TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 3, pp. 833-855, June 2013
DOI: 10.11650/tjm.17.2013.1937
This paper is available online at http://journal.taiwanmathsoc.org.tw

ROUGHLY GEODESIC B-INVEX AND OPTIMIZATION PROBLEM ON HADAMARD MANIFOLDS

Li-wen Zhou and Nan-jing Huang*

Abstract. In this paper, a new class of roughly geodesic B-invex sets, quasi roughly geodesic B-invex functions and pseudo roughly geodesic B-invex functions are introduced and studied on Hadamard manifolds by relaxing the definitions of geodesic convex sets and functions. Some properties of quasi roughly geodesic B-invex functions and pseudo roughly geodesic B-invex functions are proved on Hadamard manifolds. As applications, some sufficient and necessary conditions for optimal solution of the nonlinear programming problems involving the quasi roughly geodesic B-invex functions and the pseudo roughly geodesic B-invex functions are given on Hadamard manifolds. The Mond-weir type dual problems for the nonlinear programming problems are also considered on Hadamard manifolds.

1. INTRODUCTION

The concept of convexity for sets and functions plays a central role in nonlinear programming with continuous variables, and has various applications in the areas of mathematical economics, engineering, operations research, Riemannian manifolds, etc. [6]. Therefore, it is important to consider a wider class of generalized convex functions and also to seek practical criteria for convexity or generalized convexity. In 1981, Hanson [15] introduced the concept of invexity by generalizing the difference $(x - y)$ in the definition of convex function to any function $\eta(x, y)$. Hanson's initial results inspired a great deal of subsequent work, which has greatly expanded the role and application of invexity in nonlinear optimization and other branches of pure and applied sciences. Later, Kaul and Kaur [17] defined η-pseudoconvex and η-quasiconvex

Received April 19, 2012, accepted November 9, 2012.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: 49J40, 54H25.
Key words and phrases: Hadamard manifold, Roughly geodesic B-invex set, Roughly geodesic B-invex function, Nonlinear optimization problem, Mond-weir type dual.
This work was supported by the National Natural Science Foundation of China (11171237) and the Key Program of NSFC (Grant No. 70831005).
*Corresponding author.

functions. They studied their relations with convex, pseudoconvex, and quasiconvex functions and the interrelations between themselves. Sufficient optimality criteria for a nonlinear programming problems were also established involving these generalized functions.

Recently, another generalization of convex functions, called B-vex functions, was introduced by Bector et al. [7, 8]. Later, Suneja et al. [31] introduced a class of functions called B-preinvex functions which are generalizations of preinvex and B-vex functions. In 1997, Phu [27] introduced roughly convex functions of several kinds as ν-convex, midpoint ν-convex, lightly ν-convex functions. In Phu's opinion, many non convex functions are able to get convex by choosing a suitable condition. Mathematically, for allowing small nonconvex blips, the demand "for all $x, y \in D$" is weakened by the requirement "for all $x, y \in D$ with $\|x-y\| \geq r$," where $r > 0$ denotes the roughness degree. On the base of the researches above, Morsy [25] introduced a new class of B-vex functions, this class called roughly B-vex functions. Emam [11] introduced the concepts of roughly B-invex functions and generalized roughly B-invex functions. Some properties for roughly B-invex functions and generalized roughly B-invex functions, and the sufficient optimality criteria for nonlinear programming problems involving these functions were given.

On the other hand, in the last few years, several important concepts of nonlinear analysis and optimization problems have been extended from Euclidean space to a Riemannian manifold setting in order to go further in the study of the convex theory, the fixed point theory, the variational inequality and related topics. In general, a manifold is not a linear space, but the extension of concepts and techniques from linear spaces to Riemannian manifold are natural. Rapcsak [29] and Udriste [33] considered a generalization of convexity, called geodesic convexity and extended many result of convex analysis and optimization theory to Riemannian manifolds. For more details, readers may see [18, 29, 33] and the reference cited therein. The notion of invex functions on Riemannian manifold was introduced by Pini [28], and Mititelu [24] investigated its generalizations. Barani and Pouryayevali [4] introduced the geodesic invex set, geodesic η-invex function and geodesic η-preinvex function on Riemannian manifold and studied the relations between them.

