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OSCILLATION OF THIRD-ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the oscillatory behavior of a class of third-order nonlinear delay differential equations

$$(a(t)(b(t)y'(t))')' + q(t)y^{\gamma}(\tau(t)) = 0.$$

Some new oscillation criteria are presented by transforming this equation to the first-order delayed and advanced differential equations. Employing suitable comparison theorems we establish new results on oscillation of the studied equation. Assumptions in our theorems are less restrictive, these criteria improve those in the recent paper [Appl. Math. Comput., 202 (2008), 102-112] and related contributions to the subject. Examples are provided to illustrate new results.

1. INTRODUCTION

In the real world, one can predict dynamic behavior of solutions of third-order partial differential equations by using the qualitative behavior of the third-order differential equations. For instance, in a wide variety fascinating physical phenomena, Kuramoto–Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u^2 = 0$$

plays some roles, it is used to describe pattern formulation in reaction diffusion systems, and to model the instability of flame front propagation; see [15, 17]. To find the travelling wave solutions of this partial differential equation, one may use the substitution of the form $u(x, ct) = u(x - ct)$ with peed c and solve a third-order nonlinear differential equation

$$\lambda u'''(x) + u'(x) + f(u) = 0.$$

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Hence, it is interesting to investigate third-order differential equations of the form

$$y'''(t) + p(t)y'(t) + q(t)y^{\gamma}(\tau(t)) = 0.$$

Note that the above equation can be written as a binomial equation

$$\left(v^2(t)\left(\frac{1}{v(t)}y'(t)\right)'\right)' + v(t)q(t)y^{\gamma}(\tau(t)) = 0,$$

where v is a positive solution of an equation $v''(t) + p(t)v(t) = 0$; see the related ideas exploited in [9].

Based on the above background details, we are here concerned with the oscillation of a third-order nonlinear delay differential equation

$$(E) \qquad (a(t)(b(t)y'(t))')' + q(t)y^{\gamma}(\tau(t)) = 0.$$

Throughout this paper, it is always assumed that

(i) γ is the ratio of odd positive integers;

(ii) $a, b, q \in \mathrm{C}([t_0, \infty), \mathbb{R})$, $a(t) > 0$, $b(t) > 0$, $q(t) > 0$;

(iii) $\tau \in \mathrm{C}([t_0, \infty), \mathbb{R})$, $\tau(t) < t$, $\tau(t)$ is nondecreasing, and $\lim_{t\to\infty} \tau(t) = \infty$.

Further we will consider the following two cases

$$(1.1) \qquad \int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} < \infty, \ \int_{t_0}^{\infty} \frac{\mathrm{d}t}{b(t)} = \infty$$

and

$$(1.2) \qquad \int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} < \infty, \ \int_{t_0}^{\infty} \frac{\mathrm{d}t}{b(t)} < \infty.$$

By a solution of equation (E) we mean a nontrivial function $y \in \mathrm{C}([T_y, \infty))$, $T_y \geq t_0$, which satisfies (E) on $[T_y, \infty)$. We consider only those solutions y of (E) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$ and assume that (E) possesses such solutions. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_y, \infty)$; otherwise it is called nonoscillatory. Equation (E) is said to be oscillatory if all its solutions oscillate.

Regarding the oscillation of differential equations with deviating arguments, we refer the reader to [1-14, 16, 18-22] and the references cited therein. Baculíková and Džurina [5, 6] and Grace et al. [12] studied a third-order nonlinear delay differential equation

$$(a(t)(x''(t))^{\alpha})' + q(t)f(x(\tau(t))) = 0,$$

and established some oscillation results provided that

$$\int_{t_0}^{\infty} a^{-1/\alpha}(t)\mathrm{d}t = \infty$$

or

$$\int_{t_0}^{\infty} a^{-1/\alpha}(t)\mathrm{d}t < \infty.$$

Candan and Dahiya [8] considered equation (E) in the case where

$$\gamma = 1 \text{ and } \int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} = \int_{t_0}^{\infty} \frac{\mathrm{d}t}{b(t)} = \infty.$$

Li et al. [16], Saker and Džurina [19], and Zhang et al. [22] investigated a third-order quasilinear delay differential equation

$$(a(t)(x''(t))^{\alpha})' + q(t)x^{\beta}(\tau(t)) = 0,$$

and presented new criteria which guarantee that all non-oscillatory solutions of such equation tend to zero. Li et al. [21] studied equation (E) for the case where $\gamma = 1$, and obtained some sufficient conditions which insure that the solution x of (E) is oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

As a special case, when $b(t) = 1$, (E) reduces to

$$(a(t)y''(t))' + q(t)y^{\gamma}(\tau(t)) = 0, \tag{1.3}$$

which has been studied in [5, 6, 10, 12]. Grace et al. [12] obtained several oscillation criteria for (1.3), one of which we present below for the convenience of the reader.

