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ALMOST h-SEMI-SLANT RIEMANNIAN MAPS

Kwang-Soon Park

Abstract. As a generalization of slant Riemannian maps, semi-slant Riemannian maps, almost h-slant submersions, and almost h-semi-slant submersions, we introduce the notion of almost h-semi-slant Riemannian maps from almost quaternionic Hermitian manifolds to Riemannian manifolds. We investigate the integrability of distributions, the harmonicity of such maps, the geometry of fibers, etc. We also deal with the condition for such maps to be totally geodesic and study some decomposition theorems. Moreover, we give some examples.

1. Introduction

Let F be a C^{∞}-map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N). According to the conditions on the map F, the map F is said to be a harmonic map [1], a totally geodesic map [1], an isometric immersion [4], a Riemannian submersion ([8, 11, 19]), a Riemannian map [7], etc. As we know, if we consider the notions of an isometric immersion and a Riemannian submersion as the Riemannian generalization of the notions of an immersion and a submersion, then the notion of a Riemannian map may be the Riemannian generalization of the notion of a subimmersion [7].

The study of isometric immersions is originated from Gauss' work, which studied surfaces in the Euclidean space $\mathbb{R}^3$ and there are a lot of papers and books on this topic. In particular, B. Y. Chen introduced and studied some notions: generic submanifolds [2] and slant submanifolds [3]. The notion of generic submanifolds contains the notions of real hypersurfaces, complex submanifolds, totally real submanifolds, anti-holomorphic submanifolds, purely real submanifolds, and CR-submanifolds. And the notion of slant submanifolds has some similarities with the notions of slant submersions [17], semi-slant submersions [15], almost h-slant submersions [12], almost h-semi-slant submersions [13], slant Riemannian maps [18], and semi-slant Riemannian maps [14].

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For the Riemannian submersion F, B. O'Neill [11] and A. Gray [9] firstly studied the map F. Since then, there are several kinds of Riemannian submersions ([14], references therein). A. Fischer [7] defined a Riemannian map F, which generalizes and unifies the notions of an isometric immersion, a Riemannian submersion, and an isometry. After that, there are a lot of papers on this topic. Moreover, B. Sahin introduced a slant Riemannian map [18] and the author defined a semi-slant Riemannian map [14]. As a generalization of slant Riemannian maps [18], semi-slant Riemannian maps [14], almost h-slant submersions [12], and almost h-semi-slant submersions [13], we will define an almost h-semi-slant Riemannian map and a h-semi-slant Riemannian map. And as we know, the quaternionic Kähler manifolds have applications in physics as the target spaces for nonlinear $\sigma-$models with supersymmetry [5].

The paper is organized as follows. In section 2 we recall some notions, which are needed for later use. In section 3 we define the notions of an almost h-semi-slant Riemannian map and a h-semi-slant Riemannian map and obtain some properties on them. In section 4 using both an almost h-semi-slant Riemannian map and a h-semi-slant Riemannian map, we get some decomposition theorems. In section 5 we obtain some examples.

2. Preliminaries

Let (M, E, g) be an almost quaternionic Hermitian manifold, where M is a $4m-$dimensional differentiable manifold, g is a Riemannian metric on M, and E is a rank 3 subbundle of End(TM) such that for any point $p \in M$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{2.1}$$

$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \tag{2.2}$$

for all vector fields $X, Y \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3. The above basis $\{J_1, J_2, J_3\}$ is said to be a *quaternionic Hermitian basis*. We call (M, E, g) a *quaternionic Kähler manifold* if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2} \tag{2.3}$$

for any vector field $X \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3. If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on M, then (M, E, g) is said to be *hyperkähler*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on M and g a *hyperkähler metric*.

Let (M, g_M) and (N, g_N) be Riemannian manifolds. Let $F : (M, g_M) \mapsto (N, g_N)$ be a C^∞-map. We call the map F a *C^∞-submersion* if F is surjective and the differential $(F_*)_p$ has maximal rank for any $p \in M$. The map F is said to be a *Riemannian submersion* [11] if F is a C^∞-submersion and

$$(F_*)_p : ((\ker(F_*)_p)^\perp, (g_M)_p) \mapsto (T_{F(p)}N, (g_N)_{F(p)})$$

is a linear isometry for each $p \in M$, where $(\ker(F_*)_p)^\perp$ is the orthogonal complement of the space $\ker(F_*)_p$ in the tangent space T_pM of M at p. We call the map F a *Riemannian map* [7] if

$$(F_*)_p : ((\ker(F_*)_p)^\perp, (g_M)_p) \mapsto ((rangeF_*)_{F(p)}, (g_N)_{F(p)})$$

is a linear isometry for each $p \in M$, where $(rangeF_*)_{F(p)} := (F_*)_p((\ker(F_*)_p)^\perp)$ for $p \in M$.

Let (M, g_M, J) be an almost Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F : (M, g_M, J) \mapsto (N, g_N)$ be a C^∞-map. We call the map F a *slant submersion* [17] if F is a Riemannian submersion and the angle $\theta = \theta(X)$ between JX and the space $\ker(F_*)_p$ is constant for nonzero $X \in \ker(F_*)_p$ and $p \in M$.

We call the angle θ a *slant angle*.

The map F is said to be a *semi-slant submersion* [15] if F is a Riemannian submersion and there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call the angle θ a *semi-slant angle*.

We call the map F a *slant Riemannian map* [18] if F is a Riemannian map and the angle $\theta = \theta(X)$ between JX and the space $\ker(F_*)_p$ is constant for nonzero $X \in \ker(F_*)_p$ and $p \in M$.

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The map F is said to be a *semi-slant Riemannian map* [14] if F is a Riemannian map and there is a distribution $\mathcal{D}_1 \subset \ker F_*$ such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta = \theta(X)$ between JX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in M$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call the angle θ a *semi-slant angle*.

Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is said to be an *almost h-slant submersion* [12] if given a point $p \in M$ with a neighborhood

U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R = \theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$.

We call such a basis $\{I, J, K\}$ an *almost h-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-slant angles*.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-slant submersion* [12] if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R = \theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$, and $\theta = \theta_I = \theta_J = \theta_K$.

