TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 3, pp. 885-895, June 2013
DOI: 10.11650/tjm.17.2013.2504
This paper is available online at http://journal.taiwanmathsoc.org.tw

NEW CURVATURE INEQUALITIES FOR HYPERSURFACES IN THE EUCLIDEAN AMBIENT SPACE

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Abstract. The spread of a matrix is introduced by Mirsky in 1956 in [20]. The classical theory provides an upper bound for the spread of the shape operator in terms of the second fundamental form of a hypersurface in the Euclidean space. In the present work, we are extending our understanding of the phenomenon by proving a lower bound, inspired from an idea developed recently by X.-Q. Chang. As we study the concept of curvature on hypersurfaces, we introduce a new curvature invariant called amalgamatic curvature and explore its geometric meaning by proving an inequality related to the absolute mean curvature of the hypersurface. In our study, a new class of geometric objects is obtained: the absolutely umbilical hypersurfaces.

1. Introduction

In the classic matrix theory, *spread of a matrix* has been defined by Mirsky in [20] and then mentioned in various references, as for example [19]. Let $A \in M_n(\mathbb{C})$, $n \geq 3$, and let $\lambda_1, \ldots, \lambda_n$ be the characteristic roots of A. The *spread* of A is defined to be $s(A) = \max_{i,j} |\lambda_i - \lambda_j|$. Let us denote by $||A||$ the Euclidean norm of the matrix A, i.e.: $||A||^2 = \sum_{i,j=1}^{m,n} |a_{ij}|^2$. We use also the classical notation E_2 for the sum of all 2-square principal subdeterminants of A. If $A \in M_n(\mathbb{C})$ then we have the following inequalities (see [19]):

$$s(A) \leq (2||A||^2 - \frac{2}{n}|trA|^2)^{1/2}, \tag{1.1}$$

$$s(A) \leq \sqrt{2}||A||. \tag{1.2}$$

If $A \in M_n(\mathbb{R})$, then:

Received October 4, 2012, accepted November 19, 2012.
Communicated by Bang-Yen Chen.
2010 *Mathematics Subject Classification*: 53B25, 53B20, 53A30.
Key words and phrases: Principal curvatures, Shape operator, Extrinsic scalar curvature, Spread of shape operator, Surfaces of rotation, Absolutely umbilical hypersurfaces.

$$s(A) \leq \left[2\left(1-\frac{1}{n}\right)(trA)^2 - 4E_2(A)\right]^{1/2}, \tag{1.3}$$

with equality if and only if $n-2$ of the characteristic roots of A are equal to the arithmetic mean of the remaining two. These are upper bounds for $s(A)$.

In the present work, we address the question of a lower bound for $s(A)$ in the particular case when the matrix A is the shape operator of a hypersurface in Euclidean ambient space.

From the classical theory (see [19], 4.3.2), it is known that for a Hermitian matrix $A \in M_n(\mathbb{C})$ we have

$$s(A) \geq 2 \max_{i \neq j} |a_{ij}|,$$

and that this inequality is the best possible, in the sense that there exist hermitian matrices whose spread is equal to the absolute value of an off-diagonal element. The fact we prove in the next section is a different kind of lower bound, depending on geometric elements, and not of the entries of the shape operator (which depend on the position vector of the hypersurface).

In submanifold geometry, the spread of the shape operator is related to the study of other curvature invariants; see B.-Y. Chen's works [3, 4, 5, 6]. For the most recent comprehensive overview on the developments in the study of curvature in the last two decades, our main reference is B.-Y. Chen's monograph [8]. It is in the spirit of these developments that our present work should be viewed: the study of curvature invariants raises the most natural class of questions in Riemannian geometry.

