|u(t)|^p dt \\
&\geq \frac{1}{2p}\int_{\mathbb{R}} \left[|\dot{u}(t)|^p + a(t)|u(t)|^p\right] dt \\
&= \frac{1}{2p}\|u\|^p \\
&= \alpha.
\end{aligned} \tag{3.16}$$

(3.16) shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$, i.e., I satisfies assumption (ii) of Lemma 2.1.

Finally, it remains to show that I satisfies assumption (iii) of Lemma 2.1. Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional normed space are equivalent, so there is a constant $c > 0$ such that

$$\|u\| \leq c\|u\|_{L^\infty(\mathbb{R})} \quad \text{for } u \in E'. \tag{3.17}$$

Assume that $\dim E' = m$ and $u_1, u_2, \ldots, u_m$ are the base of E' such that

$$\|u_i\| = c, \quad i = 1, 2, \ldots, m. \tag{3.18}$$

For any $u \in E'$, there exist $\lambda_i \in \mathbb{R},\ i = 1, 2, \ldots, m$ such that

$$u(t) = \sum_{i=1}^{m} \lambda_i u_i(t) \quad \text{for } t \in \mathbb{R}. \tag{3.19}$$

Let

$$\|u\|_* = \sum_{i=1}^{m} |\lambda_i|\|u_i\|. \tag{3.20}$$

It is easy to verify that $\|\cdot\|_*$ defined by (3.20) is a norm of E'. Hence, there exists a constants $c' > 0$ such that

$$c'\|u\|_* \le \|u\| \quad \text{for } u \in E'. \tag{3.21}$$

Since $u_i \in E$, by Lemma 2.3, we can choose $R_1 > R$ such that

$$|u_i(t)| < \frac{c'\eta}{1+c'}, \qquad |t| > R_1, \quad i = 1, 2, \ldots, m, \tag{3.22}$$

where η is given in (3.13). Set

$$\begin{aligned} \Theta &= \left\{ \sum_{i=1}^{m} \lambda_i u_i(t) \ : \ \lambda_i \in \mathbb{R}, \ i = 1, 2, \ldots, m; \ \sum_{i=1}^{m} |\lambda_i| = 1 \right\} \\ &= \left\{ u \in E' \ : \ \|u\|_* = c \right\}. \end{aligned} \tag{3.23}$$

Hence, for $u \in \Theta$, let $t_0 = t_0(u) \in \mathbb{R}$ such that

$$|u(t_0)| = \|u\|_{L^\infty(\mathbb{R})}. \tag{3.24}$$

Then by (3.17)-(3.21), (3.23) and (3.24), we have

$$\begin{aligned} c'c = c'c \sum_{i=1}^{m} |\lambda_i| &= c' \sum_{i=1}^{m} |\lambda_i| \|u_i\| = c'\|u\|_* \\ &\le \|u\| \le c\|u\|_{L^\infty(\mathbb{R})} = c|u(t_0)| \\ &\le c \sum_{i=1}^{m} |\lambda_i| |u_i(t_0)|, \qquad u \in \Theta. \end{aligned} \tag{3.25}$$

This shows that $|u(t_0)| \ge c'$ and there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $|u_{i_0}(t_0)| \ge c'$, which, together with (3.22), implies that $|t_0| \le R_1$. Set $R_2 = R_1 + 1$ and

$$\gamma = \min \left\{ W_1(t, x) : -R_2 \le t \le R_2, \ \frac{c'}{2} \le |x| \le c \left(\frac{p-1}{2^q a_*} \right)^{1/pq} \right\}. \tag{3.26}$$

Since $W_1(t, x) > 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$, and $W_1 \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, it follows that $\gamma > 0$. For any $u \in E$, it follows from (2.4) and Lemma 2.5 (ii) that

