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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A CLASS OF SUBLINEAR SCHRÖDINGER-MAXWELL EQUATIONS

Zhisu Liu, Shangjiang Guo and Ziheng Zhang

Abstract. In this paper we are concerned with a class of sublinear Schrödinger-Maxwell equations

$$\begin{cases} -\triangle u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2, \ \lim\limits_{|x|\to+\infty} \phi(x) = 0, & \text{in } \mathbb{R}^3, \end{cases}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. Under certain assumptions on V and f, some new criteria on the existence and multiplicity of negative energy solutions for the above system are established via the genus properties in critical point theory. Recent results from the literature are significantly improved.

1. INTRODUCTION

Consider the following coupled nonlinear Schrödinger-Maxwell equations, also known as the nonlinear Schrödinger-Poisson equations

$$\text{(SM)} \qquad \begin{cases} -\triangle u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2, \ \lim\limits_{|x|\to+\infty} \phi(x) = 0, & \text{in } \mathbb{R}^3, \end{cases}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. Indeed, such a system and similar ones arise in many mathematical physical context, such as in quantum electrodynamics, to describe the interaction between a charge particle interacting with the electromagnetic field, and also in semiconductor theory, in nonlinear optics and in plasma physics (we refer to [10] for more details in the physics aspects). In particular, if we are looking for electrostatic-type solutions, we just have to solve (SM).

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In recent years, problem (SM) with $V(x) \equiv 1$ or being radially symmetric, has been widely studied under various conditions on f, see for example [3, 4, 5, 6, 7, 13, 19, 20]. More precisely, in [3, 20], in order to avoid the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)\,(2 < s < 6)$, the standard work space $H^1(\mathbb{R}^3)$ was replaced by the radial function space $H_r^1(\mathbb{R}^3)$ where the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)\,(2 < s < 6)$ is compact. Ruiz [20] dealt with (SM) with $V(x) \equiv 1$ and $f(u) = u^p\,(1 < p < 5)$ and got some general existence, nonexistence and multiplicity results and Ambrosetti and Ruiz [3] obtained existence of multiple bound state solutions. For the sublinear term $f(s) = \min\{|s|^r, |s|^p\}$ with $0 < r < 1 < p$, Kristály and Repovš [15] handled the form $f(x,u) = \lambda\alpha(x)f(u)$ for (SM) with $V(x) \equiv 1$. For large parameters, the system has at least two nontrivial solutions, while for small parameters, no solution exists. When $V(x)$ and $f(x,u)$ are 1-periodic in each $x_i, i = 1,2,3$ in (SM), Zhao [26] obtained the existence of infinitely many geometrically distinct solutions by using the nonlinear superposition principle established in [1]. If $V(x)$ is periodic or asymptotically periodic and $f(x,u)$ does not satisfy the Ambrosetti-Rabinowitz condition, Alves, et al [2] established the existence of positive ground state solutions by using the mountain pass theorem. We recall here that $(u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is said to be a ground state solution to problem (SM), if (u,ϕ) solves (SM) and minimizes the action functional associated to (SM) among all possible nontrivial solutions. The case of nonradial potential V has also been considered in [12, 16, 21, 25, 26] and the references mentioned therein. In particular, Wang and Zhou [25] got the existence and nonexistence results of (SM) when $f(u)$ is asymptotically linear at infinity. In [8], Azzollini and Pomponio proved the existence of ground state solutions for problem (SM) with $f(x,u) = |u|^{p-1}u, 3 < p < 5$ and Zhao [26] generalized results in [8] to the case where $2 < p \leq 3$. Chen and Tang [12] proved that (SM) had infinitely many high energy solutions under the condition that $f(x,u)$ is superlinear at infinity in u by fountain theorem. Soon after, Li, Su and Wei [16] improved their results.

For the case that $V(x)$ is nonradial potential and $f(t,u)$ is sublinear at infinity in u, as far as the authors are aware, there is only one result up to now. When $f(x,u) = (p+1)b(x)|u|^{p-1}u$, where $0 < p < 1$ is a constant and $b : \mathbb{R}^3 \to \mathbb{R}$ is a positive continuous function such that $b \in L^{\frac{2}{1-p}}(\mathbb{R}^3)$, by using variant fountain theorem [27], Sun [23] established the following theorem on the existence of infinitely many nontrivial solutions of problem (SM) under the assumption that V satisfies certain conditions.