Inspired by the concept of convexity on a linear vector space the notion of geodesic convexity on some nonlinear metric spaces has become a successful tool in optimization. In fact, some constrained optimization problems can be seen as unconstrained ones from the Riemannian geometry point of view. In addition, another advantage is that optimization problems with nonconvex objective functions can be written as convex optimization problems by endowing the space with an appropriate Riemannian metric. For instance, Rapcsak ([29] p.169) gave an example of solving the nonconvex constrained problem in Euclidean space with the Euclidean metric is equivalent to solving the unconstrained convex minimization problem on the Hadamard manifold with the affine metric.

These ideas have opened a new way to solve other related problems in optimization. Actually, in the last decades concepts and techniques which fit in Euclidean spaces have extended to the nonlinear framework of Riemannian manifolds. Nemeth [26] introduced and researched the variational inequalities on Hadamard manifolds. Ferreira et al. [12, 13, 14] studied the monotone vector fields and some proximal point algorithm on Riemannian manifolds. Weak sharp minima for constrained optimization problems and some other algorithm on Riemannian manifolds have been proposed by Li et al. [21, 22, 23, 35]. Besides, it is worth to mentioning that the extension of concepts and techniques of nonlinear analysis from Euclidean space to Riemannian manifold have been studied in a couple of papers including [3, 5, 9, 16, 19, 20, 24, 28, 32, 34, 36] and the bibliographies therein for more examples and discussions.

Motivated and inspired by the works mentioned above, in this paper, we shall introduce a new notion of roughly geodesic B-invexity on Hadamard manifolds. We generalize roughly geodesic B-invex functions, to the so called quasi roughly geodesic B-invex functions and pseudo roughly geodesic B-invex functions, and discuss their properties. The relations between roughly geodesic B-invex functions, quasi roughly geodesic B-invex functions and pseudo roughly geodesic B-invex functions are established. As applications, we study the sufficient and necessary conditions for optimal solution of programming problems which involve roughly geodesic B-invex functions on Hadamard manifolds.

2. Preliminaries

In this section we recall some notations, definitions and basic properties used throughout the paper. It can be found in many introductory books on Riemannian geometry, topology and equilibrium problems (see, for example, [10, 18, 19, 29, 33]).

Definition 2.1. A Hadamard manifold M is a simply-connected complete Riemannian manifold of non-positive sectional curvature.

Definition 2.2. The exponential mapping $exp_p : T_pM \to M$ is defined by $exp_p\nu = \gamma_\nu(1)$, where γ_ν is the geodesic defined by its position p and velocity ν at p.

Lemma 2.1. (*Cartan-Hadamard theorem*). *Let X be a connected complete metric space and suppose that X is locally convex. Then the universal cover of X is a convex geodesic space with respect to the induced length metric d. In particular, any two points of the universal cover are joined by a unique geodesic.*

Remark 2.1. The exponential mapping and its inverse are continuous on Hadamard manifolds.

From now on, let a Hadamard manifold M be endowed by a Riemannian metric $\langle \cdot, \cdot \rangle$ with corresponding norm denoted by $\| \cdot \|$, $K \subset M$ is a subset on M. A mapping $\eta : K \times K \to TM$ is a function such that for every $x, y \in K$, $\eta(x, y) \in T_yM$.

Definition 2.3. The geodesic distance $d(x, y)$ is the length of minimal geodesic segment between any two points x, y on a manifold.

Definition 2.4. A mapping $f : K \to R$ is said to be an $\eta(x, y)$-differentiable mapping at $y \in M$, if the limit

$$\lim_{\lambda \downarrow 0} \frac{f(exp_y \lambda \eta(x, y)) - f(y)}{\lambda \|\eta(x, y)\|}$$

exists. We shall denote by

$$d_{\eta(x,y)} f(y) = \lim_{\lambda \downarrow 0} \frac{f(exp_y \lambda \eta(x, y)) - f(y)}{\lambda \|\eta(x, y)\|}$$

the $\eta(x, y)$-differential of f at y.