Theorem 1. (See [12, Theorem 2.2]). *Let* $\int_{t_0}^{\infty}(a(s))^{-1}\mathrm{d}s < \infty$. *Assume that there exist two functions* $\xi, \eta \in \mathrm{C}^1([t_0, \infty), \mathbb{R})$ *such that*

$$\xi'(t) \geq 0,\ \eta'(t) \geq 0,\ \textit{and } \tau(t) < \xi(t) < \eta(t) < t \textit{ for } t \geq t_0.$$

Assume also that both first-order delay equations

$$y'(t) + cq(t)\tau^{\gamma}(t)\left(\int_{T}^{\tau(t)} \frac{s}{a(s)}\mathrm{d}s\right)^{\gamma} y^{\gamma}(\tau(t)) = 0$$

for any constant c, $0 < c < 1$ *and all* $T \geq t_0$, *and*

$$z'(t) + q(t)(\xi(t) - \tau(t))^{\gamma}\left(\int_{\xi(t)}^{\eta(t)} \frac{\mathrm{d}s}{a(s)}\right)^{\gamma} z^{\gamma}(\eta(t)) = 0$$

are oscillatory. If

$$\int_{t_0}^{\infty} \frac{1}{a(u)} \int_{t_0}^{u} q(s)\tau^{\gamma}(s)\left(\int_{\tau(s)}^{\infty} \frac{\mathrm{d}v}{a(v)}\right)^{\gamma} \mathrm{d}s\mathrm{d}u = \infty, \tag{1.4}$$

then (1.3) *is oscillatory.*

The purpose of this paper is to supplement and improve results in [5, 6, 8, 10, 12, 16, 19, 21, 22]. In what follows, all functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Main Results

In the following, we will establish some oscillation criteria for (E). To simplify our notation, let us denote $z(t) := -\omega(t) := -a(t)(b(t)y'(t))'$ and $B(t) := \int_t^\infty \frac{\mathrm{d}s}{b(s)}$.

Theorem 1. *Let* (1.1) *hold. Assume that there exist numbers $\alpha \le \gamma$, $\beta \ge \gamma$, and two functions $\xi, \sigma \in \mathrm{C}([t_0,\infty),\mathbb{R})$ such that α, β are the ratios of odd positive integers, $\xi(t) > t$, $\xi(t)$ is nondecreasing, $\tau(\xi(\xi(t))) < t$, $\sigma(t)$ is nondecreasing, and $\sigma(t) > t$. If for all sufficiently large $t_1 \ge t_0$ and for $t_2 > t_1$, the first-order delay differential equation*

$$(E_1) \qquad \omega'(t) + {c_1}^{\gamma-\alpha} q(t) \left(\int_{t_2}^{\tau(t)} \frac{\int_{t_1}^{s} \frac{\mathrm{d}u}{a(u)}}{b(s)} \mathrm{d}s \right)^{\alpha} \omega^{\alpha}(\tau(t)) = 0$$

is oscillatory for all constants $c_1 > 0$, the first-order delay differential equation

$$(E_2) \qquad \nu'(t) + \left(\frac{1}{b(t)} \int_t^{\xi(t)} \frac{1}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1)\mathrm{d}s_1 \mathrm{d}s_2 \right) \nu^{\gamma}(\tau(\xi(\xi(t)))) = 0$$

is oscillatory, the first-order advanced differential equation

$$(E_3) \qquad z'(t) - {c_2}^{\gamma-\beta} q(t) \left(\int_{\sigma(t)}^{\infty} \frac{\mathrm{d}s}{a(s)} \right)^{\beta} \left(\int_{t_1}^{\tau(t)} \frac{\mathrm{d}s}{b(s)} \right)^{\gamma} z^{\beta}(\sigma(t)) = 0$$

is oscillatory for all constants $c_2 > 0$, then (E) is oscillatory.