$$\begin{aligned}
&\int_{-R_2}^{R_2} W_2(t,u(t))dt \\
&= \int_{\{t\in[-R_2,R_2]\,:\,|u(t)|>1\}} W_2(t,u(t))dt \\
&\quad + \int_{\{t\in[-R_2,R_2]\,:\,|u(t)|\le 1\}} W_2(t,u(t))dt \\
&\le \int_{\{t\in[-R_2,R_2]\,:\,|u(t)|>1\}} W_2\left(t,\frac{u(t)}{|u(t)|}\right)|u(t)|^{\varrho}dt \\
&\quad + \int_{-R_2}^{R_2} \max_{|x|\le 1}|W_2(t,x)|dt \\
&\le \|u\|_{L^\infty(\mathbb{R})}^{\varrho}\int_{-R_2}^{R_2}\max_{|x|=1}|W_2(t,x)|dt + \int_{-R_2}^{R_2}\max_{|x|\le 1}|W_2(t,x)|dt \\
&\le \left(\frac{p-1}{2^q a_*}\right)^{\varrho/pq}\|u\|^{\varrho}\int_{-R_2}^{R_2}\max_{|x|=1}|W_2(t,x)|dt + \int_{-R_2}^{R_2}\max_{|x|\le 1}|W_2(t,x)|dt \\
&= M_1\|u\|^{\varrho} + M_2,
\end{aligned} \tag{3.27}$$

where

$$M_1 = \left(\frac{p-1}{2^q a_*}\right)^{\varrho/pq}\int_{-R_2}^{R_2}\max_{|x|=1}|W_2(t,x)|dt, \qquad M_2 = \int_{-R_2}^{R_2}\max_{|x|\le 1}|W_2(t,x)|dt.$$

Since $\dot{u}_i \in L^p(\mathbb{R}),\ i = 1,2,\ldots,m$, it follows that there exists $\epsilon \in (0,1)$ such that

$$\begin{aligned}
\int_{t-\epsilon}^{t+\epsilon}|\dot{u}_i(s)|ds &\le (2\epsilon)^{1/q}\left(\int_{t-\epsilon}^{t+\epsilon}|\dot{u}_i(s)|^p ds\right)^{1/p} \\
&\le (2\epsilon)^{1/q}\|\dot{u}_i\|_{L^p(\mathbb{R})} \\
&\le \frac{c'}{2p} \quad \text{for } \ t\in\mathbb{R},\ \ i=1,2,\ldots,m.
\end{aligned} \tag{3.28}$$

Then for $u \in \Theta$ with $|u(t_0)| = \|u\|_{L^\infty(\mathbb{R})}$ and $t \in [t_0-\epsilon, t_0+\epsilon]$, it follows from (3.19), (3.23), (3.24), (3.25) and (3.28) that

$$\begin{aligned}
|u(t)|^p &= |u(t_0)|^p + p\int_{t_0}^{t}|u(s)|^{p-2}(\dot{u}(s),u(s))ds \\
&\ge |u(t_0)|^p - p\int_{t_0-\epsilon}^{t_0+\epsilon}|\dot{u}(s)||u(s)|^{p-1}ds \\
&\ge |u(t_0)|^p - p|u(t_0)|^{p-1}\int_{t_0-\epsilon}^{t_0+\epsilon}|\dot{u}(s)|ds
\end{aligned} \tag{3.29}$$

$$\geq |u(t_0)|^p - p|u(t_0)|^{p-1}\sum_{i=1}^{m}|\lambda_i|\int_{t_0-\epsilon}^{t_0+\epsilon}|\dot{u}_i(s)|ds$$
$$\geq \frac{c'}{2}|u(t_0)|^{p-1} \geq \left(\frac{c'}{2}\right)^p.$$

On the other hand, since $\|u\| \leq c$ for $u \in \Theta$, it follows from (2.4) that

$$|u(t)| \leq c\left(\frac{p-1}{2^q a_*}\right)^{1/pq} \quad \text{for } t \in \mathbb{R}, \quad u \in \Theta. \tag{3.30}$$

Hence, from (3.26), (3.29) and (3.30), we have

$$\int_{-R_2}^{R_2} W_1(t,u(t))dt \geq \int_{t_0-\epsilon}^{t_0+\epsilon} W_1(t,u(t))dt \geq 2\epsilon\gamma \quad \text{for } u \in \Theta. \tag{3.31}$$

By (3.22) and (3.23), we have

$$|u(t)| \leq \sum_{i=1}^{m}|\lambda_i||u_i(t)| \leq \eta \quad \text{for } |t| \geq R_1, \quad u \in \Theta. \tag{3.32}$$