Remark 2.2. We would like to point out that the definition of $\eta(x, y)$-differentiable mapping is similar to the definition of directional derivative in the Euclidean space. In fact, the vector $\eta(x, y) \in T_y M$ can be considered as the direction from y to x.

Definition 2.5. [4]. Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a function such that for every $x, y \in M, \eta(x, y) \in T_y M$. A nonempty subset K of M is said to be geodesic η-invex if for every $x, y \in K$ there exists exactly one geodesic $\alpha_{x,y} : [0, 1] \to M$ such that

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \eta(x, y), \quad \alpha_{x,y}(t) \in K$$

for all $t \in [0, 1]$.

3. Geodesic B-Invex Sets on Hadamard Manifolds

Definition 3.1. Let $y \in K$. The set K is said to be a geodesic B-invex set on Hadamard manifolds with respect to η at $y \in K$, if there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that $exp_y \lambda b \eta(x, y) \in K$ for all $x \in K$ and $0 \le \lambda \le 1$.

K is said to be a geodesic B-invex set with respect to η on Hadamard manifolds, if K is B-invex at each $y \in K$ on a Hadamard manifold with respect to the same η.

Remark 3.1.

(1) Every geodesic B-invex set with respect to η is a geodesic η-invex set on Hadamard manifolds when $b = 1$;

(2) In particular, every geodesic B-invex set with respect to η is a geodesic convex set on Hadamard manifolds when $b = 1$ and $\eta(x, y) = exp_y^{-1} x$;

(3) Every geodesic η-invex set does not necessarily a geodesic B-invex set with respect to same η;

(4) If the intersection of finite (or infinite) family of geodesic B-invex sets is a geodesic B-invex with same η is nonempty, then it set but the union is not necessarily a geodesic B-invex set;
(5) The sum of geodesic B-invex sets and multiplying a geodesic B-invex set by a real number is also a geodesic B-invex set.

4. Roughly Geodesic B-Invex Functions on Hadamard Manifolds

Definition 4.1. A mapping $f : K \to TM$, defined on a geodesic B-invex set K is said to be a roughly geodesic B-invex function with respect to η with roughness degree r at $y \in K$, if there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R$ such that

$$f(exp_y \lambda b\eta(x, y)) \leq \lambda b f(x) + (1 - \lambda b)f(y)$$

for any $x \in K$ and $0 \leq \lambda \leq 1$ with $d(x, y) \geq r$. f is said to be a roughly geodesic B-invex function on K with respect to η, if it is a roughly geodesic B-invex function at any $y \in K$ with respect to the same η on K.

Remark 4.1. Every η-invex function defined in [4] is a roughly geodesic B-invex function with respect to same η, where $b(x, y, \lambda) = 1$. However, the converse does not hold in general, see Example 4.1.

Example 4.1. Let $M = \{e^{i\theta} | 0 < \theta < 1\}$ and $f : M \to R$ be defined as $f(e^{i\theta}) = \theta + \sin\theta$ with $x, y \in M, x = e^{i\alpha}$ and $y = e^{i\beta}$. If $exp_{e^{i\beta}} \lambda b\eta = e^{i((1-\lambda)\beta+\lambda\alpha)}$ with

$$\|\eta(e^{i\alpha}, e^{i\beta})\| = \frac{2(\sin\alpha - \cos\beta)}{\cos\beta}$$

and $b(e^{i\alpha}, e^{i\beta}, \lambda) = 2$, then it is easy to see that f is a roughly B-invex function with respect to η. However, f is not an η-invex function as

$$\|\eta(e^{i\alpha}, e^{i\beta})\| d_{\eta(e^{i\alpha}, e^{i\beta})} f(e^{i\beta}) > f(e^{i\alpha}) - f(e^{i\beta})$$

at $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{6}$.

Remark 4.2. In particular, every roughly geodesic B-invex function with respect to η is a geodesic convex function when $b(x, y, \lambda) = 1$ and $\eta(x, y) = exp_y^{-1} x$ for each $x, y \in K$ and $0 \leq \lambda \leq 1$.