Theorem 1.1. [23]. *Assume that the following conditions hold:*

(V_0) $V \in C(\mathbb{R}^3, \mathbb{R})$ *satisfies* $\inf\limits_{x\in\mathbb{R}^3} V(x) = a > 0$*, where* a *is a constant;*

(V_1) *For every* $M > 0$*, meas*$\{x \in \mathbb{R}^3 : V(x) \leq M\} < \infty$*, where meas denotes the Lebesgue measure in* $\mathbb{R}^3$*;*

(F_0) *$F(x,u)=b(x)|u|^{p+1}$, where $F(x,u)=\int_0^u f(x,y)dy$, $b:\mathbb{R}^3\to\mathbb{R}^+$ is a positive continuous function such that $b\in L^{\frac{2}{1-p}}(\mathbb{R}^3)$ and $0<p<1$ is a constant.*

Then system (SM) *possesses infinitely many negative energy solutions $\{(u_k,\phi_k)\}$ satisfying*

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_k|^2+V(x)u_k^2)dx-\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi_k|^2dx$$
$$+\frac{1}{2}\int_{\mathbb{R}^3}\phi_k u_k^2dx-\int_{\mathbb{R}^3}F(x,u_k)dx\to 0^-,\ \text{as } k\to\infty.$$

In the above theorem, assumptions (V_0) and (V_1) imply a coercive condition on V, which was firstly introduced in [9] and is used to overcome the lack of compactness of embedding of the working space E (see Section 2), and (F_0) contains a strict restriction on f. In fact, there are much sublinear functions in mathematical physics in problem like (SM) except for $f(x,u)=(p+1)b(x)|u|^{p-1}u$. In the present paper, motivated by paper [24], we will use the genus properties in critical theory to generalize Theorem 1.1 by removing assumption (V_1) and relaxing assumption (F_0).

Now, we state our main results.

Theorem 1.2. *Assume that* (V_0) *and the following conditions hold:*

(F_1) *$f\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$ and there exist two constants $0<p<1,\frac{1}{3}\le q<1$ and a positive function $b\in L^{\frac{2}{1-p}}(\mathbb{R}^3)$ and a nonnegative function $b_1\in L^3(\mathbb{R}^3)$ such that*

$$|f(x,u)|\le(p+1)b(x)|u|^p+(q+1)b_1(x)|u|^q,\ \forall(x,u)\in\mathbb{R}^3\times\mathbb{R};$$

(F_2) *There exist a nonzero measure open set $\Omega\subset\mathbb{R}^3$ and three constants $\delta,\eta>0$ and $p'\in(0,1)$ such that*

$$F(x,u)\ge\eta|u|^{p'+1},\ \forall(x,u)\in\Omega\times[-\delta,\delta],$$

where

$$F(x,u):=\int_0^u f(x,y)dy,\ x\in\mathbb{R}^3,u\in\mathbb{R}. \tag{1.1}$$

Then system (SM) possesses at least one nontrivial solution.

Theorem 1.3. *Assume that V and f satisfy* (V_0), (F_1), (F_2) *and the following condition:*

(F_3) $f(x,-u)=-f(x,u)$, $\forall(x,z)\in\mathbb{R}^3\times\mathbb{R}$.
Then system (SM) possesses infinitely many negative energy solutions $\{(u_k,\phi_k)\}$ satisfying

$$\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_k|^2+V(x)u_k^2)dx-\frac{1}{4}\int_{\mathbb{R}^3}|\nabla\phi_k|^2dx$$
$$+\frac{1}{2}\int_{\mathbb{R}^3}\phi_k u_k^2dx-\int_{\mathbb{R}^3}F(x,u_k)dx<0,\ \text{as } k\to\infty.$$

In fact, it is easy to see that assumption (F_2) is satisfied if the following condition holds:

(F_2') There exist a nonzero measure open set $\Omega\subset\mathbb{R}^3$ and three constants $\delta,\eta>0$ and $p'\in(0,1)$ such that

$$uf(x,u)\geq\eta|u|^{p'+1},\ \forall(x,u)\in\Omega\times[-\delta,\delta].$$

Therefore, by Theorems 1.2 and 1.3, we have the following corollary.

Corollary 1.1. *In Theorems 1.2 and 1.3, if assumption* (F_2) *is replaced by* (F_2')*, then the conclusions still hold.*

Remark 1.1. If $f(x,u)=(p+1)b(x)|u|^{p-1}u$, then $F(x,u)=b(x)|u|^{p+1}$. Hence, assumption (F_0) implies that (F_1), (F_2) and (F_3) with $p=p', b_1(x)\equiv 0$.