Proof. Let y be a non-oscillatory solution of (E). Without loss of generality, we may suppose that y is positive. Then there exist three possible cases:

$$\text{case } (1).\ y(t) > 0,\ y'(t) > 0,\ (b(t)y'(t))' > 0,\ (a(t)(b(t)y'(t))')' < 0;$$

$$\text{case } (2).\ y(t) > 0,\ y'(t) < 0,\ (b(t)y'(t))' > 0,\ (a(t)(b(t)y'(t))')' < 0;$$

and

$$\text{case } (3).\ y(t) > 0,\ y'(t) > 0,\ (b(t)y'(t))' < 0,\ (a(t)(b(t)y'(t))')' < 0$$

for $t \ge t_1$, where $t_1 \ge t_0$ is large enough. Assume that case (1) holds. Using $(a(t)\,(b(t)y'(t))')' < 0$, we have

$$b(t)y'(t) \ge \int_{t_1}^{t} \frac{a(s)(b(s)y'(s))'}{a(s)} \mathrm{d}s \ge a(t)(b(t)y'(t))' \int_{t_1}^{t} \frac{\mathrm{d}s}{a(s)}.$$

That is,

$$y'(t) \geq \frac{a(t)(b(t)y'(t))'}{b(t)} \int_{t_1}^{t} \frac{\mathrm{d}s}{a(s)}.$$

Integrating the latter inequality from t_2 ($t_2 > t_1$) to t, we get by the definition of ω that

$$y(t) \geq a(t)(b(t)y'(t))' \int_{t_2}^{t} \frac{\int_{t_1}^{s} \frac{1}{a(u)}\mathrm{d}u}{b(s)}\mathrm{d}s = \omega(t) \int_{t_2}^{t} \frac{\int_{t_1}^{s} \frac{1}{a(u)}\mathrm{d}u}{b(s)}\mathrm{d}s. \tag{2.1}$$

From (E) and the fact that $y'(t) > 0$, we see that there exists a constant $c_1 > 0$ such that

$$(a(t)(b(t)y'(t))')' + {c_1}^{\gamma-\alpha} q(t) y^{\alpha}(\tau(t)) \leq 0.$$

Using (2.1) in the above inequality, we have

$$\omega'(t) + {c_1}^{\gamma-\alpha} q(t) \left(\int_{t_2}^{\tau(t)} \frac{\int_{t_1}^{s} \frac{1}{a(u)}\mathrm{d}u}{b(s)}\mathrm{d}s \right)^{\alpha} \omega^{\alpha}(\tau(t)) \leq 0.$$

By virtue of [18, Theorem 1], the associated delay differential equation (E_1) also has a positive solution, which is a contradiction. Assume that case (2) holds. Integrating (E) from t to $\xi(t)$ implies that

$$a(t)(b(t)y'(t))' \geq \int_{t}^{\xi(t)} q(s_1) y^{\gamma}(\tau(s_1))\mathrm{d}s_1 \geq y^{\gamma}(\tau(\xi(t))) \int_{t}^{\xi(t)} q(s_1)\mathrm{d}s_1.$$

That is,

$$(b(t)y'(t))' \geq \frac{y^{\gamma}(\tau(\xi(t)))}{a(t)} \int_{t}^{\xi(t)} q(s_1)\mathrm{d}s_1.$$

Integrating the above inequality from t to $\xi(t)$, we have

$$\begin{aligned} -b(t)y'(t) &\geq \int_{t}^{\xi(t)} \frac{y^{\gamma}(\tau(\xi(s_2)))}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1)\mathrm{d}s_1\mathrm{d}s_2 \\ &\geq y^{\gamma}(\tau(\xi(\xi(t)))) \int_{t}^{\xi(t)} \frac{1}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1)\mathrm{d}s_1\mathrm{d}s_2. \end{aligned}$$

That is,

$$-y'(t) \geq \frac{y^{\gamma}(\tau(\xi(\xi(t))))}{b(t)} \int_{t}^{\xi(t)} \frac{1}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1)\mathrm{d}s_1\mathrm{d}s_2.$$

Integrating the latter inequality from t to ∞, one gets

$$y(t) \geq \int_{t}^{\infty} \frac{y^{\gamma}(\tau(\xi(\xi(s_3))))}{b(s_3)} \int_{s_3}^{\xi(s_3)} \frac{1}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3.$$

