TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 2, pp. 427-430, April 2013
DOI: 10.11650/tjm.17.2013.2282
his paper is available online at http://journal.taiwanmathsoc.org.tw

INEQUALITIES FOR GRAM MATRICES AND THEIR APPLICATIONS TO REPRODUCING KERNEL HILBERT SPACES

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Abstract. We prove elementary inequalities for the Gram matrices and their equality conditions. As an application we show that inequalities for the Gram determinants hold for general reproducing kernel Hilbert spaces.

1. Introduction

Recently, N. D. V. Nhan and D. T. Duc [3] have found interesting Gram determinant inequalities of the following form: For every $F_i \in \mathcal{H}_{K_1}$ and $G_j \in \mathcal{H}_{K_2}$ $(i, j = 1, \ldots, n)$, we have

$$\det(\langle F_i G_i, F_j G_j\rangle_{\mathcal{H}_{K_1 K_2}})_{i,j=1}^n \le C \det(\langle F_i, F_j\rangle_{\mathcal{H}_{K_1}} \langle G_i, G_j\rangle_{\mathcal{H}_{K_2}})_{i,j=1}^n,$$

where C is a positive constant, and $\mathcal{H}_{K_j}$ is a reproducing kernel Hilbert space (RKHS) defined on a set E with the reproducing kernel K_j $(j = 1, 2)$. These inequalities may be considered as an extension of the well-known norm inequalities for RKHSs [4, Appendix 2]. They, however, proved the above inequalities only for some restricted type of RKHSs whose definitions are given concretely. Moreover, they did not give the equality conditions for these inequalities. The aim of this paper is to show that their inequalities hold for general RKHSs and that they are a direct consequence of the general theory of Hermitian matrices and the tensor product Hilbert space of RKHSs. For the theory of RKHSs, the reader is referred to [1, 4].

Let $G(x_1, \ldots, x_n) = (\langle x_i, x_j\rangle)_{i,j=1}^n$ denote the Gram matrix of the vectors $\{x_1, \ldots, x_n\}$ in an inner product space. This is our main

Received August 5, 2012, accepted August 13, 2012.
Communicated by Ngai-Ching Wong.
2010 *Mathematics Subject Classification*: Primary 46E22; Secondary 15A45.
Key words and phrases: Gram matrices, Weyl's monotonicity principle, Gram determinant inequalities, Positive semidefinite matrices, Reproducing kernel Hilbert spaces.

Theorem 1.1. *Let $T\colon H_1 \to H_2$ be a bounded linear operator between inner product spaces H_1 and H_2. Then, for every $x_1, \dots, x_n \in H_1$, we have the following inequalities:*

$$G(Tx_1, \dots, Tx_n) \le \|T\|^2 G(x_1, \dots, x_n), \tag{1}$$

$$\det G(Tx_1, \dots, Tx_n) \le \|T\|^{2n} \det G(x_1, \dots, x_n). \tag{2}$$

Equality occurs in the inequality (2) if and only if one of the following conditions holds:

(i) *The set $\{x_1, \dots, x_n\}$ is linearly dependent in H_1.*
(ii) $T = 0$.
(iii) *The operator $T/\|T\|$ ($T \neq 0$) is an isometry on* $\operatorname{span}(x_1, \dots, x_n)$.

Also, equality occurs in (1) if and only if the above condition (ii) or (iii) holds.

Applying the above theorem to the tensor product Hilbert space of RKHSs (Theorem 3.1), we can immediately obtain all the Gram determinant inequalities proved in [3]. Furthermore, we also obtain the equality conditions for these inequalities.

2. PROOF OF THEOREM 1.1

It is well-known that the Gram matrix is positive semidefinite. Similarly, we have, for $(\xi_i) \in \mathbb{C}^n$,

$$\sum_{i,j=1}^{n} \xi_i \bar{\xi}_j \|T\|^2 \langle x_i, x_j \rangle - \sum_{i,j=1}^{n} \xi_i \bar{\xi}_j \langle Tx_i, Tx_j \rangle = \|T\|^2 \| \sum_{i=1}^{n} \xi_i x_i \|^2 - \|T \sum_{i=1}^{n} \xi_i x_i \|^2 \ge 0.$$

Thus, putting the Gram matrices as $A = \|T\|^2 G(x_1, \dots, x_n)$ and $B = G(Tx_1, \dots, Tx_n)$, we see that $B \le A$, so that the inequality (1) holds. Let us enumerate the eigenvalues of an $n \times n$ Hermitian matrix X as

$$\lambda_1(X) \ge \lambda_2(X) \ge \cdots \ge \lambda_n(X).$$

Then, by Weyl's monotonicity principle (cf. [2]) for eigenvalues of Hermitian matrices, we have $\lambda_j(B) \le \lambda_j(A)$ $(j = 1, \dots, n)$. Since the determinant of a matrix is the product of its eigenvalues, and since the matrices A and B are positive semidefinite, we obtain the inequality (2) immediately.