The spread of the shape operator was studied from the geometric standpoint by the last of the present authors in [28]. To mention here just one of the facts previously obtained, in [28] it is proved that for M^n a compact submanifold of a Riemannian manifold $\bar{M}^{n+s}$, the following inequality holds:

$$\left(\int_M L dV\right)^2 (vol(M))^{\frac{n}{2n-2}} \leq 2n(n-1)\left(\int_M (|H|^2 - ext)^{\frac{n}{2}} dV\right)^{\frac{2}{n}} \tag{1.4}$$

where ext stands for the extrinsic scalar curvature. The equality holds if and only if either $n=2$ or M is a totally umbilical submanifold of dimension $n \geq 3$.

The pointwise curvature inequalities have applications in integral geometry, as illustrated by Sakai's work [24].

Important developments on pointwise curvature inequalities could shed a light on other areas. One of the most important developments in the study of curvature in the last decade was Zhiqin Lu's proof of normal scalar curvature conjecture [17]. For its connection with linear algebra, of particular importance is [18]. The normal scalar curvature conjecture was also proved independently by Ge and Tang [12].

Although historically, "the idea of spectrum of an operator grew out of attempts to understand concrete problems of linear algebra involving the solutions of linear equations and their infinite dimensional generalizations", as W. Arveson writes, in differential geometry the quest to understand the behavior of the spectrum of the shape operator can be viewed as part of the evolutions of the idea of curvature. This very idea was pursued by one of the present authors in [26, 27].

2. A Rigidity Theorem for Hypersurfaces Based on X.-Q. Chang's Estimate

Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. Let p be a point on the hypersurface. Denote $\sigma_k(p) = \frac{\partial \sigma}{\partial x_k}$, for all k from 1 to n. Consider $\{\sigma_1(p), \sigma_2(p), ..., \sigma_n(p), N(p)\}$, the Gauss frame of the hypersurface, where N denotes the normal vector field. We denote by $g_{ij}(p)$ the coefficients of the first fundamental form and by $h_{ij}(p)$ the coefficients of the second fundamental form. Then we have

$$g_{ij}(p) =< \sigma_i(p), \sigma_j(p) >, \qquad h_{ij}(p) =< N(p), \sigma_{ij}(p) > .$$

The Weingarten map $L_p = -dN_p \circ d\sigma_p^{-1} : T_{\sigma(p)}\sigma \to T_{\sigma(p)}\sigma$ is linear. Denote by $(h^i_j(p))_{1 \leq i,j \leq n}$ the matrix associated to Weingarten's map, that is:

$$L_p(\sigma_i(p) = h^k_i(p)\sigma_k(p),$$

where the repeated index and upper script above indicates Einstein's summation convention. It is well-known that Weingerten's operator is self-adjoint, which implies that the roots of the algebraic equation

$$\det(h^i_j(p) - \lambda(p)\delta^i_j) = 0$$

are real. The eigenvalues of Weingarten's linear map are called principal curvatures of the hypersurface. They are the roots $\lambda_1(p), \lambda_2(p), ..., \lambda_n(p)$ of this algebraic equation. The mean curvature at the point p is

$$H(p) = \frac{1}{n}[\lambda_1(p) + ... + \lambda_n(p)],$$

and the Gauss-Kronecker curvature is

$$K(p) = \lambda_1(p)\lambda_2(p)...\lambda_n(p).$$

Without any loss of generality, in our work we assume that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. Therefore the spread of the shape operator is $s(L_p) = s(h)(p) = \lambda_n(p) - \lambda_1(p)$. This is the quantity we compare with other geometric quantities.

Define $\delta = n||h||^2 - n^2H^2$. Thus, relation (1.1) is $s(L) \leq \sqrt{\frac{2}{n}}\sqrt{\delta}$. In this context, we prove the following.

Theorem 2.1. *Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ, endowed with the second fundamental form h. Let $||h|| = \sum (h_{ij})^2$ and $\delta = n||h||^2 - n^2H^2$. Then the spread of the shape operator satisfies the double inequality:*

$$\frac{2}{n}\sqrt{\delta} \le s(L) \le \sqrt{\frac{2}{n}}\sqrt{\delta}.$$

The equality holds true everywhere at umbilics, where $s(L) = 0$.