From (3.13), (3.27), (3.31), (3.32) and Lemma 2.5, we have for $u \in \Theta$ and $\sigma > 1$

$$\begin{aligned}
&I(\sigma u)\\
&= \frac{\sigma^p}{p}\|u\|^p - \int_{\mathbb{R}} W(t,\sigma u(t))dt\\
&= \frac{\sigma^p}{p}\|u\|^p + \int_{\mathbb{R}} W_2(t,\sigma u(t))dt - \int_{\mathbb{R}} W_1(t,\sigma u(t))dt\\
&\leq \frac{\sigma^p}{p}\|u\|^p + \sigma^\varrho\int_{\mathbb{R}} W_2(t,u(t))dt - \sigma^\mu\int_{\mathbb{R}} W_1(t,u(t))dt\\
&= \frac{\sigma^p}{p}\|u\|^p + \sigma^\varrho\int_{\mathbb{R}\setminus(-R_2,R_2)} W_2(t,u(t))dt - \sigma^\mu\int_{\mathbb{R}\setminus(-R_2,R_2)} W_1(t,u(t))dt\\
&\quad +\sigma^\varrho\int_{-R_2}^{R_2} W_2(t,u(t))dt - \sigma^\mu\int_{-R_2}^{R_2} W_1(t,u(t))dt\\
&\leq \frac{\sigma^p}{p}\|u\|^p - \sigma^\varrho\int_{\mathbb{R}\setminus(-R_2,R_2)} W(t,u(t))dt\\
&\quad +\sigma^\varrho\int_{-R_2}^{R_2} W_2(t,u(t))dt - \sigma^\mu\int_{-R_2}^{R_2} W_1(t,u(t))dt\\
&\leq \frac{\sigma^p}{p}\|u\|^p + \frac{\sigma^\varrho}{2p}\int_{\mathbb{R}\setminus(-R_2,R_2)} a(t)|u(t)|^p dt + \sigma^\varrho(M_1\|u\|^\varrho + M_2) - 2\epsilon\gamma\sigma^\mu\\
&\leq \frac{\sigma^p}{p}\|u\|^p + \frac{\sigma^\varrho}{2p}\|u\|^p + \sigma^\varrho(M_1\|u\|^\varrho + M_2) - 2\epsilon\gamma\sigma^\mu\\
&\leq \frac{(c\sigma)^p}{p} + \frac{c^p\sigma^\varrho}{2p} + M_1(c\sigma)^\varrho + M_2\sigma^\varrho - 2\epsilon\gamma\sigma^\mu.
\end{aligned} \tag{3.33}$$

Since $\mu > \varrho \geq p$, we deduce that there is $\sigma_0 = \sigma_0(c, c', M_1, M_2, R_1, R_2, \epsilon, \gamma) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for } \ u \in \Theta \ \text{ and } \ \sigma \geq \sigma_0.$$

It follows that

$$I(u) < 0 \quad \text{for } \ u \in E' \ \text{ and } \ \|u\| \geq c\sigma_0.$$

This shows that (iii) of Lemma 2.1 holds. By Lemma 2.1, I possesses an unbounded sequence $\{d_k\}_{k=1}^{\infty}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. If $\{\|u_k\|\}$ is bounded, then there exists $B > 0$ such that

$$\|u_k\| \leq B \quad \text{for } \ k \in \mathbb{N}. \tag{3.34}$$

By a similar fashion for the proof of (3.5) and (3.6), for the given η in (3.13), there exists $R_3 > R$ such that

$$|u_k(t)| \leq \eta \quad \text{for } \ |t| \geq R_3, \quad k \in \mathbb{N}. \tag{3.35}$$

Thus, from (2.1), (2.4), (3.13), (3.34) and (3.35), we have

$$\begin{aligned} \frac{1}{p}\|u_k\|^p &= d_k + \int_{\mathbb{R}} W(t, u_k(t))dt \\ &= d_k + \int_{\mathbb{R}\setminus[-R_3,R_3]} W(t, u_k(t))dt + \int_{-R_3}^{R_3} W(t, u_k(t))dt \\ &\geq d_k - \frac{1}{2p}\int_{\mathbb{R}\setminus[-R_3,R_3]} a(t)|u_k(t)|^p - \int_{-R_3}^{R_3} |W(t, u_k(t))|dt \\ &\geq d_k - \frac{1}{2p}\|u_k\|^p - \int_{-R_3}^{R_3} \max_{|x|\leq[(p-1)/2^q a_*]^{1/pq}B} |W(t, x)|dt. \end{aligned} \tag{3.36}$$