Let us denote the right hand side of the last inequality by $\nu(t)$. Then $\nu(t) > 0$, and one can easily verify that ν is a solution of the differential inequality

$$\nu'(t) + \left(\frac{1}{b(t)}\int_t^{\xi(t)} \frac{1}{a(s_2)}\int_{s_2}^{\xi(s_2)} q(s_1)\mathrm{d}s_1\mathrm{d}s_2\right)\nu^\gamma(\tau(\xi(\xi(t)))) \le 0.$$

By [18, Theorem 1], the associated delay differential equation (E_2) also has a positive solution, which is a contradiction. Assume that case (3) holds. As $(a(t)\,(b(t)y'(t))')' < 0$, we see that $a(t)\,(b(t)y'(t))'$ is decreasing. Thus, we get

$$a(s)(b(s)y'(s))' \le a(t)(b(t)y'(t))' \text{ for } s \ge t \ge t_1.$$

Dividing the above inequality by $a(s)$ and integrating the resulting inequality from t to l, we obtain

$$b(l)y'(l) \le b(t)y'(t) + a(t)(b(t)y'(t))'\int_t^l \frac{\mathrm{d}s}{a(s)}.$$

Letting $l \to \infty$, we have

$$b(t)y'(t) \ge -a(t)(b(t)y'(t))'\int_t^\infty \frac{\mathrm{d}s}{a(s)}. \tag{2.2}$$

Using conditions $y(t) > 0$ and $(b(t)y'(t))' < 0$, we have

$$y(t) \ge b(t)y'(t)\int_{t_1}^t \frac{\mathrm{d}s}{b(s)}. \tag{2.3}$$

Thus,

$$\left(\frac{y(t)}{\int_{t_1}^t \frac{\mathrm{d}s}{b(s)}}\right)' \le 0. \tag{2.4}$$

Combining (2.2) and (2.3), we get

$$y(t) \ge -a(t)(b(t)y'(t))'\int_t^\infty \frac{\mathrm{d}s}{a(s)}\int_{t_1}^t \frac{\mathrm{d}s}{b(s)}. \tag{2.5}$$

On the other hand, we have by (2.4) and $\sigma(t) \ge \tau(t)$ that

$$y^\gamma(\tau(t)) \ge \left(\frac{\int_{t_1}^{\tau(t)} \frac{\mathrm{d}s}{b(s)}}{\int_{t_1}^{\sigma(t)} \frac{\mathrm{d}s}{b(s)}}\right)^\gamma y^\gamma(\sigma(t)) = \left(\frac{\int_{t_1}^{\tau(t)} \frac{\mathrm{d}s}{b(s)}}{\int_{t_1}^{\sigma(t)} \frac{\mathrm{d}s}{b(s)}}\right)^\gamma y^\beta(\sigma(t))y^{\gamma-\beta}(\sigma(t)). \tag{2.6}$$

By virtue of (2.4), we have that there exists a constant c_2 such that $y(t) \le c_2\int_{t_1}^t \frac{\mathrm{d}s}{b(s)}$. Hence by (2.6), we get

$$\begin{aligned} y^\gamma(\tau(t)) &\ge {c_2}^{\gamma-\beta}\left(\frac{\int_{t_1}^{\tau(t)} \frac{\mathrm{d}s}{b(s)}}{\int_{t_1}^{\sigma(t)} \frac{\mathrm{d}s}{b(s)}}\right)^\gamma y^\beta(\sigma(t))\left(\int_{t_1}^{\sigma(t)} \frac{\mathrm{d}s}{b(s)}\right)^{\gamma-\beta} \\ &= {c_2}^{\gamma-\beta}\left(\int_{t_1}^{\sigma(t)} \frac{\mathrm{d}s}{b(s)}\right)^{-\beta}\left(\int_{t_1}^{\tau(t)} \frac{\mathrm{d}s}{b(s)}\right)^\gamma y^\beta(\sigma(t)). \end{aligned} \tag{2.7}$$

Combining (2.5) and (2.7), we obtain

$$y^{\gamma}(\tau(t)) \geq c_2{}^{\gamma-\beta}\left(\int_{\sigma(t)}^{\infty}\frac{\mathrm{d}s}{a(s)}\right)^{\beta}\left(\int_{t_1}^{\tau(t)}\frac{\mathrm{d}s}{b(s)}\right)^{\gamma}(-\omega(\sigma(t)))^{\beta}. \tag{2.8}$$