Proposition 4.1. *If functions $f_i : K \to R, i = 1, 2, \cdots, k$ are roughly geodesic B-invex functions with respect to same $\eta : K \times K \to TM$ at $y \in K$ with roughness degree r, then for $a_i \geq 0, i = 1, 2, \cdots, k$, the function defined by*

$$h(x) = \sum_{i=1}^{k} a_i f_i(x)$$

is a roughly geodesic B-invex function at $y \in K$.

Proof. Since $f_i : K \to R, i = 1, 2, \cdots, k$ are roughly geodesic B-invex functions with respect to same $\eta : K \times K \to TM$ at $y \in K$, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that, for some i,

$$f_i(exp_y \lambda b\eta(x, y)) \le \lambda b f_i(x) + (1 - \lambda b) f_i(y)$$

for any $x \in K$ and $0 \le \lambda \le 1$ with $d(x, y) \ge r$. Thus, we get

$$\begin{aligned} h(exp_y \lambda b\eta(x, y)) &= \sum_{i=1}^{k} a_i f_i(exp_y \lambda b\eta(x, y)) \\ &\le \lambda b \sum_{i=1}^{k} a_i f_i(x) + (1 - \lambda b) \sum_{i=1}^{k} a_i f_i(y) \\ &= \lambda b h(x) + (1 - \lambda b) h(y). \end{aligned}$$

Thus, $h(x)$ is a roughly geodesic B-invex function with respect to η at $y \in K$. This completes the proof. ∎

Proposition 4.2. *If $f : K \to R$ is a roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ at $y \in K$ with roughness degree r on a geodesic B-invex set K, then for any real number $\gamma \in R$, the level set $K_\gamma = \{x | x \in K, f(x) \le \gamma\}$ is a geodesic B-invex set.*

Proof. For any $x, y \in K_\gamma$ and $0 \le \lambda \le 1$, we have

$$f(x) \le \gamma, \quad f(y) \le \gamma.$$

Since f is a roughly geodesic B-invex function with respect to $\lambda : K \times K \to TM$ at $y \in K$ with roughness degree r on K, there exists $b(x, y, \lambda) : M \times M \times [0, 1] \to R_+$ such that

$$f(exp_y \lambda b\eta(x, y)) \le \lambda b f(x) + (1 - \lambda b) f(y),$$

where $exp_y \lambda b\eta(x, y) \in K$ and $d(x, y) \ge r$. Thus, we have

$$f(exp_y \lambda b\eta(x, y)) \le \lambda b\gamma + (1 - \lambda b)\gamma = \gamma,$$

which implies that K_γ is a geodesic B-invex set. This completes the proof. ∎

Theorem 4.1. *Let K be a geodesic B-invex set. Then a function $f : K \to R$ is roughly geodesic B-invex with respect to $\eta : K \times K \to TM$ with roughness degree r on K if and only if $epi(f)$ is geodesic B-invex on $K \times R$.*

Proof. Let f be a roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ with roughness degree r on K. Then there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_y \lambda b\eta(x, y)) \le \lambda b f(x) + (1 - \lambda b) f(y),$$

where $exp_y\lambda b\eta(x,y)\in K$ and $d(x,y)\geq r$. Assume that $(x,\gamma),(y,\beta)\in epi(f)$. Then it is easy to see that

$$f(x)\leq\gamma,\quad f(y)\leq\beta.$$

From above inequalities, we have

$$\begin{aligned} f(exp_y\lambda b\eta(x,y)) &\leq \lambda\, b\gamma+(1-\lambda b)\beta \\ &= \beta\, +(\gamma-\beta)\lambda b \\ &= \beta\, +\eta_0(\gamma,\beta)\lambda b, \end{aligned}$$

where $\eta_0(\gamma,\beta)=\gamma-\beta$. Thus,

$$(exp_y\lambda\, b\,\eta\,(x,y),\beta+\lambda\, b\,\eta_0\,(\gamma,\beta))\in epi(f),$$

which implies that $epi(f)$ is a geodesic B-invex set on $K\times R$.