It follows that

$$d_k \leq \frac{3}{2p}\|u_k\|^p + \int_{-R_3}^{R_3} \max_{|x|\leq[(p-1)/2^q a_*]^{1/pq}B} |W(t, x)|dt < +\infty.$$

This contradicts to the fact that $\{d_k\}_{k=1}^{\infty}$ is unbounded, and so $\{\|u_k\|\}$ is unbounded. The proof is complete.

Proof of Theorem 1.2. In the proof of Theorem 1.1, the condition that $W_2(t, x) \geq 0$ in (W1) is only used in the proofs of (3.2) and assumption (ii) of Lemma 2.1. Therefore, we only prove (3.2) and assumption (ii) of Lemma 2.1 still holds use (W1′) instead of (W1). We first prove that (3.2) still holds. From (2.1), (2.2), (3.1) and (W1$'$), we obtain

$$\begin{aligned} & pc + \frac{pc\mu}{\varrho}\|u_k\| \\ \geq\ & pI(u_k) - \frac{p}{\varrho}\langle I'(u_k), u_k\rangle \\ =\ & \frac{\varrho - p}{\varrho}\|u_k\|^p + p\int_{\mathbb{R}}\left[W_2(t, u_k(t)) - \frac{1}{\varrho}(\nabla W_2(t, u_k(t)), u_k(t))\right]dt \\ & -p\int_{\mathbb{R}}\left[W_1(t, u_k(t)) - \frac{1}{\varrho}(\nabla W_1(t, u_k(t)), u_k(t))\right]dt \\ \geq\ & \frac{\varrho - p}{\varrho}\|u_k\|^p, \qquad k \in \mathbb{N}. \end{aligned}$$

It follows that there exists a constant $A > 0$ such that (3.2) holds. Next, we prove that assumption (ii) of Lemma 2.1 still holds. By (W2$'$), there exists $\eta \in (0, 1)$ such that

$$|\nabla W(t, x)| \leq \frac{1}{2}a(t)|x|^{p-1} \quad \text{for } t \in \mathbb{R}, \quad |x| \leq \eta. \tag{3.37}$$

Since $W(t, 0) = 0$, it follows that

$$|W(t, x)| \leq \frac{1}{2p}a(t)|x|^p \quad \text{for } t \in \mathbb{R}, \quad |x| \leq \eta. \tag{3.38}$$

If $\|u\| = \left(\frac{2^q a_*}{p-1}\right)^{1/pq}\eta := \rho$, then by (2.4), $|u(t)| \leq \eta$ for $t \in \mathbb{R}$. Set

$$\alpha = \left(\frac{2^q a_*}{p-1}\right)^{1/q}\frac{\eta^p}{2p}.$$

Hence, from (2.1) and (3.38), we have

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \int_{\mathbb{R}} W(t, u(t))dt \\ &\geq \frac{1}{p}\|u\|^p - \frac{1}{2p}\int_{\mathbb{R}} a(t)|u(t)|^p dt \\ &= \frac{1}{p}\int_{\mathbb{R}} |\dot{u}(t)|^p dt + \frac{1}{2p}\int_{\mathbb{R}} a(t)|u(t)|^p dt \\ &\geq \frac{1}{2p}\int_{\mathbb{R}} \left[|\dot{u}(t)|^p + a(t)|u(t)|^p\right]dt \\ &= \frac{1}{2p}\|u\|^p \\ &= \alpha. \end{aligned} \tag{3.39}$$

(3.39) shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$, i.e., assumption (ii) of Lemma 2.1 holds. The proof of Theorem 1.2 is completed.