We call such a basis $\{I, J, K\}$ a *h-slant basis* and the angle θ a *h-slant angle*.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-semi-slant submersion* [13] if given a point $p \in M$ with a neighborhood U, there exist a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U and a distribution $\mathcal{D}_1 \subset \ker F_*$ on U such that for any $R \in \{I, J, K\}$,

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ R(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in U$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ a *h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *h-semi-slant angles*.

Furthermore, if we have

$$\theta = \theta_I = \theta_J = \theta_K,$$

then we call the map $F : (M, E, g_M) \mapsto (N, g_N)$ a *strictly h-semi-slant submersion*, $\{I, J, K\}$ a *strictly h-semi-slant basis*, and the angle θ a *strictly h-semi-slant angle*.

A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-semi-slant submersion* [13] if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ an *almost h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-semi-slant angles*.

Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds. A map $F : M \mapsto N$ is called a $(E_M, E_N)-$*holomorphic map* if given a point $x \in M$, for any $J \in (E_M)_x$ there exists $J' \in (E_N)_{F(x)}$ such that

$$F_* \circ J = J' \circ F_*. \tag{2.4}$$

A Riemannian submersion $F : M \mapsto N$ which is a $(E_M, E_N)-$holomorphic map is called a *quaternionic submersion*. Moreover, if (M, E_M, g_M) is a quaternionic Kähler manifold (or a hyperkähler manifold), then we say that F is a *quaternionic Kähler submersion* (or a *hyperkähler submersion*) [10].

Let $F : (M, g_M) \mapsto (N, g_N)$ be a C^∞-map. The second fundamental form of F is given by

$$(2.5) \qquad (\nabla F_*)(X, Y) := \nabla^F_X F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N [1]. Recall that F is said to be *harmonic* if we have the tension field $\tau(F) := trace(\nabla F_*) = 0$ and we call the map F a *totally geodesic* map if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$ [1]. Denote the range of F_* by $rangeF_*$ as a subset of the pullback bundle $F^{-1}TN$. With its orthogonal complement $(rangeF_*)^\perp$ we obtain the following decomposition

$$(2.6) \qquad F^{-1}TN = rangeF_* \oplus (rangeF_*)^\perp.$$

Moreover, we have

$$(2.7) \qquad TM = \ker F_* \oplus (\ker F_*)^\perp.$$

Then we easily get

Lemma 2.1. [16]. *Let F be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N). Then*

$$(2.8) \qquad (\nabla F_*)(X, Y) \in \Gamma((rangeF_*)^\perp) \quad \textit{for } X, Y \in \Gamma((\ker F_*)^\perp).$$

Lemma 2.2. [7]. *Let F be a Riemannian map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N). Then the map F satisfies a generalized eikonal equation*

$$(2.9) \qquad 2e(F) = ||F_*||^2 = rank\, F.$$

As we can see, $||F_*||^2$ is continuous on M and rank F is an integer-valued function on M so that rank F is locally constant. Hence, if M is connected, then rank F is a constant function [7]. In [7], A. Fischer suggested that using (2.9), we may build a quantum model of nature. And if we can do it, then there will be an interesting relationship between the mathematical side from Riemannian maps, harmonic maps, and Lagrangian field theory and the physical side from Maxwell's equation, Schrödinger's equation, and their proposed generalization.

3. Almost h-Semi-slant Riemannian Maps

Definition 3.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian map $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-semi-slant Riemannian map* if given a point $p \in M$ with a neighborhood U, there exist a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U and a distribution $\mathcal{D}_1 \subset \ker F_*$ on U such that for any $R \in \{I, J, K\}$,

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \ \ R(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2)_q$ is constant for nonzero $X \in (\mathcal{D}_2)_q$ and $q \in U$, where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ a *h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *h-semi-slant angles*.

Furthermore, if we have

$$\theta = \theta_I = \theta_J = \theta_K,$$

then we call the map $F : (M, E, g_M) \mapsto (N, g_N)$ a *strictly h-semi-slant Riemannian map*, $\{I, J, K\}$ a *strictly h-semi-slant basis*, and the angle θ a *strictly h-semi-slant angle*.

Definition 3.2. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian map $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-semi-slant Riemannian map* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$.

We call such a basis $\{I, J, K\}$ an *almost h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-semi-slant angles*.

Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-semi-slant Riemannian map. Then given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \ \ R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$.

Then for $X \in \Gamma(\ker F_*)$, we write

$$X = P_R X + Q_R X, \tag{3.1}$$

where $P_R X \in \Gamma(\mathcal{D}_1^R)$ and $Q_R X \in \Gamma(\mathcal{D}_2^R)$.

For $X \in \Gamma(\ker F_*)$, we obtain

$$RX = \phi_R X + \omega_R X, \tag{3.2}$$

where $\phi_R X \in \Gamma(\ker F_*)$ and $\omega_R X \in \Gamma((\ker F_*)^\perp)$.

For $Z \in \Gamma((\ker F_*)^\perp)$, we have

$$RZ = B_R Z + C_R Z, \tag{3.3}$$

where $B_R Z \in \Gamma(\ker F_*)$ and $C_R Z \in \Gamma((\ker F_*)^\perp)$.

For $U \in \Gamma(TM)$, we get

$$U = \mathcal{V}U + \mathcal{H}U, \tag{3.4}$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

For $W \in \Gamma(F^{-1}TN)$, we write

$$W = \bar{P}W + \bar{Q}W, \tag{3.5}$$

where $\bar{P}W \in \Gamma(rangeF_*)$ and $\bar{Q}W \in \Gamma((rangeF_*)^\perp)$.

Then

$$(\ker F_*)^\perp = \omega_R \mathcal{D}_2^R \oplus \mu_R, \tag{3.6}$$

where μ_R is the orthogonal complement of $\omega_R \mathcal{D}_2^R$ in $(\ker F_*)^\perp$ and is invariant under R.

Furthermore,

$$\phi_R \mathcal{D}_1^R = \mathcal{D}_1^R,\ \omega_R \mathcal{D}_1^R = 0,\ \phi_R \mathcal{D}_2^R \subset \mathcal{D}_2^R,\ B_R((\ker F_*)^\perp) = \mathcal{D}_2^R$$

$$\phi_R^2 + B_R \omega_R = -id,\ C_R^2 + \omega_R B_R = -id,\ \omega_R \phi_R + C_R \omega_R = 0,\ B_R C_R + \phi_R B_R = 0.$$

Define the tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \tag{3.7}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{3.8}$$

for $E, F \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of g_M.