Remark 1.2. Our results can be applied to some indefinite sign sublinear functions which can not been implied by the sublinear term in [23]. For example, let

$$f(x,u)=\frac{4\cos x_1}{3e^{|x|}}|u|^{-2/3}u+\frac{3\sin x_2}{2e^{3|x|}}|u|^{-1/2}u,\ \forall(x,u)\in\mathbb{R}^3\times\mathbb{R},$$

where $x=(x_1,x_2,x_3)^\top$. Clearly,

$$\begin{aligned}|f(x,u)|&\leq\frac{4}{3e^{|x|}}|u|^{1/3}+\frac{3}{2e^{3|x|}}|u|^{1/2},\ \forall(x,u)\in\mathbb{R}^3\times\mathbb{R},\\ F(x,u)&=\frac{\cos x_1}{e^{|x|}}|u|^{4/3}+\frac{\sin x_2}{e^{3|x|}}|u|^{3/2}\\ &\geq\frac{\cos 1}{e}|u|^{4/3},\ \forall(x,u)\in(0,1)^3\times[-1,1].\end{aligned}$$

Thus (F_1), (F_2) and (F_3) are satisfied with

$$\begin{aligned}&\frac{1}{3}=p=p',\ q=\frac{1}{2},\ b(x)=\frac{4}{3e^{|x|}},\ b_1(x)=\frac{3}{2e^{3|x|}},\\ &\delta=1,\quad \eta=\frac{\cos 1}{e},\quad \Omega=(0,1)^3.\end{aligned}$$

Throughout this paper, $C>0$ denotes various positive generic constants. The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. The proofs of our main results are given in Section 3.

2. Notation and Preliminaries

Hereafter, we recall the following notations. For any $1 \leq s \leq +\infty$, we denote by $\|\cdot\|_s$ the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$. Let

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx < +\infty \right\}$$

equipped with the inner product

$$(u, v) := \int_{\mathbb{R}^3} [\nabla u \nabla v + V(x)u(x)v(x)]dx, \ u, v \in E,$$

and the norm

$$\|u\| = (u, u)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx \right)^{\frac{1}{2}}.$$

Then E is a Hilbert space with the above inner product. $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{D^{1,2}} := \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Note that E is continuously embedded into $L^s(\mathbb{R}^3)$ for all $s \in [2, 2^*]$, where $2^* = 6$ is the critical exponent for the Sobolev embeddings in dimension 3. Therefore, there exists a constant $C > 0$ such that

$$\|u\|_s \leq C\|u\|, \quad \forall u \in E. \tag{2.1}$$

For every $u \in E$, by the Lax-Milgram theorem, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$-\triangle \phi_u = u^2$$

and

$$\int_{\mathbb{R}^3} u^2 v dx = \int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla v dx, \quad \forall v \in D^{1,2}(\mathbb{R}^3). \tag{2.2}$$

Moreover, ϕ_u can be expressed by (see [14]):

$$\phi_u = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|} dy = \frac{1}{|x|} * u^2. \tag{2.3}$$

Now we collect some properties of the functions ϕ_u (see [5, 11, 23]).

Lemma 2.1. *For any $u \in E$, we have*

(1) $\|\phi_u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} \phi_u u^2 dx \leq C\|u\|^4_{12/5} \leq C\|u\|^4$;

(2) $\phi_u \geq 0$;

(3) *for any* $t > 0$, $\phi_{u_t} = t^2\phi_u$, *where* $u_t = tu$;

(4) *if* $u_n \rightharpoonup u$ *in* E, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* $D^{1,2}(\mathbb{R}^3)$.

Now we define the following integral momentums

$$\Phi_1(u) := \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx, \qquad \Phi_2(u) := \int_{\mathbb{R}^3} F(x,u)dx. \tag{2.4}$$

Lemma 2.2. $\Phi_1' : E \to E^*$ *is weakly continuous, where* E^* *is the dual space of* E.

The proof is similar to Lemma 2.3 (i) in [26], so we omit the details.

Lemma 2.3. *Assume that* (V_0) *and* (F_1) *hold. Then the functional* $I : E \to \mathbb{R}$ *defined by*

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx \tag{2.5}$$

is well defined and of class $C^1(E;\mathbb{R})$ *and*

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv + \phi_u uv - f(x,u)v)\, dx, v \in E. \tag{2.6}$$

Furthermore, if $u \in E$ *is a critical point of the functional* I, *then the pair* $(u, \phi_u) \in E \times D^{1,2}(\mathbb{R}^3)$, *with* ϕ_u *defined as in* (2.3), is a solution of system (SM).