Proof of Theorem 1.3. We first show that I satisfies condition (C). Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a (C) sequence of I, that is, $\{I(u_k)\}$ is bounded and $(1 + \|u_k\|)\|I'(u_k)\| \to 0$ as $k \to \infty$. Then it follows from (2.1) and (2.2) that

$$\begin{aligned} C_1 &\geq pI(u_k) - \langle I'(u_k), u_k\rangle \\ &= \int_{\mathbb{R}} [(\nabla W(t, u_k(t)), u_k(t)) - pW(t, u_k(t))]\, dt. \end{aligned} \tag{3.40}$$

It follows from (W2′) that there exists $\eta \in (0, 1)$ such that (3.38) holds. By (W4), we have

$$(\nabla W(t, x), x) \geq pW(t, x) \geq 0 \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^N, \tag{3.41}$$

and

$$\begin{aligned} W(t, x) \leq{}& (\alpha + \beta|x|^{\nu})[(\nabla W(t, x), x) \\ & -pW(t, x)] \quad \text{for } (t, x) \in \mathbb{R} \times \{x \in \mathbb{R}^N : |x| > \eta\}. \end{aligned} \tag{3.42}$$

It follows from (2.1), (2.4), (3.38), (3.40), (3.41) and (3.42) that

$$\begin{aligned} &\frac{1}{p}\|u_k\|^p \\ &= I(u_k) + \int_{\mathbb{R}} W(t, u_k(t))dt \\ &= I(u_k) + \int_{\{t\in\mathbb{R}\,:\,|u_k(t)|\leq\eta\}} W(t, u_k(t))dt + \int_{\{t\in\mathbb{R}\,:\,|u_k(t)|>\eta\}} W(t, u_k(t))dt \\ &\leq I(u_k) + \frac{1}{2p}\int_{\{t\in\mathbb{R}\,:\,|u_k(t)|\leq\eta\}} a(t)|u_k(t)|^p dt \\ &\quad + \int_{\{t\in\mathbb{R}\,:\,|u_k(t)|>\eta\}} (\alpha+\beta|u_k(t)|^{\nu})[(\nabla W(t, u_k(t)), u_k(t))-pW(t, u_k(t))]dt \\ &\leq C_2+\frac{1}{2p}\|u_k\|^p+\int_{\mathbb{R}}(\alpha+\beta|u_k(t)|^{\nu})[(\nabla W(t, u_k(t)), u_k(t))-pW(t, u_k(t))]dt \\ &\leq C_2+\frac{1}{2p}\|u_k\|^p+(\alpha+\beta\|u_k\|_{\infty}^{\nu})\int_{\mathbb{R}}[(\nabla W(t, u_k(t)), u_k(t))-pW(t, u_k(t))]dt \\ &\leq C_2+\frac{1}{2p}\|u_k\|^p + C_1(\alpha + \beta\|u_k\|_{\infty}^{\nu}) \\ &\leq C_2+\frac{1}{2p}\|u_k\|^p + C_1\left[\alpha + \left(\frac{p-1}{2^q a_*}\right)^{\nu/pq}\beta\|u_k\|^{\nu}\right]. \end{aligned} \tag{3.43}$$

Since $\nu < p$, it follows that $\{\|u_k\|\}$ is bounded. Similar to the proof of Theorem 1.1, we can prove that $\{u_k\}$ has a convergent subsequence in E. Hence, I satisfies condition (C).

It is obvious that I is even and $I(0) = 0$ and so assumption (i) of Lemma 2.1 holds. The proof of assumption (ii) of Lemma 2.1 is the same as in the proof of Theorem 1.2.

Now, we prove condition (iii) of Lemma 2.1. Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional normed space are equivalent, so there is a constants $c > 0$ such that (3.17) holds. Assume that $\dim E' = m$ and $u_1, u_2, \ldots, u_m$ are the base of E' such that (3.18) holds. Let $c', \eta, \epsilon, R_1, R_2$ and Θ be the same as in the proof of Theorem 1.1. Then (3.21) and (3.22) hold. For $u \in \Theta$, let $t_0 = t_0(u) \in \mathbb{R}$ such (3.24) holds. Then $\|u\|_{L^\infty(\mathbb{R})} = |u(t_0)| \geq c'$, and so (3.29) holds. For R_2 and $\epsilon \in (0,1)$ given in the proof of Theorem 1.1, by (W5), there exists $\sigma_0 = \sigma_0(\epsilon, R_2) > 1$ such that

$$(3.44) \qquad s^{-p} \int_{t-\epsilon}^{t+\epsilon} \min_{|x|\geq 1} W(\tau, sx)d\tau \geq \left(\frac{2c}{c'}\right)^p \qquad \text{for } \; s \geq c'\sigma_0/2, \quad t \in [-R_2, R_2].$$