For $X, Y \in \Gamma(\ker F_*)$, define

$$\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y \tag{3.9}$$

$$(\nabla_X \phi)Y := \widehat{\nabla}_X \phi Y - \phi \widehat{\nabla}_X Y \tag{3.10}$$

$$(\nabla_X \omega)Y := \mathcal{H}\nabla_X \omega Y - \omega \widehat{\nabla}_X Y. \tag{3.11}$$

Then we easily obtain

Lemma 3.3. *Let F be an almost h-semi-slant Riemannian map from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Then we get*

(1)

$$\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \omega_R Y = \phi_R \widehat{\nabla}_X Y + B_R \mathcal{T}_X Y$$
$$\mathcal{T}_X \phi_R Y + \mathcal{H}\nabla_X \omega_R Y = \omega_R \widehat{\nabla}_X Y + C_R \mathcal{T}_X Y$$

for $X, Y \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$.

(2)

$$\mathcal{V}\nabla_Z B_R W + \mathcal{A}_Z C_R W = \phi_R \mathcal{A}_Z W + B_R \mathcal{H}\nabla_Z W$$
$$\mathcal{A}_Z B_R W + \mathcal{H}\nabla_Z C_R W = \omega_R \mathcal{A}_Z W + C_R \mathcal{H}\nabla_Z W$$

for $Z, W \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$.

(3)

$$\widehat{\nabla}_X B_R Z + \mathcal{T}_X C_R Z = \phi_R \mathcal{T}_X Z + B_R \mathcal{H}\nabla_X Z$$
$$\mathcal{T}_X B_R Z + \mathcal{H}\nabla_X C_R Z = \omega_R \mathcal{T}_X Z + C_R \mathcal{H}\nabla_X Z$$
$$\mathcal{V}\nabla_Z \phi_R X + \mathcal{A}_Z \omega_R X = \phi_R \mathcal{V}\nabla_Z X + B_R \mathcal{A}_Z X$$
$$\mathcal{A}_Z \phi_R X + \mathcal{H}\nabla_Z \omega_R X = \omega_R \mathcal{V}\nabla_Z X + C_R \mathcal{A}_Z X$$

for $X \in \Gamma(\ker F_*)$, $Z \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$.

Using the h-semi-slant Riemannian map F, we have

Theorem 3.4. *Let F be a h-semi-slant Riemannian map from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-slant basis. Then the following conditions are equivalent:*

(a) *the complex distribution $\mathcal{D}_1$ is integrable.*

(b) $Q_I(\widehat{\nabla}_X \phi_I Y - \widehat{\nabla}_Y \phi_I X) = 0$ *and* $\mathcal{T}_X \phi_I Y = \mathcal{T}_Y \phi_I X$ *for $X, Y \in \Gamma(\mathcal{D}_1)$.*

(c) $Q_J(\widehat{\nabla}_X\phi_J Y - \widehat{\nabla}_Y\phi_J X) = 0$ *and* $\mathcal{T}_X\phi_J Y = \mathcal{T}_Y\phi_J X$ *for* $X, Y \in \Gamma(\mathcal{D}_1)$.

(d) $Q_K(\widehat{\nabla}_X\phi_K Y - \widehat{\nabla}_Y\phi_K X) = 0$ *and* $\mathcal{T}_X\phi_K Y = \mathcal{T}_Y\phi_K X$ *for* $X, Y \in \Gamma(\mathcal{D}_1)$.

Proof. Given $X, Y \in \Gamma(\mathcal{D}_1)$ and $R \in \{I, J, K\}$, we obtain

$$\begin{aligned} R[X,Y] &= R(\nabla_X Y - \nabla_Y X) = \nabla_X RY - \nabla_Y RX \\ &= \widehat{\nabla}_X\phi_R Y - \widehat{\nabla}_Y\phi_R X + \mathcal{T}_X\phi_R Y - \mathcal{T}_Y\phi_R X. \end{aligned}$$

Hence, we get

$$a) \Leftrightarrow b), \quad a) \Leftrightarrow c), \quad a) \Leftrightarrow d).$$

Therefore, the result follows. ■

Theorem 3.5. *Let F be a h-semi-slant Riemannian map from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-slant basis. Then the following conditions are equivalent:*

(a) *the slant distribution $\mathcal{D}_2$ is integrable.*

(b) $P_I(\widehat{\nabla}_X\phi_I Y - \widehat{\nabla}_Y\phi_I X + \mathcal{T}_X\omega_I Y - \mathcal{T}_Y\omega_I X) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_2)$.

(c) $P_J(\widehat{\nabla}_X\phi_J Y - \widehat{\nabla}_Y\phi_J X + \mathcal{T}_X\omega_J Y - \mathcal{T}_Y\omega_J X) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_2)$.

(d) $P_K(\widehat{\nabla}_X\phi_K Y - \widehat{\nabla}_Y\phi_K X + \mathcal{T}_X\omega_K Y - \mathcal{T}_Y\omega_K X) = 0$ *for* $X, Y \in \Gamma(\mathcal{D}_2)$.

Proof. Given $X, Y \in \Gamma(\mathcal{D}_2)$, $Z \in \Gamma(\mathcal{D}_1)$, and $R \in \{I, J, K\}$, we obtain

$$\begin{aligned} g_M(R[X,Y], Z) &= g_M(\nabla_X RY - \nabla_Y RX, Z) \\ &= g_M(\widehat{\nabla}_X\phi_R Y + \mathcal{T}_X\phi_R Y + \mathcal{T}_X\omega_R Y + \mathcal{H}\nabla_X\omega_R Y - \widehat{\nabla}_Y\phi_R X \\ &\quad - \mathcal{T}_Y\phi_R X - \mathcal{T}_Y\omega_R X - \mathcal{H}\nabla_Y\omega_R X, Z) \\ &= g_M(\widehat{\nabla}_X\phi_R Y + \mathcal{T}_X\omega_R Y - \widehat{\nabla}_Y\phi_R X - \mathcal{T}_Y\omega_R X, Z). \end{aligned}$$

Since $[X, Y] \in \Gamma(\ker F_*)$, we get

$$a) \Leftrightarrow b), \quad a) \Leftrightarrow c), \quad a) \Leftrightarrow d).$$

Therefore, we have the result. ■

In the same way as in the proof of Proposition 2.6 in [13], we can show

Proposition 3.6. *Let F be an almost h-semi-slant Riemannian map from an almost quaternionic Hermitian manifold (M, E, g_M) to a Riemannian manifold (N, g_N). Then we have*

$$\phi_R^2 X = -\cos^2\theta_R X \quad \text{for } X \in \Gamma(\mathcal{D}_2^R) \text{ and } R \in \{I, J, K\}, \tag{3.12}$$

where $\{I, J, K\}$ is an almost h-semi-slant basis with the almost h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$.