Proof. It is clear that (SM) is the Euler-Lagrange equations of the functional $\Phi : E \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\Phi(u,\phi) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx.$$

Evidently, the action functional Φ exhibits a strong indefiniteness, namely it is unbounded both from below and above in infinite dimensional subspaces. In fact, using the reduction method described in [4,6], one gets

$$\Phi(u,\phi_u) = I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx,$$

which is a variable functional that does not present such a strongly indefinite nature. By (F_1) and (1.1), one has

$$|F(x,u)| \leq b(x)|u|^{p+1} + b_1(x)|u|^{q+1},\ \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.7}$$

For any $u \in E$, it follows from (V_0), (2.1), (2.7) and the Hölder inequality that

$$
\begin{aligned}
\int_{\mathbb{R}^3} |F(x,u)|dx \le & \int_{\mathbb{R}^3} [b(x)|u|^{p+1} + b_1(x)|u|^{q+1}]dx \\
\le & a^{\frac{-(p+1)}{2}} \left(\int_{\mathbb{R}^3} |b(x)|^{\frac{2}{1-p}} dx\right)^{\frac{1-p}{2}} \left(\int_{\mathbb{R}^3} V(x)u^2 dx\right)^{\frac{p+1}{2}} \\
& + \left(\int_{\mathbb{R}^3} |b_1(x)|^3 dx\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} u^{\frac{3}{2}(q+1)} dx\right)^{\frac{2}{3}} \\
\le & C(\|u\|^{p+1} + \|u\|^{q+1}).
\end{aligned} \tag{2.8}
$$

Hence, combining Lemma 2.1 with (2.8), we see that I is well defined on E.

Next, we prove that (2.6) holds. For any function $\theta : \mathbb{R}^3 \to (0,1)$ and $u, v \in E$, using (F_1) and the Hölder inequality we have

$$
\begin{aligned}
& \int_{\mathbb{R}^3} \max_{h\in[0,1]} |f(x, u + \theta(x)hv)v|dx \\
\le & \int_{\mathbb{R}^3} \max_{h\in[0,1]} |f(x, u + \theta(x)hv)||v|dx \\
\le & C \int_{\mathbb{R}^3} [b(x)(|u| + |v|)^p + b_1(x)(|u| + |v|)^q]|v|dx \\
\le & C \int_{\mathbb{R}^3} [b(x)(|u|^p + |v|^p) + b_1(x)(|u|^q + |v|^q)]|v|dx \\
\le & a^{\frac{-2p-1}{2}} C \left(\int_{\mathbb{R}^3} |b(x)|^{\frac{2}{1-p}} dx\right)^{\frac{1-p}{2}} \left(\int_{\mathbb{R}^3} V(x)|u|^2 dx\right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^3} V(x)|v|^2 dx\right)^{\frac{1}{2}} \\
& + Ca^{\frac{-(p+1)}{2}} \left(\int_{\mathbb{R}^3} |b(x)|^{\frac{2}{1-p}} dx\right)^{\frac{1-p}{2}} \left(\int_{\mathbb{R}^3} V(x)v^2 dx\right)^{\frac{p+1}{2}} \\
& + C \left(\int_{\mathbb{R}^3} |b_1(x)|^3 dx\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} |u|^{6q} dx\right)^{\frac{1}{6}} \left(\int_{\mathbb{R}^3} |v|^2 dx\right)^{\frac{1}{2}} \\
& + C \left(\int_{\mathbb{R}^3} |b_1(x)|^3 dx\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} v^{\frac{3}{2}(q+1)} dx\right)^{\frac{2}{3}} \\
\le & C(\|u\|^p + \|v\|^p + \|v\|^q + \|u\|^q)\|v\| \\
< & +\infty.
\end{aligned} \tag{2.9}
$$

Then by (2.4), (2.9) and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}\langle\Phi_2'(u),v\rangle &= \lim_{h\to0^+}\frac{\Phi_2(u+hv)-\Phi_2(u)}{h}\\ &= \lim_{h\to0^+}\frac{1}{h}\int_{\mathbb{R}^3}[F(x,u+hv)-F(x,u)]dx\\ &= \lim_{h\to0^+}\int_{\mathbb{R}^3}[f(x,u+\theta(x)hv)vdx\\ &= \int_{\mathbb{R}^3}f(x,u)vdx.\end{aligned}\tag{2.10}$$

In addition, it is clear that $\Phi_1' \in C^1(E,\mathbb{R})$. Therefore, (2.6) holds. Let us prove now that Φ_2' is continuous. Let $u_k \to u$ in E, then

$$u_k \to u,\ \text{ in } L^s(\mathbb{R}^3), s\in[2,6], \quad u_k\to u \quad a.e. \text{ in } \mathbb{R}^3.\tag{2.11}$$

We show that

$$\int_{\mathbb{R}^3}|f(x,u_k)-f(x,u)|^2dx \to 0, \quad \text{as} \quad k\to+\infty.\tag{2.12}$$