Using (2.8) in (E), we have

$$\omega'(t) + c_2{}^{\gamma-\beta}q(t)\left(\int_{\sigma(t)}^{\infty}\frac{\mathrm{d}s}{a(s)}\right)^{\beta}\left(\int_{t_1}^{\tau(t)}\frac{\mathrm{d}s}{b(s)}\right)^{\gamma}(-\omega(\sigma(t)))^{\beta} \leq 0.$$

Writing the latter inequality in the form

$$z'(t) - c_2{}^{\gamma-\beta}q(t)\left(\int_{\sigma(t)}^{\infty}\frac{\mathrm{d}s}{a(s)}\right)^{\beta}\left(\int_{t_1}^{\tau(t)}\frac{\mathrm{d}s}{b(s)}\right)^{\gamma}z^{\beta}(\sigma(t)) \geq 0,$$

we deduce from [2, Lemma 2.3] that the associated advanced differential equation (E_3) also has a positive solution, which is a contradiction. This completes the proof. ■

Corollary 1. *Let* (1.1) *hold and* $\gamma = 1$. *Assume that there exist two functions* $\xi, \sigma \in \mathrm{C}([t_0,\infty),\mathbb{R})$ *such that* $\xi(t) > t$, $\xi(t)$ *is nondecreasing,* $\tau(\xi(\xi(t))) < t$, $\sigma(t)$ *is nondecreasing, and* $\sigma(t) > t$. *If for all sufficiently large* $t_1 \geq t_0$ *and for* $t_2 > t_1$,

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}q(s)\int_{t_2}^{\tau(s)}\frac{\int_{t_1}^{\mu}\frac{\mathrm{d}u}{a(u)}}{b(\mu)}\mathrm{d}\mu\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.9}$$

$$\liminf_{t\to\infty}\int_{\tau(\xi(\xi(t)))}^{t}\frac{1}{b(s)}\int_{s}^{\xi(s)}\frac{1}{a(s_2)}\int_{s_2}^{\xi(s_2)}q(s_1)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.10}$$

and

$$\liminf_{t\to\infty}\int_{t}^{\sigma(t)}q(s)\int_{\sigma(s)}^{\infty}\frac{\mathrm{d}u}{a(u)}\int_{t_1}^{\tau(s)}\frac{\mathrm{d}u}{b(u)}\mathrm{d}s > \frac{1}{\mathrm{e}}, \tag{2.11}$$

then (E) *is oscillatory.*

Proof. Let $\alpha = \beta = \gamma = 1$. Then equations (E_1), (E_2), and (E_3) reduce to

$$\omega'(t) + q(t)\left(\int_{t_2}^{\tau(t)}\frac{\int_{t_1}^{s}\frac{1}{a(u)}\mathrm{d}u}{b(s)}\mathrm{d}s\right)\omega(\tau(t)) = 0,$$

$$\nu'(t) + \left(\frac{1}{b(t)}\int_{t}^{\xi(t)}\frac{1}{a(s_2)}\int_{s_2}^{\xi(s_2)}q(s_1)\mathrm{d}s_1\mathrm{d}s_2\right)\nu(\tau(\xi(\xi(t)))) = 0,$$

and

$$z'(t) - q(t)\left(\int_{\sigma(t)}^{\infty}\frac{\mathrm{d}s}{a(s)}\right)\left(\int_{t_1}^{\tau(t)}\frac{\mathrm{d}s}{b(s)}\right)z(\sigma(t)) = 0,$$

respectively. Applications of Theorem 1 and results of [11] complete the proof. ■

Example 1. For $t \geq 1$, consider a third-order linear delay differential equation

$$(\mathrm{e}^t y''(t))' + \sqrt{2}\mathrm{e}^t y\left(t - \frac{15\pi}{4}\right) = 0. \tag{2.12}$$