Remark 3.7. In particular, it is easy to obtain that the converse of Proposition 3.6 is also true.

Since

$$\begin{aligned} g_M(\phi_R X, \phi_R Y) &= \cos^2\theta_R \, g_M(X,Y) \\ g_M(\omega_R X, \omega_R Y) &= \sin^2\theta_R \, g_M(X,Y) \end{aligned}$$

for $X, Y \in \Gamma(\mathcal{D}_2^R)$, if $\theta_R \in (0, \frac{\pi}{2})$, then we can locally choose an orthonormal frame $\{f_1, \sec\theta_R\phi_R f_1, \cdots, f_s, \sec\theta_R\phi_R f_s\}$ of $\mathcal{D}_2^R$.

Using (3.12), in a similar way with Lemma 3.5 in [14], we can obtain

Lemma 3.8. *Let F be an almost h-semi-slant Riemannian map from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis with the almost h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$. If the tensor ω_R is parallel, then we have*

$$\mathcal{T}_{\phi_R X}\phi_R X = -\cos^2\theta_R \cdot \mathcal{T}_X X \quad \textit{for } X \in \Gamma(\mathcal{D}_2^R), \tag{3.13}$$

where $R \in \{I, J, K\}$.

Given an almost h-semi-slant Riemannian map F from an almost quaternionic Hermitian manifold (M, E, g_M) to a Riemannian manifold (N, g_N), for some $R \in \{I, J, K\}$ with $\theta_R \in [0, \frac{\pi}{2})$, we can define an endomorphism $\widehat{R}$ of $\ker F_*$ by

$$\widehat{R} := RP_R + \sec\theta_R\phi_R Q_R.$$

Then

$$\widehat{R}^2 = -id \quad \text{on } \ker F_*. \tag{3.14}$$

Note that the distribution $\ker F_*$ is integrable. But its dimension may be odd. With the endomorphism $\widehat{R}$ we get

Theorem 3.9. *Let F be an almost h-semi-slant Riemannian map from an almost quaternionic Hermitian manifold (M, E, g_M) to a Riemannian manifold (N, g_N) with the almost h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$ not all $\frac{\pi}{2}$. Then the fibers $F^{-1}(x)$ are even dimensional submanifolds of M for $x \in M$.*

Now, we consider the harmonicity of such maps. Let F be a C^∞-map from a Riemannian manifold (M, g_M) to a Riemannian manifold (N, g_N). We can canonically define a function $e(F) : M \mapsto [0, \infty)$ given by

$$e(F)(x) := \frac{1}{2}|(F_*)_x|^2, \quad x \in M, \tag{3.15}$$

where $|(F_*)_x|$ denotes the Hilbert-Schmidt norm of $(F_*)_x$ [1]. Then the function $e(F)$ is said to be the *energy density* of F. Let D be a compact domain of M, i.e., D is the compact closure $\bar{U}$ of a non-empty connected open subset U of M. The *energy integral* of F over D is the integral of its energy density:

$$E(F;D) := \int_D e(F)v_{g_M} = \frac{1}{2}\int_D |F_*|^2 v_{g_M}, \tag{3.16}$$

where v_{g_M} is the volume form on (M, g_M). Let $C^\infty(M,N)$ denote the space of all C^∞-maps from M to N. A C^∞-map $F: M \mapsto N$ is said to be *harmonic* if it is a critical point of the energy functional $E(\ ;D): C^\infty(M,N) \mapsto \mathbb{R}$ for any compact domain $D \subset M$. By the result of J. Eells and J. Sampson [6], we see that the map F is harmonic if and only if the tension field $\tau(F) := trace\nabla F_* = 0$.

Theorem 3.10. *Let F be an almost h-semi-slant Riemannian map from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Assume that $\widetilde{H} = 0$, where $\widetilde{H}$ denotes the mean curvature vector field of $rangeF_*$. Then each of the following conditions implies that F is harmonic.*

(a) $\mathcal{D}_1^I$ is integrable and $trace(\nabla F_*) = 0$ on $\mathcal{D}_2^I$.
(b) $\mathcal{D}_1^J$ is integrable and $trace(\nabla F_*) = 0$ on $\mathcal{D}_2^J$.
(c) $\mathcal{D}_1^K$ is integrable and $trace(\nabla F_*) = 0$ on $\mathcal{D}_2^K$.

Proof. Using Lemma 2.1, we get $trace\nabla F_*|_{\ker F_*} \in \Gamma(rangeF_*)$ and $trace\nabla F_*|_{(\ker F_*)^\perp} \in \Gamma((rangeF_*)^\perp)$ so that from (2.7), we have

$$trace(\nabla F_*) = 0 \quad \Leftrightarrow \quad trace\nabla F_*|_{\ker F_*} = 0 \text{ and } trace\nabla F_*|_{(\ker F_*)^\perp} = 0.$$

Moreover, we easily obtain

$$trace\nabla F_*|_{(\ker F_*)^\perp} = l\widetilde{H} \quad \text{for } l := \dim(\ker F_*)^\perp$$

so that

$$trace\nabla F_*|_{(\ker F_*)^\perp} = 0 \quad \Leftrightarrow \quad \widetilde{H} = 0.$$

Given $R \in \{I, J, K\}$, since $\mathcal{D}_1^R = R(\mathcal{D}_1^R)$, we can choose locally an orthonormal frame $\{e_1, Re_1, \cdots, e_k, Re_k\}$ of $\mathcal{D}_1^R$ so that

$$\begin{aligned}(\nabla F_*)(Re_i, Re_i) &= -F_*\nabla_{Re_i}Re_i = -F_*R(\nabla_{e_i}Re_i + [Re_i, e_i]) \\ &= F_*\nabla_{e_i}e_i - F_*R[Re_i, e_i] = -(\nabla F_*)(e_i, e_i) - F_*R[Re_i, e_i]\end{aligned}$$

for $1 \leq i \leq k$.