In fact, since $u_k \to u$ in $L^2(\mathbb{R}^3)$ and $u_k \to u$ in $L^{6q}(\mathbb{R}^3)$, passing to a subsequence if necessary, there exist $w\in L^2(\mathbb{R}^3)$ and $w'\in L^{6q}(\mathbb{R}^3)$ such that, for all $k\in\mathbb{N}$,

$$\begin{aligned}|u_k(x)| &\le w(x) \quad a.e. \text{ in } \mathbb{R}^3,\\ |u_k(x)| &\le w'(x) \quad a.e. \text{ in } \mathbb{R}^3.\end{aligned}$$

Note that, for all $k\in\mathbb{N}$,

$$\begin{aligned}|f(x,u_k(x))-f(x,u(x))|^2 \le& 2|f(x,u_k(x))|^2+2|f(x,u(x))|^2\\ \le& 4(p+1)^2|b(x)|^2[|u_k(x)|^{2p}+|u(x)|^{2p}]\\ &+4(q+1)^2|b_1(x)|^2[|u_k(x)|^{2q}+|u(x)|^{2q}]\\ \le& C|b(x)|^2[|w(x)|^{2p}+|u(x)|^{2p}]\\ &+C|b_1(x)|^2[|w'(x)|^{2q}+|u(x)|^{2q}]\\ :=& g(x),\ a.e \text{ in } \mathbb{R}^3\end{aligned}\tag{2.13}$$

and

$$\begin{aligned}\int_{\mathbb{R}^3}g(x)dx =& \int_{\mathbb{R}^3}C|b(x)|^2[|w|^{2p}+|u|^{2p}]dx\\ &+\int_{\mathbb{R}^3}C|b_1(x)|^2[|w'|^{2q}+|u|^{2q}]dx\\ \le& C\|b\|^2_{\frac{2}{1-p}}(\|w\|_2^{2p}+\|u\|_2^{2p})+C\|b_1\|_3^2(\|w'\|_{6q}^{2q}+\|u\|_{6q}^{2q})\\ <& +\infty.\end{aligned}\tag{2.14}$$

Then, by (2.11), (2.13), (2.14) and Lebesgue's Dominated Convergence Theorem, we know that (2.12) holds. By (2.10), (2.12) and the Hölder inequality, for all given $v \in E$ we have

$$\begin{aligned}|\langle \Phi_2'(u_k) - \Phi_2'(u), v\rangle| &= |\int_{\mathbb{R}^3} [f(x,u_k) - f(x,u)]v dx| \\ &\le C\|v\|(\int_{\mathbb{R}^3} |f(x,u_k) - f(x,u)|^2 dx)^{\frac{1}{2}} \to 0, \quad k \to +\infty,\end{aligned}$$

which implies the continuity of Φ_2'. Hence, $I \in C^1(E,\mathbb{R})$. Furthermore, It can be proved that $(u,\phi) \in E \times D^{1,2}(\mathbb{R}^3)$ is a solution of (SM) if and only if $u \in E$ is a critical point of the functional I, and $\phi = \phi_u$, see for instance [10]. The proof is complete. ∎

Definition 2.1. $I \in C^1(E,\mathbb{R})$ is said to satisfy the (PS)-condition if any sequence $\{u_j\}_{j\in\mathbb{N}} \subset E$, for which $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$, possesses a convergent subsequence in E.

Let E be a Banach space, $I \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$. Set

$$\Sigma = \{A \subset E - \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\},$$

$$K_c = \{u \in E : I(u) = c, I'(u) = 0\}, \qquad I^c = \{u \in E : I(u) \le c\}.$$

Definition 2.2. ([18]). For $A \in \Sigma$, we say genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\varphi \in C(A, \mathbb{R}^n \backslash \{0\})$ and n is the smallest integer with this property.

As a conclusion of this section, we state the following theorems which are crucial to our arguments in Section 3.

Theorem 2.4. ([17]). *Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfy the* (PS)*-condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

Theorem 2.5. ([22]). *Let I be an even C^1 functional on E and satisfy the* (PS)*-condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \ge n\}, \qquad c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} I(u).$$

(*1*) *If $\Sigma_n \neq \emptyset$ and $c_n \in \mathbb{R}$, then c_n is a critical value of I;*

(*2*) *If there exists $r \in \mathbb{N}$ such that*

$$c_n = c_{n+1} = \cdots = c_{n+r} = c \in \mathbb{R},$$

and $c \neq I(0)$, *then* $\gamma(K_c) \geq r+1$.

3. Proofs of Main Results

In order to make use of Theorem 2.4 to prove Theorem 1.2, we need the following Lemma.