Conversely, suppose that $epi(f)$ is geodesic B-invex on $K\times R$, where $\eta: K\times K\to TM$ and $\eta_0(p,q)=p-q$ for all $p,q\in R$. Let $x,y\in K$ and $0\leq\lambda\leq 1$. Then

$$(x,f(x))\in epi(f) \text{ and } (y,f(y))\in epi(f).$$

Since $epi(f)$ is geodesic B-invex on $K\times R$ with respect to $\eta\times\eta_0$, there exists $b(x,y,\lambda): K\times K\times[0,1]\to R_+$ such that

$$\big(exp_y\lambda b\eta(x,y),f(y)+\lambda b\eta_0\big(f(x),f(y)\big)\big)\in epi(f),$$

which implies that

$$f\big(exp_y\lambda b\eta(x,y)\big)\leq f(y)+\lambda b\eta_0\big(f(x),f(y)\big)=\lambda bf(x)+(1-\lambda b)f(y).$$

It follows that f is roughly geodesic B-invex with respect to η on K. This completes the proof. ■

Theorem 4.2. *If $(f_i)_{i\in I}$ is a family of roughly geodesic B-invex functions with respect to the same $\eta: K\times K\to TM$ with roughness degree r and bounded from above on a geodesic B-invex set K. Then the function defined by $f(x)=sup_{i\in I}f_i(x)$ is a roughly geodesic B-invex function with respect to η on K.*

Proof. Since each f_i is a roughly geodesic B-invex function on K,

$$epi(f_i)=\{(x,\gamma)|x\in K,\gamma\in R,f_i(x)\leq\gamma\}$$

is a geodesic B-invex set on $K\times R$. Thus,

$$\begin{aligned} \bigcap_{i\in I} epi(f_i) &= \bigcap_{i\in I}\{(x,\gamma)|x\in K,\gamma\in R,f_i(x)\leq\gamma\} \\ &= \{(x,\gamma)|x\in K,\gamma\in R,sup_{i\in I}f_i(x)\leq\gamma\} \\ &= \{(x,\gamma)|x\in K,\gamma\in R,f(x)\leq\gamma\} \\ &= epi(f). \end{aligned}$$

Since $epif = \cap_{i\in I} epi(f_i)$ is geodesic B-invex on $K \times R$, by Theorem 4.1, we know that f is a roughly geodesic B-invex function on K. This completes the proof. ■

Remark 4.3. Theorems 4.1 and 4.2 generalize Theorems 3.2 and 3.4 of [1] from geodesic η-invexity to roughly geodesic B-invexity, respectively.

Proposition 4.3. *Let $f : K \to R$ be a roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ at $y \in K$ with roughness degree r on K and let $\varphi : R \to R$ be a positively homogenous nondecreasing function. Then the composite function $\varphi \circ f$ is a roughly geodesic B-invex function with respect to η at y with roughness degree r.*

Proof. Since f is a roughly geodesic B-invex function, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_y \lambda b\eta(x, y)) \leq \lambda b f(x) + (1 - \lambda b)f(y)$$

for each $x \in K$ and $0 \leq \lambda \leq 1$ with $d(x, y) \geq r$. Since φ is positively homogenous nondecreasing, we have

$$\varphi \circ f(exp_y \lambda b\eta(x, y)) \leq \varphi \circ \big(f(y) + \lambda b(f(x) - f(y))\big)$$

and so

$$(\varphi \circ f)(exp_y \lambda b\eta(x, y)) \leq (\varphi \circ f)(y) + \lambda b\big((\varphi \circ f)(x) - (\varphi \circ f)(y)\big).$$

It follows that $\varphi \circ f$ is a roughly geodesic B-invex function at $y \in K$. This completes the proof. ■

Theorem 4.3. *If $g_i : K \to R$ are roughly geodesic B-invex functions with respect to the same $\eta : M \times M \to TM$ with roughness degree r on K for $i = 1, 2, \ldots, m$, then the set defined by*

$$M = \{x \in K | g_i(x) \leq 0, i = 1, 2, \ldots, m\}$$

is a geodesic B-invex set with respect to η.