Let $\xi(t) = t+1$ and $\sigma(t) = t+1$. Then one can obtain that

$$\begin{aligned}
&\liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s) \int_{t_2}^{\tau(s)} \frac{\int_{t_1}^{\mu} \frac{\mathrm{d}u}{a(u)}}{b(\mu)} \mathrm{d}\mu \mathrm{d}s \\
&= \sqrt{2} \liminf_{t\to\infty} \int_{t-\frac{15\pi}{4}}^{t} \mathrm{e}^s \int_{t_2}^{s-\frac{15\pi}{4}} (\mathrm{e}^{-t_1} - \mathrm{e}^{-\mu}) \mathrm{d}\mu \mathrm{d}s = \infty, \\
&\liminf_{t\to\infty} \int_{\tau(\xi(\xi(t)))}^{t} \frac{1}{b(s)} \int_{s}^{\xi(s)} \frac{1}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1) \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s \\
&= \sqrt{2}(\mathrm{e}-1) \liminf_{t\to\infty} \int_{t+2-\frac{15\pi}{4}}^{t} \mathrm{d}s = \sqrt{2}(\mathrm{e}-1)\left(\frac{15\pi}{4} - 2\right) > \frac{1}{\mathrm{e}},
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{t\to\infty} \int_{t}^{\sigma(t)} q(s) \int_{\sigma(s)}^{\infty} \frac{\mathrm{d}u}{a(u)} \int_{t_1}^{\tau(s)} \frac{\mathrm{d}u}{b(u)} \mathrm{d}s \\
&= \sqrt{2} \liminf_{t\to\infty} \int_{t}^{t+1} \mathrm{e}^s \int_{s+1}^{\infty} \mathrm{e}^{-u} \mathrm{d}u \int_{t_1}^{s-\frac{15\pi}{4}} \mathrm{d}u \mathrm{d}s = \infty.
\end{aligned}$$

Hence equation (2.12) is oscillatory when using Corollary 1. As a matter of fact, one such solution is $y(t) = \sin t$. Note that

$$\begin{aligned}
&\int_{t_0}^{\infty} \frac{1}{a(u)} \int_{t_0}^{u} q(s)\tau^{\gamma}(s) \left(\int_{\tau(s)}^{\infty} \frac{\mathrm{d}v}{a(v)}\right)^{\gamma} \mathrm{d}s \mathrm{d}u \\
&= \sqrt{2} \int_{1}^{\infty} \mathrm{e}^{-u} \int_{1}^{u} \mathrm{e}^s \left(s - \frac{15\pi}{4}\right) \int_{s-\frac{15\pi}{4}}^{\infty} \mathrm{e}^{-v} \mathrm{d}v \mathrm{d}s \mathrm{d}u < \infty.
\end{aligned}$$

Thus, Theorem 1 cannot be applied to equation (2.12) since condition (1.4) does not hold.

Corollary 2. *Let* (1.1) *hold and* $\gamma < 1$. *Assume that there exist a number* $\beta > 1$ *and two functions* $\xi, \sigma \in \mathrm{C}([t_0, \infty), \mathbb{R})$ *such that* β *is the ratio of odd positive integers,* $\xi(t) > t$, $\xi(t)$ *is nondecreasing,* $\tau(\xi(\xi(t))) < t$, $\sigma(t)$ *is nondecreasing, and* $\sigma(t) > t$. *Suppose further that for all sufficiently large* $t_1 \geq t_0$ *and for* $t_3 > t_2 > t_1$,

$$\int_{t_3}^{\infty} q(t) \left(\int_{t_2}^{\tau(t)} \frac{\int_{t_1}^{s} \frac{1}{a(u)} \mathrm{d}u}{b(s)} \mathrm{d}s\right)^{\gamma} \mathrm{d}t = \infty, \tag{2.13}$$

$$\int_{t_0}^{\infty} \frac{1}{b(t)} \int_{t}^{\xi(t)} \frac{1}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t = \infty, \tag{2.14}$$

and

$$\int_{t_2}^{\infty} q(t) \left(\int_{\sigma(t)}^{\infty} \frac{\mathrm{d}s}{a(s)} \right)^{\beta} \left(\int_{t_1}^{\tau(t)} \frac{\mathrm{d}s}{b(s)} \right)^{\gamma} \mathrm{d}t = \infty. \tag{2.15}$$

Then (E) *is oscillatory.*

Proof. Let $\alpha = \gamma$. Applications of Theorem 1 and results of [14] complete the proof. ■