It follows from (2.1), (W4), (3.29), (3.41) and (3.44) that

$$(3.45) \qquad \begin{aligned} I(\sigma u) &= \frac{\sigma^p}{p}\|u\|^p - \int_{\mathbb{R}} W(t, \sigma u(t))dt \\ &\leq \frac{\sigma^p}{p}\|u\|^p - \int_{t_0-\epsilon}^{t_0+\epsilon} W(t, \sigma u(t))dt \\ &\leq \frac{(c\sigma)^p}{p} - \int_{t_0-\epsilon}^{t_0+\epsilon} \min_{|x|\geq 1} W(t, 2^{-1}c'\sigma x)dt \\ &\leq \frac{(c\sigma)^p}{p} - (c\sigma)^p \\ &= -\frac{(p-1)(c\sigma)^p}{p} \qquad \text{for } \; u \in \Theta \; \text{ and } \; \sigma \geq \sigma_0. \end{aligned}$$

That is

$$I(\sigma u) < 0 \qquad \text{for } \; u \in \Theta \; \text{ and } \; \sigma \geq \sigma_0,$$

where $\sigma_0 = \sigma_0(\epsilon, R_2) = \sigma_0(E') > 1$. It follows that

$$I(u) < 0 \qquad \text{for } \; u \in E' \; \text{ and } \; \|u\| \geq c\sigma_0.$$

This shows that condition (iii) of Lemma 2.1 holds. The rest proof is the same as that in Theorem 1.1. The proof is complete.

4. Examples

In this section, we give three examples to illustrate our results.

Example 4.1. Consider the second-order ordinary p-Laplacian system

$$\frac{d}{dt}\left(|\dot{u}(t)|\dot{u}(t)\right) - a(t)|u(t)|u(t) + \nabla W(t,u(t)) = 0, \tag{4.1}$$

where $p = 3$, $t \in \mathbb{R}, u \in \mathbb{R}^N$, $a \in C(\mathbb{R},(0,\infty))$ such that $a(t) \to +\infty$ as $|t| \to \infty$. Let

$$W(t,x) = a(t)\left(\sum_{i=1}^{m} a_i|x|^{\mu_i} - \sum_{j=1}^{n} b_j|x|^{\varrho_j}\right),$$

where $\mu_1 > \mu_2 > \cdots > \mu_m > \varrho_1 > \varrho_2 > \cdots > \varrho_n > 3,\ a_i, b_j > 0,\ i = 1,2,\ldots,m; j = 1,2,\ldots,n$. Let $\mu = \mu_m,\ \varrho = \varrho_1$, and

$$W_1(t,x) = a(t)\sum_{i=1}^{m} a_i|x|^{\mu_i}, \qquad W_2(t,x) = a(t)\sum_{j=1}^{n} b_j|x|^{\varrho_j}.$$

Then it is easy to verify that all conditions of Theorem 1.1 are satisfied. By Theorem 1.1, system (4.1) has an unbounded sequence of homoclinic solutions.

Example 4.2. Consider the second-order ordinary p-Laplacian system

$$\frac{d}{dt}\left(|\dot{u}(t)|^2\dot{u}(t)\right) - a(t)|u(t)|^2u(t) + \nabla W(t,u(t)) = 0, \tag{4.2}$$

where $p = 4$, $t \in \mathbb{R}, u \in \mathbb{R}^N$, $a \in C(\mathbb{R},(0,\infty))$ such that $a(t) \to +\infty$ as $|t| \to \infty$. Let

$$W(t,x) = a(t)\left[a_1|x|^{\mu_1} + a_2|x|^{\mu_2} - b_1(\sin t)|x|^{\varrho_1} - b_2|x|^{\varrho_2}\right],$$

where $\mu_1 > \mu_2 > \varrho_1 > \varrho_2 > 4$, $a_1, a_2 > 0, b_1, b_2 > 0$. Let $\mu = \mu_2,\ \varrho = \varrho_1$, and

$$W_1(t,x) = a(t)\left(a_1|x|^{\mu_1} + a_2|x|^{\mu_2}\right), \qquad W_2(t,x) = a(t)\left[b_1(\sin t)|x|^{\varrho_1} + b_2|x|^{\varrho_2}\right].$$

Then it is easy to verify that all conditions of Theorem 1.2 are satisfied. By Theorem 1.2, system (4.2) has an unbounded sequence of homoclinic solutions.