Proof. The second inequality is known. We need to prove the first inequality. X.-Q. Chang's [2] idea is to write δ as a combination of square and of a positive quantity whose sign we fully control. Namely, we have

$$\begin{aligned} \delta(L) &= n||h||^2 - n^2H^2 = n\ trace(h^2) - (trace\ h)^2 \\ &= \frac{n^2}{4}(\lambda_n - \lambda_1)^2 - n\sum_{j=2}^{n-1}(\lambda_n - \lambda_j)(\lambda_j - \lambda_1) - \left[\sum_{j=2}^{n-1}\lambda_j - \frac{n-2}{2}(\lambda_1 + \lambda_n)\right]^2. \end{aligned} \tag{2.1}$$

Before proving (2.1), remark that our assumption $\lambda_1 \le \lambda_2 \le ... \le \lambda_n$ yields

$$\sum_{j=2}^{n-1}(\lambda_n - \lambda_j)(\lambda_j - \lambda_1) \ge 0.$$

By multiplying in the left hand side term, we obtain:

$$\begin{aligned} \sum_{j=2}^{n-1}(\lambda_n - \lambda_j)(\lambda_j - \lambda_1) &= \sum_{j=2}^{n-1}(\lambda_n\lambda_j + \lambda_j\lambda_1 - \lambda_1\lambda_n - \lambda_j^2) \\ &= (\lambda_n + \lambda_1)\left(\sum_{j=2}^{n-1}\lambda_j\right) - \sum_{j=2}^{n-1}\lambda_j^2 - (n-2)\lambda_1\lambda_n. \end{aligned}$$

Here we solve for $\sum_{j=2}^{n-1}\lambda_j^2$ and we obtain:

$$\sum_{j=2}^{n-1}\lambda_j^2 = (\lambda_n + \lambda_1)\left(\sum_{j=2}^{n-1}\lambda_j\right) - (n-2)\lambda_1\lambda_n - \sum_{j=2}^{n-1}(\lambda_n - \lambda_j)(\lambda_j - \lambda_1). \tag{2.2}$$

We use (2.2) to prove (2.1). Now we are ready to prove (2.1). We start by writing:

$$\begin{aligned} \delta(L) &= n\ trace(h^2) - (trace\ h)^2 = n\sum_{j=1}^{n}\lambda_j^2 - \left(\sum_{j=1}^{n}\lambda\right)^2 \\ &= (n-1)\sum_{j=1}^{n}\lambda_j^2 - 2\sum_{1\le i<j\le n}\lambda_i\lambda_j. \end{aligned}$$

Now we separate the terms of index 1 and n, as we are aiming to reach $s(L) = \lambda_n - \lambda_1$. We continue further our computation:

$$\begin{aligned}\delta(L) &= (n-1)(\lambda_1^2 + \lambda_n^2) - 2\lambda_1\lambda_n + (n-1)\left(\sum_{j=2}^{n-1} \lambda_j^2\right)\\ &\quad -2\sum_{1\le i<j\le n} \lambda_i\lambda_j - 2(\lambda_1 + \lambda_n)\left(\sum_{j=2}^{n-1} \lambda_j\right).\end{aligned}$$

We complete the square for the sum of the principal curvatures for indices $2 \le i, j \le n-1$, to obtain $\left(\sum_{j=2}^{n-1} \lambda_j\right)^2$, as follows:

$$\delta(L) = (n-1)(\lambda_1^2+\lambda_n^2) - 2\lambda_1\lambda_n + n\left(\sum_{j=2}^{n-1} \lambda_j^2\right) - \left(\sum_{j=2}^{n-1} \lambda_j\right)^2 - 2(\lambda_1+\lambda_n)\left(\sum_{j=2}^{n-1} \lambda_j\right).$$