Example 2. For $t \geq 1$, consider a third-order sublinear delay differential equation

$$(t^2 y''(t))' + t^{-1/21} y^{1/3} \left(\frac{t}{8} \right) = 0. \tag{2.16}$$

Let $\beta = 9/7$, $\xi(t) = 2t$, and $\sigma(t) = 2t$. Then one can get

$$\int_{t_3}^{\infty} q(t) \left(\int_{t_2}^{\tau(t)} \frac{\int_{t_1}^{s} \frac{1}{a(u)}\mathrm{d}u}{b(s)} \mathrm{d}s \right)^{\gamma} \mathrm{d}t$$
$$= \int_{t_3}^{\infty} t^{-1/21} \left(\int_{t_2}^{\frac{t}{8}} \int_{t_1}^{s} \frac{1}{u^2} \mathrm{d}u\mathrm{d}s \right)^{1/3} \mathrm{d}t = \infty,$$

$$\int_{t_0}^{\infty} \frac{1}{b(t)} \int_{t}^{\xi(t)} \frac{1}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t$$
$$= \int_{1}^{\infty} \int_{t}^{2t} \frac{1}{s_2^2} \int_{s_2}^{2s_2} s_1^{-1/21}\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}t = \infty,$$

and

$$\int_{t_2}^{\infty} q(t) \left(\int_{\sigma(t)}^{\infty} \frac{\mathrm{d}s}{a(s)} \right)^{\beta} \left(\int_{t_1}^{\tau(t)} \frac{\mathrm{d}s}{b(s)} \right)^{\gamma} \mathrm{d}t$$
$$= \int_{t_2}^{\infty} t^{-1/21} \left(\int_{2t}^{\infty} \frac{\mathrm{d}s}{s^2} \right)^{9/7} \left(\int_{t_1}^{\frac{t}{8}} \mathrm{d}s \right)^{1/3} \mathrm{d}t = \infty.$$

Hence conditions (2.13), (2.14), and (2.15) are satisfied, and so equation (2.16) is oscillatory due to Corollary 2. Note that

$$\int_{t_0}^{\infty} \frac{1}{a(u)} \int_{t_0}^{u} q(s)\tau^{\gamma}(s) \left(\int_{\tau(s)}^{\infty} \frac{\mathrm{d}v}{a(v)} \right)^{\gamma} \mathrm{d}s\mathrm{d}u$$
$$= \int_{1}^{\infty} \frac{1}{u^2} \int_{1}^{u} s^{-1/21} \left(\frac{s}{8} \right)^{1/3} \left(\int_{\frac{s}{8}}^{\infty} \frac{\mathrm{d}v}{v^2} \right)^{1/3} \mathrm{d}s\mathrm{d}u < \infty.$$

Thus, Theorem 1 cannot be applied to equation (2.16) since condition (1.4) does not hold.

Corollary 3. *Let* (1.1) *hold,* $\gamma > 1$*, and* $\tau \in \mathrm{C}^1([t_0, \infty), \mathbb{R})$*. Assume that there exist a number* $\alpha < 1$ *and two functions* $\xi, \sigma \in \mathrm{C}([t_0, \infty), \mathbb{R})$ *such that* α *is the ratio of odd positive integers,* $\xi(t) > t$*,* $\xi(t)$ *is nondecreasing,* $\tau(\xi(\xi(t))) < t$*,* $\sigma(t)$ *is nondecreasing, and* $\sigma(t) > t$*. Suppose also that there exists a function* $\varphi \in \mathrm{C}^1([t_0, \infty), \mathbb{R})$ *such that* $\varphi'(t) > 0$*,* $\lim_{t\to\infty} \varphi(t) = \infty$*,*

$$\limsup_{t\to\infty} \frac{\varphi'(\tau(\xi(\xi(t))))(\tau(\xi(\xi(t))))'}{\varphi'(t)} < \frac{1}{\gamma}, \tag{2.17}$$

and

$$\liminf_{t\to\infty} \frac{\left(\frac{1}{b(t)} \int_t^{\xi(t)} \frac{1}{a(s_2)} \int_{s_2}^{\xi(s_2)} q(s_1)\mathrm{d}s_1\mathrm{d}s_2\right) \mathrm{e}^{-\varphi(t)}}{\varphi'(t)} > 0. \tag{2.18}$$

If for all sufficiently large $t_1 \geq t_0$ *and for* $t_3 > t_2 > t_1$*,*

$$\int_{t_3}^{\infty} q(t) \left(\int_{t_2}^{\tau(t)} \frac{\int_{t_1}^{s} \frac{1}{a(u)}\mathrm{d}u}{b(s)}\mathrm{d}s\right)^{\alpha} \mathrm{d}t = \infty \tag{2.19}$$

and

$$\int_{t_2}^{\infty} q(t) \left(\int_{\sigma(t)}^{\infty} \frac{\mathrm{d}s}{a(s)}\right)^{\gamma} \left(\int_{t_1}^{\tau(t)} \frac{\mathrm{d}s}{b(s)}\right)^{\gamma} \mathrm{d}t = \infty, \tag{2.20}$$

then (E) *is oscillatory.*

Proof. Let $\beta = \gamma$. Applications of Theorem 1 and results of [14, 20] complete the proof. ■

Next, we establish an oscillation criterion for (E) under the case where (1.2) holds.