Example 4.3. Consider the second-order ordinary p-Laplacian system

$$\frac{d}{dt}\left(|\dot{u}(t)|^{-1/2}\dot{u}(t)\right) - a(t)|u(t)|^{-1/2}u(t) + \nabla W(t,u(t)) = 0, \tag{4.3}$$

where $p = 3/2$, $t \in \mathbb{R}, u \in \mathbb{R}^N$, $a \in C(\mathbb{R},(0,\infty))$ such that $a(t) \to +\infty$ as $|t| \to \infty$. Let

$$W(t,x) = a(t)(1+\sin t)|x|^{3/2}\ln(1+|x|).$$

Since

$$\begin{aligned}(\nabla W(t,x),x) &= a(t)(1+\sin t)\left[\frac{3}{2}|x|^{3/2}\ln(1+|x|)+\frac{|x|^{5/2}}{1+|x|}\right]\\ &\geq \left(\frac{3}{2}+\frac{1}{1+|x|}\right)W(t,x)\geq 0\end{aligned}$$

for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^N$. This shows that (W4) holds with $\alpha=\beta=\nu=1$. In addition, for any $\gamma>0$ and $\varepsilon>0$

$$\begin{aligned}&s^{-3/2}\int_{t-\varepsilon}^{t+\varepsilon}\min_{|x|\geq 1}W(\tau,sx)d\tau\\ &= s^{-3/2}\int_{t-\varepsilon}^{t+\varepsilon}\min_{|x|\geq 1}\left[a(\tau)(1+\sin\tau)|sx|^{3/2}\ln(1+|sx|)\right]d\tau\\ &\geq \left[\min_{\tau\in[t-\varepsilon,t+\varepsilon]}a(\tau)\int_{t-\varepsilon}^{t+\varepsilon}(1+\sin\tau)d\tau\right]\ln(1+s)\\ &= 2(\varepsilon-\sin\varepsilon\sin t)\left[\min_{\tau\in[t-\varepsilon,t+\varepsilon]}a(\tau)\right]\ln(1+s)\\ &\geq 2(\varepsilon-|\sin\varepsilon|)\left[\min_{\tau\in[-\gamma-\varepsilon,\gamma+\varepsilon]}a(\tau)\right]\ln(1+s)\\ &\to +\infty,\qquad s\to+\infty\end{aligned}$$

uniformly with respect to $t\in[-\gamma,\gamma]$. This shows that (W5) also holds. It is easy to verify that assumptions (A), (W3) and (W2$''$) of Theorem 1.3 are satisfied. By Theorem 1.3, system (4.3) has an unbounded sequence of homoclinic solutions.

References

1. G. Arioli and A. Szulkin, Homoclinic solutions of Hamiltonian systems with symmetry, *J. Differential Equations*, **158** (1999), 291-313.

2. P. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity, *Nonlinear Anal.*, **7** (1983), 241-273.

3. T. Bartsch and Y. H. Ding, Homoclinic solutions of an infinite-dimensional Hamiltonian system, *Math. Z.*, **240** (2002), 289-310.

4. T. Bartsch and Y. H. Ding, Deformation theorems on non-metrizable vector spaces and applications to critical point theory, *Math. Nachr.*, **279** (2006), 1267-1288.

5. V. Coti Zelati, I. Ekeland and E. Sere, A variational approach to homoclinic orbits in Hamiltonian systems, *Math. Ann.*, **288(1)** (1990), 133-160.

6. V. Coti Zelati and P. H. RabinoWitz, Homoclinic orbits for second second order Hamiltonian systems possessing superquadratic potentials, *J. Amer. Math. Soc.*, **4** (1991), 693-727.

7. Y. Ding and M. Girardi, Periodic and homoclinic solutions to a class of Hamiltonian systems With the potentials changing sign, *Dynam. Systems Appl.*, **2(1)** (1993), 131-145.