Lemma 3.1. *Under the conditions of Theorem 1.1, I is bounded from below and satisfies the* (PS)-*condition.*

Proof. In what follows, we first show that I is bounded from below. By (2.1), (2.5) and the Hölder inequality, one has

$$
\begin{aligned}
I(u) =& \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} F(x,u)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx \\
\geq& \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} b(x)|u|^{p+1}dx - \int_{\mathbb{R}^3} b_1(x)|u|^{q+1}dx \\
\geq& \frac{1}{2}\|u\|^2 - \left(\int_{\mathbb{R}^3} |b(x)|^{\frac{2}{1-p}}dx\right)^{\frac{1-p}{2}} \left(\int_{\mathbb{R}^3} |u|^2 dx\right)^{\frac{1+p}{2}} \\
& - \left(\int_{\mathbb{R}^3} |b_1(x)|^3 dx\right)^{\frac{1}{3}} \left(\int_{\mathbb{R}^3} |u|^{\frac{3}{2}(q+1)}dx\right)^{\frac{2}{3}} \\
\geq& \frac{1}{2}\|u\|^2 - \|b\|_{\frac{2}{1-p}}\|u\|_2^{1+p} - \|b_1\|_3 \|u\|_{\frac{3}{2}(q+1)}^{q+1} \\
\geq& \frac{1}{2}\|u\|^2 - C(\|u\|^{p+1} + \|u\|^{q+1}).
\end{aligned} \tag{3.1}
$$

Since $0 < p < 1, \frac{1}{3} \leq q < 1$, (3.1) implies that $I(u) \to +\infty$ as $\|u\| \to +\infty$. Consequently, I is bounded from below. Next, we prove that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then by (3.1), there exists a constant $A > 0$ such that

$$\|u_k\| \leq A, \quad \forall k \in \mathbb{N}. \tag{3.2}$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E. Hence

$$u_k \to u_0, \quad \text{in} \quad L^s_{loc}(\mathbb{R}^3) \quad s \in [2,6). \tag{3.3}$$

For any given number $\varepsilon > 0$, by (F_1), we can choose $\rho > 0$ such that

$$\left(\int_{|x|\geq\rho} |b(x)|^{\frac{2}{1-p}}dx\right)^{\frac{1-p}{2}} < \varepsilon, \qquad \left(\int_{|x|\geq\rho} |b_1(x)|^3 dx\right)^{\frac{1}{3}} < \varepsilon. \tag{3.4}$$

It follows from (2.1), (2.13), (3.2) and (3.3) that there exists $k_0 \in \mathbb{N}$ such that

$$
\begin{aligned}
&\int_{|x|\le\rho} |f(x,u_k)-f(x,u_0)||u_k-u_0|dx \\
&\le \left(\int_{|x|\le\rho} |f(x,u_k)-f(x,u_0)|^2dx\right)^{\frac12}\left(\int_{|x|\le\rho}|u_k-u_0|^2dx\right)^{\frac12} \\
&\le \varepsilon\left(\int_{|x|\le\rho} 2(|f(x,u_k)|^2+|f(x,u_0)|^2)dx\right)^{\frac12} \\
&\le \varepsilon C\left(\int_{|x|\le\rho} |b(x)|^2[|u_k|^{2p}+|u_0|^{2p}]+|b_1(x)|^2[|u_k|^{2q}+|u_0|^{2q}]dx\right)^{\frac12} \\
&\le \varepsilon C[\|b\|^2_{\frac{2}{1-p}}(\|u_k\|_2^{2p}+\|u_0\|_2^{2p})+\|b_1\|_3^2(\|u_k\|_{6q}^{2q}+\|u_0\|_{6q}^{2q})] \\
&\le \varepsilon C(\|b\|^2_{\frac{2}{1-p}}A^{2p}+\|u_0\|_2^{2p}+\|u_0\|_{6q}^{2q}+C\|b_1\|_3^2A^{2q}),
\end{aligned}
\tag{3.5}
$$

for $k \ge k_0$. On the other hand, it follows from (F_1), (2.1), (3.2) and the Hölder inequality that