In this last relation we substitute the left hand side term obtained in (2.2) and we have:

$$\begin{aligned}\delta(L) &= (n-1)(\lambda_1^2 + \lambda_n^2) - 2\lambda_1\lambda_n\\ &\quad + n\left\{(\lambda_n + \lambda_1)\left(\sum_{j=2}^{n-1} \lambda_j\right) - (n-2)\lambda_1\lambda_n - \sum_{j=2}^{n-1}(\lambda_n - \lambda_j)(\lambda_j - \lambda_1)\right\}\\ &\quad - \left(\sum_{j=2}^{n-1} \lambda_j\right)^2 - 2(\lambda_1 + \lambda_n)\left(\sum_{j=2}^{n-1} \lambda_j\right).\end{aligned}$$

We collect the like terms:

$$\begin{aligned}\delta(L) &= (n-1)(\lambda_1^2 + \lambda_n^2) - (2 + n(n-2))\lambda_1\lambda_n - \left(\sum_{j=2}^{n-1} \lambda_j\right)^2\\ &\quad + (n-2)(\lambda_1 + \lambda_n)\left(\sum_{j=2}^{n-1} \lambda_j\right) - n\sum_{j=2}^{n-1}(\lambda_n - \lambda_j)(\lambda_j - \lambda_1).\end{aligned}$$

In the third term, we are completing a perfect square and rewrite the whole expression in the form:

$$\begin{aligned}\delta(L) &= (n-1)(\lambda_1^2 + \lambda_n^2) - (2 + n(n-2))\lambda_1\lambda_n\\ &\quad - \left[\left(\sum_{j=2}^{n-1} \lambda_j\right) - \frac{(n-2)}{2}(\lambda_1+\lambda_n)\right]^2 + \frac{n-2}{4}(\lambda_1+\lambda_n)^2 - n\sum_{j=2}^{n-1}(\lambda_n-\lambda_j)(\lambda_j-\lambda_1).\end{aligned}$$

The terms in λ_1 and λ_n can be collected together into a single square:

$$\delta(L) = \frac{n^2}{4}(\lambda_n - \lambda_1)^2 - \left[\left(\sum_{j=2}^{n-1}\lambda_j\right) - \frac{(n-2)}{2}(\lambda_1 + \lambda_n)\right]^2 - n\sum_{j=2}^{n-1}(\lambda_n - \lambda_j)(\lambda_j - \lambda_1).$$

This proves finally (2.1). From (2.1) we get immediately that $\delta(L) - \frac{n^2}{4}s(h) \geq 0$, which was the inequality we wanted to prove. The equality case holds when the sum

$$\sum_{j=2}^{n-1}(\lambda_n - \lambda_j)(\lambda_j - \lambda_1)$$

vanishes, and this corresponds to the equality of all the principal curvatures, i.e. p is an umbilical point. ■

3. Amalgamatic Curvature and Absolutely Umbilical Hypersurfaces

In the classical geometry of curves, a curve satisfying the property that the ratio between curvature and torsion is constant is called a generalized helix. It's natural to think if it is possible to extend this idea to higher dimensional geometric objects.

For example, would it make sense to study surfaces that are satisfying a similar relationship between the mean curvature H and the Gaussian curvature K? One could consider both the ratio $\frac{K}{H}$ or $\frac{K}{H^2}$ and derive some analogies with the theory of curves. The history of the original idea can be traced back to Weingarten's original papers [29, 30], as some authors suggest, e.g. [16]. After WW II, one of the first contributions to the study of this problem, along with [13], is S.-S. Chern's paper [9]. There are many substantive recent works focused on the study of linear Weingarten surfaces, i.e. of the surfaces in $\mathbb{R}^3$ satisfying the relation $aH^2 + bK = c$, where a, b, c are real constants; see e.g. [15, 16, 22]. This is the context in which we look at the idea of the ratio $\frac{K}{H}$ seen in the geometry of surfaces. In general, the ratio $\frac{K}{H}$ is a function that depends on the point of the surface, everywhere where it is defined. If we denote the principal curvatures by k_1 and k_2, then

$$\frac{K}{H} = \frac{2k_1k_2}{k_1 + k_2}.$$

Note that this term could be also viewed as the harmonic ratio of the real numbers k_1 and k_2. We would like to define a geometric quantity that encodes the same information as this ratio. This discussion is our motivation to introduce the following.