8. Y. H. Ding, Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, *Nonlinear Anal.*, **25(11)** (1995), 1095-1113.

9. G. H. Fei, The existence of homoclinic orbits for Hamiltonian systems with the potential changing sign, *Chinese Ann. Math. Ser. A*, **17(4)** (1996), 651, (A Chinese summary); *Chinese Ann. Math. Ser. B*, **4** (1996), 403-410.

10. P. L. Felmer, E. A. De and B. E. Silva, Homoclinic and periodic orbits for Hamiltonian systems, *Ann. Sc. Norm. Super. Pisa CI. Sci.* (*4*), **26(2)** (1998), 285-301.

11. E. Hewitt and K. Stromberg, *Real and Absract Analysis*, Spinger, New York, 1965.

12. M. Izydorek and J. Janczewska, Homoclinic solutions for a class of second order Hamiltonian systems, *J. Differential Equations*, **219(2)** (2005), 375-389.

13. P. Korman and A. C. Lazer, Homoclinic orbits for a class of symmetric Hamiltonian systems, *Electron. J. Differential Equations*, **1994(1)** (1994), 1-10.

14. S. Lu, Homoclinic solutions for a nonlinear second order differential system with Laplacian operator, *Nonlinear Analysis: Real World Applications*, **12** (2011), 525-534.

15. X. Lv and S. Lu, Homoclinic solutions for ordinary p-Laplacian systems, *Appl. Math. Comput.*, **218** (2012), 5682-5692.

16. R. Manásevich and J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian-like operators, *J. Differential Equations*, **145(2)** (1998), 367-393.

17. J. Mawhin, Periodic solutions of systems with p-Laplacian-like operators, in: *Nonlinear Analysis and Its Applications to Differential Equations*, (Lisbon, 1998), 43 of Progress in Nonlinear Differential Equations and Applications, 37-63, Birkhauser, Boston, Mass, USA, 2001.

18. J. Mawhin, Some boundary value problems for Hartman-type perturbations of the ordinary vector p-Laplacian, *Nonlinear Anal. TMA*, **40(1-8)** (2000), 497-503.

19. J. Mawhin and M. Willem, Critical point theory and Hamiltonian systems, in: *Applied Mathematical Sciences*, Vol. 74, Springer-Verlag, New York, 1989.

20. W. Omana and M. Willem, Homoclinic orbits for a class of Hamiltonian systems, *Differential Integral Equations*, **5(5)** (1992), 1115-1120.

21. P. H. Rabinowitz, Periodic and Heteroclinic orbits for a periodic Hamiltonian systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **6(5)** (1989), 331-346.

22. P. H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, *Proc. Roy. Soc. Edinburgh Sect. A*, **114(1-2)** (1990), 33-38.

23. P. H. Rabinowitz and K. Tanaka, Some results on connecting orbits for a class of Hamiltonian systems, *Math. Z.*, **206(3)** (1991), 473-499.

24. P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, in: *CBMS Reg. Conf. Ser. Math.*, Vol. 65, Amer. Math. Soc., Providence, RI, 1986.

25. E. Séré, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.*, **209** (1992), 27-42.

26. A. Szulkin and W. M. Zou, Homoclinic orbits for asymptotically linear Hamiltonian systems, *J. Funct. Anal.*, **187** (2001), 25-41.

27. K. Tanaka, Homoclinic orbits in a first order superquadratic Hamiltonian system: Convergence of subharmonic orbits, *J. Differential Equations*, **94** (1991), 315-339.

28. X. H. Tang and L. Xiao, Homoclinic solutions for ordinary p-Laplacian systems with a coercive potential, *Nonlinear Anal.*, **71(3-4)** (2009), 1124-1132.

29. X. H. Tang and L. Xiao, Homoclinic solutions for nonautonomous second order Hamiltonian systems with a coercive potential, *J. Math. Anal. Appl.*, **351** (2009), 586-594.

30. X. H. Tang and L. Xiao, Homoclinic solutions for a class of second-order Hamiltonian systems, *Nonlinear Anal.*, **71** (2009), 1140-1152.

31. X. H. Tang and X. Y. Lin, Homoclinic solutions for a class of second-order Hamiltonian systems, *J. Math. Anal. Appl.*, **354** (2009), 539-549.

32. X. H. Tang and X. Y. Lin, Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Proc. Royal Soc. Edinburgh, A*, **141** (2011), 103-1119.

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