Thus,

$$\mathcal{D}_1^R \text{ is integrable} \quad \Rightarrow \quad trace\nabla F_*|_{\mathcal{D}_1^R} = 0.$$

Since $\mathcal{D}_2^R$ is the orthogonal complement of $\mathcal{D}_1^R$ in $\ker F_*$, we have the result. ■

Using Lemma 3.8, we obtain

Corollary 3.11. *Let F be an almost h-semi-slant Riemannian map from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis with the almost h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$. Assume that $\widetilde{H} = 0$. Then each of the following conditions implies that F is harmonic.*

(a) *$\mathcal{D}_1^I$ is integrable, the tensor ω_I is parallel, and $\theta_I \in [0, \frac{\pi}{2})$.*

(b) *$\mathcal{D}_1^J$ is integrable, the tensor ω_J is parallel, and $\theta_J \in [0, \frac{\pi}{2})$.*

(c) *$\mathcal{D}_1^K$ is integrable, the tensor ω_K is parallel, and $\theta_K \in [0, \frac{\pi}{2})$.*

We now investigate the condition for such a map F to be totally geodesic.

Theorem 3.12. *Let F be an almost h-semi-slant Riemannian map from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Assume that $\bar{Q}(\nabla^F_{Z_1} F_* Z_2) = 0$ for $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$. Then the following conditions are equivalent:*

(a) *F is a totally geodesic map.*

(b)

$$\omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0$$
$$\omega_I(\widehat{\nabla}_X B_I Z + \mathcal{T}_X C_I Z) + C_I(\mathcal{T}_X B_I Z + \mathcal{H}\nabla_X C_I Z) = 0$$

for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.

(c)

$$\omega_J(\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y) + C_J(\mathcal{T}_X \phi_J Y + \mathcal{H}\nabla_X \omega_J Y) = 0$$
$$\omega_J(\widehat{\nabla}_X B_J Z + \mathcal{T}_X C_J Z) + C_J(\mathcal{T}_X B_J Z + \mathcal{H}\nabla_X C_J Z) = 0$$

for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.

(d)

$$\omega_K(\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y) + C_K(\mathcal{T}_X \phi_K Y + \mathcal{H}\nabla_X \omega_K Y) = 0$$
$$\omega_K(\widehat{\nabla}_X B_K Z + \mathcal{T}_X C_K Z) + C_K(\mathcal{T}_X B_K Z + \mathcal{H}\nabla_X C_K Z) = 0$$

for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$.

Proof. If $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$, then by Lemma 2.1, we get

$$(\nabla F_*)(Z_1, Z_2) = 0 \quad \Leftrightarrow \quad \bar{Q}((\nabla F_*)(Z_1, Z_2)) = \bar{Q}(\nabla^F_{Z_1} F_* Z_2) = 0.$$

For $X, Y \in \Gamma(\ker F_*)$, we obtain

$$\begin{aligned}(\nabla F_*)(X, Y) = -F_*(\nabla_X Y) = F_*(I\nabla_X(\phi_I Y + \omega_I Y))\\ = F_*(\phi_I \widehat{\nabla}_X \phi_I Y + \omega_I \widehat{\nabla}_X \phi_I Y + B_I \mathcal{T}_X \phi_I Y + C_I \mathcal{T}_X \phi_I Y + \phi_I \mathcal{T}_X \omega_I Y\\ + \omega_I \mathcal{T}_X \omega_I Y + B_I \mathcal{H}\nabla_X \omega_I Y + C_I \mathcal{H}\nabla_X \omega_I Y).\end{aligned}$$

Thus,

$$(\nabla F_*)(X, Y) = 0 \Leftrightarrow \omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0.$$

Given $X \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$, since $(\nabla F_*)(X, Z) = (\nabla F_*)(Z, X)$, it is sufficient to consider the following case:

$$\begin{aligned}(\nabla F_*)(X, Z) = -F_*(\nabla_X Z) = F_*(I\nabla_X(B_I Z + C_I Z))\\ = F_*(\phi_I \widehat{\nabla}_X B_I Z + \omega_I \widehat{\nabla}_X B_I Z + B_I \mathcal{T}_X B_I Z + C_I \mathcal{T}_X B_I Z + \phi_I \mathcal{T}_X C_I Z\\ + \omega_I \mathcal{T}_X C_I Z + B_I \mathcal{H}\nabla_X C_I Z + C_I \mathcal{H}\nabla_X C_I Z)\end{aligned}$$

so that

$$(\nabla F_*)(X, Z) = 0 \Leftrightarrow \omega_I(\widehat{\nabla}_X B_I Z + \mathcal{T}_X C_I Z) + C_I(\mathcal{T}_X B_I Z + \mathcal{H}\nabla_X C_I Z) = 0.$$

Hence,

$$a) \Leftrightarrow b).$$

Similarly,

$$a) \Leftrightarrow c) \quad \text{and} \quad a) \Leftrightarrow d).$$

Therefore, we get the result. ■

Let $F : (M, g_M) \mapsto (N, g_N)$ be a Riemannian map. The map F is called a Riemannian map *with totally umbilical fibers* if

$$\mathcal{T}_X Y = g_M(X, Y)H \quad \text{for } X, Y \in \Gamma(\ker F_*), \tag{3.17}$$

where H is the mean curvature vector field of the fiber.

In a similar way with Lemma 2.17 in [13], we have

Lemma 3.13. *Let F be an almost h-semi-slant Riemannian map with totally umbilical fibers from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Then we get*

$$H \in \Gamma(\omega_R \mathcal{D}_2^R) \quad \textit{for } R \in \{I, J, K\}. \tag{3.18}$$

Using Lemma 3.13, we obtain

Corollary 3.14. *Let F be an almost h-semi-slant Riemannian map with totally umbilical fibers from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis with the almost h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$. Assume that $\theta_R = 0$ for some $R \in \{I, J, K\}$. Then the fibers of F are minimal submanifolds of M.*