Theorem 3. *Let all conditions of Theorem 1 hold with* (1.1) *replaced by* (1.2)*. If*

$$\int_{t_0}^{\infty} \frac{1}{b(v)} \int_{t_0}^{v} \frac{1}{a(u)} \int_{t_0}^{u} q(s)B^{\gamma}(\tau(s))\mathrm{d}s\mathrm{d}u\mathrm{d}v = \infty, \tag{2.21}$$

then (E) *is oscillatory.*

Proof. Let y be a non-oscillatory solution of (E). Without loss of generality, we may suppose that y is positive. Then there exist four possible cases (1), (2), (3) (as those of Theorem 1), and

$$\text{case (4)}.\ y(t) > 0,\ y'(t) < 0,\ (b(t)y'(t))' < 0,\ (a(t)(b(t)y'(t))')' < 0$$

for $t \geq t_1$, where $t_1 \geq t_0$ is large enough. From the proof of Theorem 1, we can eliminate cases (1), (2), and (3). Consider now the case (4). Since $(b(t)y'(t))' < 0$, we get

$$y'(s) \leq \frac{b(t)y'(t)}{b(s)} \text{ for } s \geq t.$$

Integrating this inequality from t to l and letting $l \to \infty$ implies that

$$y(t) \geq -B(t)b(t)y'(t) \geq LB(t) \tag{2.22}$$

for some constant $L > 0$. From (E), we have

$$(a(t)(b(t)y'(t))')' + L^{\gamma} q(t) B^{\gamma}(\tau(t)) \leq 0.$$

Integrating the above inequality from t_1 to t, we get

$$a(t)(b(t)y'(t))' + L^{\gamma} \int_{t_1}^{t} q(s) B^{\gamma}(\tau(s)) \mathrm{d}s \leq 0.$$

Integrating again, we have

$$b(t)y'(t) + L^{\gamma} \int_{t_1}^{t} \frac{1}{a(u)} \int_{t_1}^{u} q(s) B^{\gamma}(\tau(s)) \mathrm{d}s \mathrm{d}u \leq 0.$$

Integrating again, we obtain

$$y(t_1) \geq L^{\gamma} \int_{t_1}^{t} \frac{1}{b(v)} \int_{t_1}^{v} \frac{1}{a(u)} \int_{t_1}^{u} q(s) B^{\gamma}(\tau(s)) \mathrm{d}s \mathrm{d}u \mathrm{d}v + y(t),$$

which contradicts (2.21). This completes the proof. ■

Remark 1. Based on Theorem 3, similar as Corollary 1, Corollary 2, and Corollary 3, one can obtain some oscillation criteria for (E). The details are left to the reader.

3. Conclusions

To achieve oscillation criteria for the case where (1.1) holds, we consider three possible cases. In order to eliminate case (3), the papers [5, 6, 10, 12] gave condition (1.4) which differs from assumptions provided in this paper. Note that in Example 1 and Example 2, condition (1.4) cannot hold. Therefore, our results are new.

Results given in this paper supplement and improve those reported in [5, 6, 8, 12, 16, 19, 21, 22] since them can be applied to the case when $\int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} < \infty$, $\int_{t_0}^{\infty} \frac{\mathrm{d}t}{b(t)} < \infty$, $\gamma \neq 1$, and we present sufficient conditions which guarantee that all solutions of (E) are oscillatory.

We stress that the study of oscillatory properties of (E) brings additional difficulties. In particular, in order to establish conditions which ensure that (E) oscillates, we are forced to require that, as in [6, Theorem 3], τ and ξ are nondecreasing, $\xi(t) > t$, and $\tau(\xi(\xi(t))) < t$.

Therefore, an interesting problem for future research can be formulated as follows.

(P): is it possible to establish oscillation criteria for (E) without requiring conditions that τ and ξ are nondecreasing, $\xi(t) > t$, and $\tau(\xi(\xi(t))) < t$?

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