$$
\begin{aligned}
&\int_{|x|>\rho} |f(x,u_k)-f(x,u_0)||u_k-u_0|dx \\
&\le \int_{|x|>\rho} [(p+1)b(x)(|u_k|^p+|u_0|^p)+(q+1)b_1(x)(|u_k|^q+|u_0|^q)](|u_k|+|u_0|)dx \\
&\le 2(p+1)\int_{|x|>\rho} b(x)(|u_k|^{p+1}+|u_0|^{p+1})dx \\
&\quad + 2(q+1)\int_{|x|>\rho} b_1(x)(|u_k|^{q+1}+|u_0|^{q+1})dx \\
&\le C\left(\int_{|x|>\rho}|b(x)|^{\frac{2}{1-p}}dx\right)^{\frac{1-p}{2}}\left[\left(\int_{|x|>\rho}|u_k|^2\right)^{\frac{1+p}{2}}+\left(\int_{|x|>\rho}|u_0|^2\right)^{\frac{1+p}{2}}\right] \\
&\quad + C\left(\int_{|x|>\rho}|b_1(x)|^3dx\right)^{\frac13}\left[\left(\int_{|x|>\rho}|u_k|^{\frac32(q+1)}\right)^{\frac23}+\left(\int_{|x|>\rho}|u_0|^{\frac32(q+1)}\right)^{\frac23}\right] \\
&\le C\varepsilon[\|u_k\|_2^{p+1}+\|u_0\|_2^{p+1}]+C\varepsilon[\|u_k\|^{q+1}_{\frac32(q+1)}+\|u_0\|^{q+1}_{\frac32(q+1)}] \\
&\le C\varepsilon[A^{p+1}+A^{q+1}+\|u_0\|^{p+1}+\|u_0\|^{q+1}].
\end{aligned}
\tag{3.6}
$$

Since ε is arbitrary, it follows from (3.5) and (3.6) that

$$
\int_{\mathbb{R}^3} |f(x,u_k)-f(x,u_0)||u_k-u_0|dx \to 0,
\tag{3.7}
$$

as $k \to +\infty$. Since Φ_1' is the weakly continuous by Lemma 2.2, we conclude that

$$
\begin{aligned}
(3.8) \qquad |\int_{\mathbb{R}^3} (\phi_{u_k} u_k - \phi_{u_0} u_0)(u_k - u_0)dx| &= |\langle \Phi_1'(u_k) - \Phi_1'(u_0), u_k - u_0 \rangle| \\
&\leq \|\Phi_1'(u_k) - \Phi_1'(u_0)\|_{E^*} \|u_k - u_0\| \to 0.
\end{aligned}
$$

It follows from (2.6) that

$$
\begin{aligned}
(3.9) \qquad \langle I'(u_k) - I'(u_0), u_k - u_0 \rangle = \|u_k - u_0\|^2 - &\int_{\mathbb{R}^3} (f(x, u_k) - f(x, u_0))(u_k - u_0)dx \\
+ &\int_{\mathbb{R}^3} (\phi_{u_k} u_k - \phi_{u_0} u_0)(u_k - u_0)dx.
\end{aligned}
$$

Obviously, $\langle I'(u_k) - I'(u_0), u_k - u_0 \rangle \to 0$ as $k \to +\infty$. Combining (3.7), (3.8) and (3.9), one knows that $u_k \to u_0$ in E. Hence, I satisfies (PS)-condition. The proof is complete. ■

The proof of Theorem 1.2. In view of Lemma 2.3, $I \in C^1(E, \mathbb{R}^3)$. By Theorem 2.4 and Lemma 3.1, we get $c = \inf_E I(u)$ is a critical value of I, that is, there exists a critical point $u^* \in E$ such that $I(u^*) = c$.

Finally, we show that $u^* \neq 0$. Let $u_0 \in (W_0^{1,2}(\Omega) \cap E) \setminus \{0\}$, then by (2.5) and Lemma 2.1, we infer

$$
\begin{aligned}
I(su_0) &= \frac{s^2}{2}\|u_0\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{su_0}(su_0)^2 dx - \int_{\mathbb{R}^3} F(x, su_0)dx \\
(3.10) \qquad &\leq \frac{s^2}{2}\|u_0\|^2 + C\|su_0\|^4 - \int_{\Omega} \eta |su_0|^{p'+1} dx \\
&\leq \frac{s^2}{2}\|u_0\|^2 + Cs^4\|u_0\|^4 - \eta s^{p'+1} \int_{\Omega} |u_0|^{p'+1} dx, \quad 0 < s < 1.
\end{aligned}
$$

Since $0 < p' < 1$, it follows from (3.10) that $I(su_0) < 0$ for $s > 0$ small enough. Hence $I(u^*) = c < 0$, therefore u^* is nontrivial critical point of I with $I(u^*) = \inf_E I(u)$ and is a nontrivial solution of (SM). The proof is complete. ■