4. Decomposition Theorems

Let (M, g_M) be a Riemannian manifold and $\mathcal{D}$ a (C^∞-) distribution on M. The distribution $\mathcal{D}$ is said to be *autoparallel* (or a *totally geodesic foliation*) if $\nabla_X Y \in \Gamma(\mathcal{D})$ for $X, Y \in \Gamma(\mathcal{D})$. Given an autoparallel distribution $\mathcal{D}$ on M, it is easy to see that $\mathcal{D}$ is integrable and its leaves are totally geodesic in M. Moreover, we call the distribution $\mathcal{D}$ *parallel* if $\nabla_Z Y \in \Gamma(\mathcal{D})$ for $Y \in \Gamma(\mathcal{D})$ and $Z \in \Gamma(TM)$. Given a parallel distribution $\mathcal{D}$ on M, we easily obtain that its orthogonal complementary distribution $\mathcal{D}^\perp$ is also parallel. In this case, M is locally a Riemannian product manifold of the leaves of $\mathcal{D}$ and $\mathcal{D}^\perp$. We can also obtain that if the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are simultaneously autoparallel, then they are also parallel. Using this fact, we have

Theorem 4.1. *Let F be an almost h-semi-slant Riemannian map from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-semi-slant basis. Then the following conditions are equivalent:*

(a) *(M, g_M) is locally a Riemannian product manifold of the leaves of* $\ker F_*$ *and* $(\ker F_*)^\perp$

(b)

$$\omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0 \quad \textit{for } X, Y \in \Gamma(\ker F_*),$$

$$\phi_I(\mathcal{V}\nabla_Z B_I W + \mathcal{A}_Z C_I W) + B_I(\mathcal{A}_Z B_I W + \mathcal{H}\nabla_Z C_I W) = 0 \quad \textit{for } Z, W \in \Gamma((\ker F_*)^\perp).$$

(c)

$$\omega_J(\widehat{\nabla}_X \phi_J Y + \mathcal{T}_X \omega_J Y) + C_J(\mathcal{T}_X \phi_J Y + \mathcal{H}\nabla_X \omega_J Y) = 0 \quad \textit{for } X, Y \in \Gamma(\ker F_*),$$

$$\phi_J(\mathcal{V}\nabla_Z B_J W + \mathcal{A}_Z C_J W) + B_J(\mathcal{A}_Z B_J W + \mathcal{H}\nabla_Z C_J W) = 0 \quad \textit{for } Z, W \in \Gamma((\ker F_*)^\perp).$$

(d)

$$\omega_K(\widehat{\nabla}_X \phi_K Y + \mathcal{T}_X \omega_K Y) + C_K(\mathcal{T}_X \phi_K Y + \mathcal{H}\nabla_X \omega_K Y) = 0 \quad \textit{for } X, Y \in \Gamma(\ker F_*),$$

$$\phi_K(\mathcal{V}\nabla_Z B_K W + \mathcal{A}_Z C_K W) + B_K(\mathcal{A}_Z B_K W + \mathcal{H}\nabla_Z C_K W) = 0 \; \textit{for } Z, W \in \Gamma((\ker F_*)^\perp).$$

Proof. Given $R \in \{I, J, K\}$, for $X, Y \in \Gamma(\ker F_*)$, we get

$$\begin{aligned}\nabla_X Y &= -R\nabla_X RY = -R(\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \phi_R Y + \mathcal{T}_X \omega_R Y + \mathcal{H}\nabla_X \omega_R Y)\\ &= -(\phi_R \widehat{\nabla}_X \phi_R Y + \omega_R \widehat{\nabla}_X \phi_R Y + B_R \mathcal{T}_X \phi_R Y + C_R \mathcal{T}_X \phi_R Y + \phi_R \mathcal{T}_X \omega_R Y\\ &\quad + \omega_R \mathcal{T}_X \omega_R Y + B_R \mathcal{H}\nabla_X \omega_R Y + C_R \mathcal{H}\nabla_X \omega_R Y).\end{aligned}$$

Thus,

$$\nabla_X Y \in \Gamma(\ker F_*) \Leftrightarrow \omega_R(\widehat{\nabla}_X \phi_R Y + \mathcal{T}_X \omega_R Y) + C_R(\mathcal{T}_X \phi_R Y + \mathcal{H}\nabla_X \omega_R Y) = 0.$$

For $Z, W \in \Gamma((\ker F_*)^\perp)$, we have

$$\begin{aligned}\nabla_Z W &= -R\nabla_Z RW = -R(\mathcal{V}\nabla_Z B_R W + \mathcal{A}_Z B_R W + \mathcal{A}_Z C_R W + \mathcal{H}\nabla_Z C_R W)\\ &= -(\phi_R \mathcal{V}\nabla_Z B_R W + \omega_R \mathcal{V}\nabla_Z B_R W + B_R \mathcal{A}_Z B_R W + C_R \mathcal{A}_Z B_R W\\ &\quad + \phi_R \mathcal{A}_Z C_R W + \omega_R \mathcal{A}_Z C_R W + B_R \mathcal{H}\nabla_Z C_R W + C_R \mathcal{H}\nabla_Z C_R W).\end{aligned}$$

Thus,

$$\begin{aligned}\nabla_Z W \in \Gamma((\ker F_*)^\perp) \Leftrightarrow\ & \phi_R(\mathcal{V}\nabla_Z B_R W + \mathcal{A}_Z C_R W)\\ &+ B_R(\mathcal{A}_Z B_R W + \mathcal{H}\nabla_Z C_R W) = 0.\end{aligned}$$

Hence, we obtain

$$a) \Leftrightarrow b), \quad a) \Leftrightarrow c), \quad a) \Leftrightarrow d).$$

Therefore, the result follows. ■

Theorem 4.2. *Let F be a h-semi-slant Riemannian map from a hyperkähler manifold (M, I, J, K, g_M) to a Riemannian manifold (N, g_N) such that (I, J, K) is a h-semi-slant basis. Then the following conditions are equivalent:*

(a) *the fibers of F are locally Riemannian product manifolds of the leaves of $\mathcal{D}_1$ and $\mathcal{D}_2$*

(b)

$$Q_I(\phi_I \widehat{\nabla}_U \phi_I V + B_I \mathcal{T}_U \phi_I V) = 0 \text{ and } \omega_I \widehat{\nabla}_U \phi_I V + C_I \mathcal{T}_U \phi_I V = 0$$

for $U, V \in \Gamma(\mathcal{D}_1)$,

$$\begin{aligned}&P_I(\phi_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + B_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y)) = 0\\ &\omega_I(\widehat{\nabla}_X \phi_I Y + \mathcal{T}_X \omega_I Y) + C_I(\mathcal{T}_X \phi_I Y + \mathcal{H}\nabla_X \omega_I Y) = 0\end{aligned}$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