Next, we proceed to determine the equality condition for the inequality (2). It is well-known that the set $\{x_1, \dots, x_n\}$ is linearly independent if and only if the Gram matrix $G(x_1, \dots, x_n)$ is nonsingular. Thus, it is clear that, if one of the conditions (i), (ii) and (iii) holds, then equality holds in (2). Conversely, if equality holds in (2), we need only to show that (iii) holds by assuming that $T \neq 0$ and that $G(x_1, \dots, x_n)$ is nonsingular. We may assume without loss of generality that $\|T\| = 1$. Then, since both Gram matrices are positive definite, we have $\lambda_j(A) = \lambda_j(B)$ $(j = 1, \ldots, n)$. Since $A - B \geq 0$ and

$$\mathrm{tr}(A - B) = \sum_{j=1}^{n} \lambda_j(A) - \sum_{j=1}^{n} \lambda_j(B) = 0,$$

we conclude that $A = B$. Therefore, T is an isometry on the subspace span$\{x_1, \ldots, x_n\}$, so we have proved that (iii) holds.

Similarly, it is easy to see that equality occurs in (1) if and only if the condition (ii) or (iii) holds.

3. Applications to RKHSs

Let $\mathcal{H}_{K_j}$ be a RKHS on a set E with the reproducing kernel K_j $(j = 1, 2)$. Then, their tensor product Hilbert space (direct product) $\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}$ is a RKHS on $E \times E$ whose reproducing kernel is given by $(K_1 \otimes K_2)((x, y), (x', y')) = K_1(x, x')K_2(y, y')$. By setting $(f \otimes g)(x, y) = f(x)g(y)$, the linear span of the functions $\{f \otimes g \colon f \in \mathcal{H}_{K_1},\ g \in \mathcal{H}_{K_2}\}$ is dense in $\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}$. Also, we have an identity for the inner products (cf. [1, p. 357]):

$$\langle f \otimes g, f' \otimes g' \rangle_{\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}} = \langle f, f' \rangle_{\mathcal{H}_{K_1}} \langle g, g' \rangle_{\mathcal{H}_{K_2}}.$$

Let $\iota \colon E \to E \times E$, $\iota(x) = (x, x)$ be the diagonal embedding from the set E into $E \times E$. Then, the operator range of the linear map $Tf = f \circ \iota$ defined on $\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}$ is a RKHS on E with the reproducing kernel $K_1 K_2$. Moreover, the induced operator $T \colon \mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2} \to \mathcal{H}_{K_1 K_2}$ is a contraction with $T(f \otimes g) = fg$ (cf. [1], [4], [5]):

$$\|fg\|_{\mathcal{H}_{K_1 K_2}} \leq \|f \otimes g\|_{\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}} = \|f\|_{\mathcal{H}_{K_1}} \|g\|_{\mathcal{H}_{K_2}}.$$

Applying Theorem 1.1 to the contraction T, we obtain

Theorem 3.1. *Let $\mathcal{H}_{K_j}$ be a RKHS on E with the reproducing kernel K_j $(j = 1, 2)$. Then, for any $F_i \in \mathcal{H}_{K_1}$ and $G_j \in \mathcal{H}_{K_2}$ $(i, j = 1, \ldots, n)$, we have*

$$\det(\langle F_i G_i, F_j G_j \rangle_{\mathcal{H}_{K_1 K_2}})_{i,j=1}^{n} \leq \det(\langle F_i, F_j \rangle_{\mathcal{H}_{K_1}} \langle G_i, G_j \rangle_{\mathcal{H}_{K_2}})_{i,j=1}^{n}.$$

Equality holds in the above inequality if and only if one of the following conditions holds:

(*i*) $\{F_i \otimes G_i \colon i = 1, \ldots, n\}$ *is linearly dependent in* $\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}$, *or*

(*ii*) $\{F_i \otimes G_i \colon i = 1, \ldots, n\} \subset (\mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2}) \ominus \ker T$.

Proof. We need only to prove the equality condition. The operator $T\colon \mathcal{H}_{K_1} \otimes \mathcal{H}_{K_2} \to \mathcal{H}_{K_1 K_2}$ is a coisometry (i.e. the adjoint T^* is an isometry) by definition of the operator range. Hence, the subspace on which T is isometry is the orthogonal complement of $\ker T$. Thus, from Theorem 1.1 we easily conclude the equality conditions of Theorem 3.1. ■

From Theorem 3.1 we immediately obtain all the Gram determinant inequalities proved in [3].

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