Definition 3.1. Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. In the generic case, the amalgamatic curvature at point p is

$$A(p) = \binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \frac{2|k_i||k_j|}{|k_i| + |k_j|}.$$

First, we show why $A(p)$ is defined everywhere on the hypersurface. Suppose $|k_i| + |k_j|$ vanishes at some point p. Remark that the inequality $\frac{2|k_i||k_j|}{|k_i|+|k_j|} \leq |k_i| + |k_j|$ insures the existence of the limit of the function $A(p)$ at p. For a hyperplane in $\mathbb{R}^{n+1}$, we have $A(p) \equiv 0$, for all p.

We can describe the amalgamatic curvature as the arithmetic mean of the harmonic means of all the pairs of absolute values of principal curvatures. Working with algebraic means is the reason why we consider the absolute value of the principal curvatures in this definition.

Definition 3.2. Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. The point p on the hypersurface is called *absolutely umbilical* if the principal curvatures satisfy $|k_1| = |k_2| = ... = |k_n|$. If all the points of a hypersurface are absolutely umbilical, then the hypersurface is called absolutely umbilical.

Example 3.1. All the points of a minimal surface in $\mathbb{R}^3$ are absolutely umbilical.

Example 3.2. On the cylinder $\mathbb{S}^1(a) \times \mathbb{R} \subset \mathbb{R}^3$ there are no absolutely umbilical points.

Example 3.3. Any totally umbilical hypersurface in the Euclidean ambient space is absolutely umblical hypersurface. For a more comprehensive discussion, see section 12.8. in [7].

Example 3.4. Consider the torus $\sigma : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$, parametrized by

$$\sigma(u, v) = ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v), \quad a > b > 0.$$

It is well known that

$$k_1 = -\frac{\cos v}{a + b\cos v}, \qquad k_2 = -\frac{1}{b}.$$

The umbilical points should satisfy $k_1 = k_2$, and this equality yields that there are no umbilics on the torus, since $a \neq 0$. On the other hand, there are absolutely umbilical points, provided $2b \geq a$. To prove this assertion, set $k_1 = -k_2$, and get $-2b\cos v = a$. Thus, absolutely umbilical points exist if and only if $\cos v = -\frac{a}{2b} \in [-1, 1]$. The existence condition $2b \geq a \geq -2b$ is actually $2b \geq a > b$. Under this condition, the curve $\gamma(u) = \sigma(u, v_0)$, for $v_0 = \arccos\left(-\frac{a}{2b}\right)$ on the torus is a set of absolutely umbilical points.

We prove the following.

Theorem 3.1. *Let $\sigma : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a hypersurface given by the smooth map σ. Let $k_1, k_2, ..., k_n$ be the principal curvatures at p. Denote by $\bar{H}$ its absolute mean curvature, i.e.*

$$\bar{H} = \frac{1}{n}\left(|k_1| + |k_2| + ... + |k_n|\right).$$

Then the absolute mean curvature and the amalgamatic curvature satisfy

$$\bar{H}(p) \geq A(p),$$

with equality being satisfied at all the points where the hypersurface is absolutely umbilical.

Proof. Denote by $a_i = |k_i|$. Then we need to prove the following:

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \frac{2 \cdot 2}{n(n-1)} \sum_{1 \leq i < j \leq n} \frac{a_i a_j}{a_i + a_j},$$

Thus, we need to prove that for any $a_1, a_2, ..., a_n > 0$, we have

$$\frac{(n-1)}{4}\left(\sum_{i=1}^{n} a_i\right) \geq \sum_{1 \leq i < j \leq n} \frac{a_i a_j}{a_i + a_j}, \tag{3.1}$$

with equality if and only if $a_1 = a_2 = ... = a_n$. (See also [1].)