(c)

$$Q_J(\phi_J\widehat{\nabla}_U\phi_JV + B_J\mathcal{T}_U\phi_JV) = 0 \textit{ and } \omega_J\widehat{\nabla}_U\phi_JV + C_J\mathcal{T}_U\phi_JV = 0$$

for $U, V \in \Gamma(\mathcal{D}_1)$,

$$P_J(\phi_J(\widehat{\nabla}_X\phi_JY + \mathcal{T}_X\omega_JY) + B_J(\mathcal{T}_X\phi_JY + \mathcal{H}\nabla_X\omega_JY)) = 0$$

$$\omega_J(\widehat{\nabla}_X\phi_JY + \mathcal{T}_X\omega_JY) + C_J(\mathcal{T}_X\phi_JY + \mathcal{H}\nabla_X\omega_JY) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

(d)

$$Q_K(\phi_K\widehat{\nabla}_U\phi_KV + B_K\mathcal{T}_U\phi_KV) = 0 \textit{ and } \omega_K\widehat{\nabla}_U\phi_KV + C_K\mathcal{T}_U\phi_KV = 0$$

for $U, V \in \Gamma(\mathcal{D}_1)$,

$$P_K(\phi_K(\widehat{\nabla}_X\phi_KY + \mathcal{T}_X\omega_KY) + B_K(\mathcal{T}_X\phi_KY + \mathcal{H}\nabla_X\omega_KY)) = 0$$

$$\omega_K(\widehat{\nabla}_X\phi_KY + \mathcal{T}_X\omega_KY) + C_K(\mathcal{T}_X\phi_KY + \mathcal{H}\nabla_X\omega_KY) = 0$$

for $X, Y \in \Gamma(\mathcal{D}_2)$.

Proof. Given $R \in \{I, J, K\}$, for $U, V \in \Gamma(\mathcal{D}_1)$, we get

$$\begin{aligned}\nabla_U V &= -J\nabla_U JV = -J(\widehat{\nabla}_U\phi V + \mathcal{T}_U\phi V)\\ &= -(\phi\widehat{\nabla}_U\phi V + \omega\widehat{\nabla}_U\phi V + B\mathcal{T}_U\phi V + C\mathcal{T}_U\phi V).\end{aligned}$$

Thus,

$$\nabla_U V \in \Gamma(\mathcal{D}_1) \Leftrightarrow Q(\phi\widehat{\nabla}_U\phi V + B\mathcal{T}_U\phi V) = 0 \text{ and } \omega\widehat{\nabla}_U\phi V + C\mathcal{T}_U\phi V = 0.$$

For $X, Y \in \Gamma(\mathcal{D}_2)$, we have

$$\begin{aligned}\nabla_X Y &= -R\nabla_X RY = -R(\widehat{\nabla}_X\phi_RY + \mathcal{T}_X\phi_RY + \mathcal{T}_X\omega_RY + \mathcal{H}\nabla_X\omega_RY)\\ &= -(\phi_R\widehat{\nabla}_X\phi_RY + \omega_R\widehat{\nabla}_X\phi_RY + B_R\mathcal{T}_X\phi_RY + C_R\mathcal{T}_X\phi_RY + \phi_R\mathcal{T}_X\omega_RY\\ &\quad + \omega_R\mathcal{T}_X\omega_RY + B_R\mathcal{H}\nabla_X\omega_RY + C_R\mathcal{H}\nabla_X\omega_RY).\end{aligned}$$

Thus,

$\nabla_X Y \in \Gamma(\mathcal{D}_2) \Leftrightarrow$

$$\begin{cases} P_R(\phi_R(\widehat{\nabla}_X\phi_RY + \mathcal{T}_X\omega_RY) + B_R(\mathcal{T}_X\phi_RY + \mathcal{H}\nabla_X\omega_RY)) = 0, \\ \omega_R(\widehat{\nabla}_X\phi_RY + \mathcal{T}_X\omega_RY) + C_R(\mathcal{T}_X\phi_RY + \mathcal{H}\nabla_X\omega_RY) = 0. \end{cases}$$

Hence, we have

$$a) \Leftrightarrow b), \quad a) \Leftrightarrow c), \quad a) \Leftrightarrow d).$$

Therefore, we obtain the result. ∎

5. Examples

Note that given an Euclidean space $\mathbb{R}^{4m}$ with coordinates $(x_1, x_2, \cdots, x_{4m})$, we can canonically choose complex structures I, J, K on $\mathbb{R}^{4m}$ as follows:

$$I(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+2}}, I(\tfrac{\partial}{\partial x_{4k+2}}) = -\tfrac{\partial}{\partial x_{4k+1}}, I(\tfrac{\partial}{\partial x_{4k+3}}) = \tfrac{\partial}{\partial x_{4k+4}}, I(\tfrac{\partial}{\partial x_{4k+4}}) = -\tfrac{\partial}{\partial x_{4k+3}},$$
$$J(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+3}}, J(\tfrac{\partial}{\partial x_{4k+2}}) = -\tfrac{\partial}{\partial x_{4k+4}}, J(\tfrac{\partial}{\partial x_{4k+3}}) = -\tfrac{\partial}{\partial x_{4k+1}}, J(\tfrac{\partial}{\partial x_{4k+4}}) = \tfrac{\partial}{\partial x_{4k+2}},$$
$$K(\tfrac{\partial}{\partial x_{4k+1}}) = \tfrac{\partial}{\partial x_{4k+4}}, K(\tfrac{\partial}{\partial x_{4k+2}}) = \tfrac{\partial}{\partial x_{4k+3}}, K(\tfrac{\partial}{\partial x_{4k+3}}) = -\tfrac{\partial}{\partial x_{4k+2}}, K(\tfrac{\partial}{\partial x_{4k+4}}) = -\tfrac{\partial}{\partial x_{4k+1}}$$

for $k \in \{0, 1, \cdots, m-1\}$. Then it is easy to check that $(I, J, K, <\,,\,>)$ is a hyperkähler structure on $\mathbb{R}^{4m}$, where $<\,,\,>$ denotes the Euclidean metric on $\mathbb{R}^{4m}$. Throughout this section, we will use these notations.