The proof of Theorem 1.3. In view of Lemma 3.1, $I \in C^1(E, \mathbb{R})$ is bounded from below and satisfies the (PS)-condition. It is clear that I is even and $I(0) = 0$. In order to apply Theorem 2.5, we now prove that for any $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that

$$
(3.11) \qquad \gamma(I^{-\varepsilon}) \geq n.
$$

For any $n \in \mathbb{N}$, we take n disjoint open sets Ω_i such that

$$\bigcup_{i=1}^{n} \Omega_i \subset \Omega.$$

For each $i \in \{1, 2, \cdots, n\}$, let $u_i \in (W_0^{1,2}(\Omega_i) \cap E) \setminus \{0\}$ and $\|u_i\| = 1$, and

$$E_n = \text{span}\{u_1, u_2, \cdots, u_n\}, \qquad S_n = \{u \in E_n : \|u\| = 1\}.$$

So for any $u \in E_n$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$ such that

$$u(x) = \sum_{i=1}^{n} \lambda_i u_i, \ \text{ for } x \in \mathbb{R}^3. \tag{3.12}$$

Then, we have

$$\|u\|_{p'+1} = \left(\int_{\mathbb{R}^3} |u|^{p'+1} dx\right)^{\frac{1}{p'+1}} = \left(\sum_{i=1}^{n} |\lambda_i|^{p'+1} \int_{\Omega_i} |u_i(x)|^{p'+1} dx\right)^{\frac{1}{p'+1}}, \tag{3.13}$$

and

$$\begin{aligned}
\|u\|^2 &= \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx \\
&= \sum_{i=1}^{n} \lambda_i^2 \int_{\Omega_i} (|\nabla u_i|^2 + V(x)u_i^2) dx \\
&= \sum_{i=1}^{n} \lambda_i^2 \int_{\mathbb{R}^3} (|\nabla u_i|^2 + V(x)u_i^2) dx \\
&= \sum_{i=1}^{n} \lambda_i^2 \|u_i\|^2 \\
&= \sum_{i=1}^{n} \lambda_i^2.
\end{aligned} \tag{3.14}$$

Since all norms of a finite dimensional normed space are equivalent, there is a constant $c' > 0$ such that

$$c'\|u\| \leq \|u\|_{p'+1}, \quad for \quad u \in E_n. \tag{3.15}$$

By Lemma 2.1, (2.5), (3.13) and (3.15), we obtain

$$\begin{aligned}
I(su) &= \frac{s^2}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{su}(su)^2 dx - \int_{\mathbb{R}^3} F(x, su) dx \\
&\leq \frac{s^2}{2}\|u\|^2 + C\|su\|^4 - \sum_{i=1}^{n} \int_{\Omega_i} \eta |s\lambda_i u_i|^{p'+1} dx \\
&\leq \frac{s^2}{2}\|u\|^2 + C\|su\|^4 - \eta s^{p'+1} \sum_{i=1}^{n} |\lambda_i|^{p'+1} \int_{\Omega_i} |u_i|^{p'+1} dx \\
&= \frac{s^2}{2}\|u\|^2 + C\|su\|^4 - \eta s^{p'+1} \|u\|_{p'+1}^{p'+1}
\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\le \frac{s^2}{2}\|u\|^2 + C\|su\|^4 - \eta s^{p'+1}c'^{p'+1}\|u\|^{p'+1}\\&= \frac{s^2}{2} + s^4 C - \eta s^{p'+1}c'^{p'+1}, \quad \forall u \in S_n,\ 0 < s < 1.\end{aligned}$$

Hence, $0 < p' < 1$ and (3.16) imply that there exist $\varepsilon > 0$ and $\sigma > 0$ such that

$$I(\sigma u) < -\varepsilon \quad \text{for } u \in S_n. \tag{3.17}$$

Let

$$S_n^\sigma = \{\sigma u : u \in S_n\}, \quad \Omega = \{(\lambda_1, \lambda_2, \cdots, \lambda_n) : \sum_{i=1}^n \lambda_i^2 < \sigma^2\}.$$

It follows from (3.17) that $I(u) < -\varepsilon$ for $u \in S_n^\sigma$, which, together with the fact that $I \in C^1(E, R)$ and is even, implies that

$$S_n^\sigma \subset I^{-\varepsilon} \in \Sigma.$$

On the other hand, it follows from (3.12) and (3.14) that there exists an odd homeomorphism mapping $\psi \in C(S_n^\sigma, \partial\Omega)$. By some properties of the genus (see 3° of Propositions 7.5 and 7.7 in [18], we deduce

$$\gamma(I^{-\varepsilon}) \ge \gamma(S_n^\sigma) = n, \tag{3.18}$$

so (3.11) holds. Set

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} I(u).$$

It follows from (3.18) and the fact that I is bounded from below on E that $-\infty < c_n \le -\varepsilon < 0$, that is for any $n \in \mathbb{N}$, c_n is a real negative number. By Theorem 2.5, I has infinitely many nontrivial critical points, and so (SM) possesses infinitely many nontrivial negative energy solutions. ■

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