Multiply the inequality above by 2 and collect all the terms on one side:

$$\frac{(n-1)}{2}\left(\sum_{i=1}^{n} a_i\right) - \sum_{1 \leq i < j \leq n} \frac{2a_i a_j}{a_i + a_j} \geq 0.$$

Add to both sides the quantity $\frac{(n-1)}{2}\left(\sum_{i=1}^{n} a_i\right)$, and the inequality we need to prove turns out to be:

$$(n-1)\left(\sum_{i=1}^{n} a_i\right) - \sum_{1 \leq i < j \leq n} \frac{2a_i a_j}{a_i + a_j} \geq \frac{(n-1)}{2}\left(\sum_{i=1}^{n} a_i\right). \tag{3.2}$$

We group the a_i's two by two in the left hand side term:

$$\begin{aligned} \textstyle\sum_{1 \leq i < j \leq n}\left(a_i - \frac{2a_i a_j}{a_i + a_j} + a_j\right) &= \sum_{1 \leq i < j \leq n} \frac{a_i^2 + a_j^2}{a_i + a_j} \\ &= \sum_{1 \leq i < j \leq n}\left(\frac{a_i^2}{a_i + a_j} + \frac{a_j^2}{a_i + a_j}\right). \end{aligned} \tag{3.3}$$

At this point we apply the so-called Engel's Lemma, which is a fundamental algebraic argument that yields Cauchy-Schwarz inequality. Specifically, we apply the following: If $a_1, a_2, ..., a_n$ are any real numbers and $x_1, x_2, ..., x_n > 0$, then

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + ... + \frac{a_n^2}{x_n} \geq \frac{(a_1 + a_2 + ... + a_n)^2}{x_1 + x_2 + ... + x_n},$$

with equality if and only if $\frac{a_1}{x_1} = ... = \frac{a_n}{x_n}$. Thus, from (3.3), we obtain:

$$\sum_{1\leq i<j\leq n} \left(\frac{a_i^2}{a_i + a_j} + \frac{a_j^2}{a_i + a_j} \right) \geq \frac{[(n-1)\sum a_i]^2}{2(n-1)\sum a_i} = \frac{n-1}{2} \sum_{i=1}^{n} a_i.$$

This last relation proves (3.2). The equality case follows immediately from $a_1 = a_2 = ... = a_n$, as proved in (3.1). ■

Example 3.5. An even dimensional austere hypersurface $M^{2n} \subset \mathbb{R}^{2n+1}$ with principal curvatures $\{-k, -k, ..., -k, k, k, ..., k\}$ is absolutely umbilical, if the multiplicities of the eigenvalues $\lambda_1 = -k$ and $\lambda_2 = k$ are the same.

Note that austere hypersurfaces were studied recently in [10] and served as class of examples that are satisfying Walker identities in Riemannian space forms of dimension at least four.

The origin of the study of hypersurfaces with two disjoint principal curvatures can be found in Otsuki's work [21]. The absolute umbilical hypersurfaces that are not umbilical have values of their principal curvatures as k and $-k$. Recently, in [25], Shu and Liu studied hypersurfaces with two distinct principal curvatures in a real space form, under the assumption that the extrinsic invariant ρ^2 is constant. Also recently, Wu [31] studied hypersurfaces with two distinct non-simple principal curvatures in space forms, without the assumption that the (high order) mean curvature is constant.

ACKNOWLEDGMENTS

The authors express their thanks to Xiang-Qian Chang for his advice and his valuable insight in the algebraic phenomenon explored in the second section of our present paper. Professor Chang mentioned to us that his inspiration for the algebraic estimate in [2] was triggered by an interesting idea from [14]. Additionally, the authors would like to express their thanks to the referee and the editor for their suggestions.

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