Example 5.1. [12]. Let F be an almost h-slant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Then the map $F : (M, E, g_M) \mapsto (N, g_N)$ is a h-semi-slant Riemannian map with $\mathcal{D}_2 = \ker F_*$.

Example 5.2. [13]. Let F be an almost h-semi-slant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Then the map $F : (M, E, g_M) \mapsto (N, g_N)$ is an almost h-semi-slant Riemannian map.

Example 5.3. [10]. Let (M, E, g) be an almost quaternionic Hermitian manifold. Let $\pi : TM \mapsto M$ be the natural projection. Then the map π is a strictly h-semi-slant Riemannian map such that $\mathcal{D}_1 = \ker \pi_*$ and the strictly h-semi-slant angle $\theta = 0$.

Example 5.4. [10]. Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds. Let $F : M \mapsto N$ be a quaternionic submersion. Then the map F is a strictly h-semi-slant Riemannian map such that $\mathcal{D}_1 = \ker F_*$ and the strictly h-semi-slant angle $\theta = 0$.

Example 5.5. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, \cdots, x_8) = (x_2, x_1 \sin\alpha - x_3 \cos\alpha, 1968, x_4),$$

where α is constant. Then the map F is a strictly h-semi-slant Riemannian map such that

$$\mathcal{D}_1 =< \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} > \text{ and } \mathcal{D}_2 =< \cos\alpha \frac{\partial}{\partial x_1} + \sin\alpha \frac{\partial}{\partial x_3} >$$

with the strictly h-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 5.6. Let (M, E, g_M) be a $4m-$dimensional almost quaternionic Hermitian manifold and (N, g_N) a $(4m-1)-$dimensional Riemannian manifold. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be a Riemannian map with $\operatorname{rank} F = 4m - 1$. Then the map F is a strictly h-semi-slant Riemannian map such that $\mathcal{D}_2 = \ker F_*$ and the strictly h-semi-slant angle $\theta = \frac{\pi}{2}$.

Example 5.7. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^5$ by

$$F(x_1, \cdots, x_{12}) = (x_6, \frac{x_1 - x_3}{\sqrt{2}}, c, x_4, \frac{x_5 - x_7}{\sqrt{2}}),$$

where c is constant. Then the map F is a h-semi-slant Riemannian map such that

$$\mathcal{D}_1 =< \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} > \text{ and } \mathcal{D}_2 =< \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7} >$$

with the h-semi-slant angles $\{\theta_I = \frac{\pi}{4}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{4}\}$.

Example 5.8. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^7$ by

$$F(x_1, \cdots, x_{12}) = (x_5 \cos\alpha - x_7 \sin\alpha, \gamma, x_6 \sin\beta - x_8 \cos\beta, x_9, x_{11}, x_{12}, x_{10}),$$

where α, β, and γ are constant. Then the map F is a h-semi-slant Riemannian map such that

$$\mathcal{D}_1 =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >$$

and

$$\mathcal{D}_2 =< \sin\alpha \frac{\partial}{\partial x_5} + \cos\alpha \frac{\partial}{\partial x_7}, \cos\beta \frac{\partial}{\partial x_6} + \sin\beta \frac{\partial}{\partial x_8} >$$

with the h-semi-slant angles $\{\theta_I, \theta_J = \frac{\pi}{2}, \theta_K\}$ such that $\cos\theta_I = |\sin(\alpha + \beta)|$ and $\cos\theta_K = |\cos(\alpha + \beta)|$.

Example 5.9. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^7$ by

$$F(x_1, \cdots, x_{12}) = (x_3, x_4, 0, x_7, x_5, x_6, x_8).$$

Then the map F is an almost h-semi-slant Riemannian map such that

$$\mathcal{D}_1^I =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >,$$
$$\mathcal{D}_1^J = \mathcal{D}_1^K =< \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >,$$
$$\mathcal{D}_2^I = 0, \quad \mathcal{D}_2^J = \mathcal{D}_2^K =< \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} > .$$

with the almost h-semi-slant angles $\{\theta_I = 0, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$.

Example 5.10. Define a map $F : \mathbb{R}^{12} \mapsto \mathbb{R}^6$ by

$$F(x_1, \cdots, x_{12}) = (x_2, x_5, \alpha, x_1, \beta, x_7),$$

where α and β are constant. Then the map F is an almost h-semi-slant Riemannian map such that

$$\begin{aligned}
\mathcal{D}_1^I &=< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >, \\
\mathcal{D}_1^J &=< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >, \\
\mathcal{D}_1^K &=< \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} >, \\
\mathcal{D}_2^I &=< \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} >, \quad \mathcal{D}_2^J =< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} >, \\
\mathcal{D}_2^K &=< \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} >
\end{aligned}$$

with the almost h-semi-slant angles $\{\theta_I = \frac{\pi}{2}, \theta_J = \frac{\pi}{2}, \theta_K = \frac{\pi}{2}\}$.

Example 5.11. Let $\widetilde{F}$ be a h-semi-slant Riemannian map from an almost quaternionic Hermitian manifold (M_1, E_1, g_{M_1}) to a Riemannian manifold (N, g_N) with $\mathcal{D}_2 = \ker \widetilde{F}_*$. Let (M_2, E_2, g_{M_2}) be an almost quaternionic Hermitian manifold. Denote by (M, E, g_M) the warped product of (M_1, E_1, g_{M_1}) and (M_2, E_2, g_{M_2}) by a positive function g on M_1 [8], where $E = E_1 \times E_2$.

Define a map $F : (M, E, g_M) \mapsto (N, g_N)$ by

$$F(x, y) = \widetilde{F}(x) \quad \text{for } x \in M_1 \text{ and } y \in M_2.$$

Then the map F is a h-semi-slant Riemannian map such that

$$\mathcal{D}_1 = TM_2 \text{ and } \mathcal{D}_2 = \ker \widetilde{F}_*$$

with the h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$, where $\{I, J, K\}$ is a h-slant basis for the map $\widetilde{F}$ with the h-semi-slant angles $\{\theta_I, \theta_J, \theta_K\}$.

Note that as a generalization of an almost h-slant submersion [12], we call the map $\widetilde{F}$ an *almost h-slant Riemannian map*.

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Kwang-Soon Park
Department of Mathematical Sciences
Seoul National University
Seoul 151-747
Republic of Korea
E-mail: parkksn@gmail.com