Proof. Since $g_i(x), i = 1, 2, \ldots, m$, are roughly geodesic B-invex functions, there exists $b(x, y, \lambda) : M \times M \times [0, 1] \to R_+$ such that

$$g_i(exp_y \lambda b\eta(x, y)) \leq \lambda b g_i(x) + (1 - \lambda b)g_i(y)$$

for each $x, y \in M$ and $0 \leq \lambda \leq 1$ with $d(x, y) \geq r$. It follows that

$$g_i(exp_y \lambda b\eta(x, y)) \leq 0, i = 1, 2, \ldots, m$$

and so $exp_y \lambda b\eta(x, y) \in M$. Thus, M is a geodesic B-invex set. This completes the proof. ■

Theorem 4.4. *Let K be a geodesic B-invex set. If $f : K \to R$ is an η-differentiable roughly geodesic B-invex function with respect to $\eta : K \times K \to R$ with roughness degree r at $y \in K$, then there exists a function $\bar{b}(x, y) : K \times K \to R_+$ such that*

$$\|\eta(x,y)\| d_{\eta(x,y)} f(y) \leq \bar{b}(x,y)\big(f(x) - f(y)\big)$$

for each $x \in K$ with $d(x, y) \geq r$ and $\bar{b}(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda)$.

Proof. Since f is a roughly geodesic B-invex function at y, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_y \lambda b \eta(x,y)) \leq \lambda b f(x) + (1 - \lambda b) f(y)$$

for each $x \in K$ and $0 \leq \lambda \leq 1$ with $d(x, y) \geq r$. Since f is η-differentiable at y, we have

$$d_{\eta(x,y)} f(y) = \lim_{\lambda \downarrow 0} \frac{f(exp_y \lambda b \eta(x,y)) - f(y)}{\lambda \|\eta(x,y)\|}$$

and so

$$\begin{aligned} f(y) + d_{\eta(x,y)} f(y) \lambda \|\eta(x,y)\| + o^2(\lambda b) &= f(exp_y \lambda b \eta(x,y)) \\ &\leq \lambda b f(x) + (1 - \lambda b) f(y) \\ &= f(y) + \lambda b (f(x) - f(y)). \end{aligned}$$

Dividing the above inequality by $\lambda \geq 0$ and taking $\lambda \to 0$, we get

$$\|\eta(x,y)\| d_{\eta(x,y)} f(y) \leq \bar{b}(x,y)(f(x) - f(y))$$

for each $x \in K$ with $\bar{b}(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda)$. This completes the proof. ■

5. Generalized Roughly Geodesic B-Invex Functions on Hadamard Manifolds

In this section, we generalize roughly geodesic B-invex functions, to what is called quasi roughly geodesic B-invex and pseudo roughly geodesic B-invex functions on Hadamard manifolds.

Definition 5.1. A function f, defined on a geodesic B-invex set K, is said to be a quasi roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ at $y \in K$ with roughness degree r if there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_y \lambda b \eta(x,y)) \leq \max\{f(x), f(y)\}$$

for each $x \in K$ and $0 \leq \lambda \leq 1$ with $d(x, y) \geq r$. f is said to be a quasi roughly geodesic B-invex function on K with respect to $\eta : K \times K \to TM$ if it is quasi roughly geodesic B-invex at each $y \in K$ with respect to the same η.

Proposition 5.1. *Let K be a geodesic B-invex set. Then $f : K \to R$ is a quasi roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ with roughness degree r on K if and only if the level set $K_\gamma = \{x \in K | f(x) \leq \gamma\}$ is a geodesic B-invex set with respect to the same η.*

Proof. For any $x, y \in K_\gamma$ and $0 \leq \lambda \leq 1$, we have

$$f(x) \leq \gamma, \quad f(y) \leq \gamma.$$

Since f is a quasi roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ with roughness degree r on K, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_y \lambda b\eta(x, y)) \leq \max\{f(x), f(y)\} \leq \gamma,$$

where $exp_y \lambda b\eta(x, y) \in K$ and $d(x, y) \geq r$. This implies that K_γ is a geodesic B-invex set.

Conversely, assume that K and K_γ are geodesic B-invex sets for each $\gamma \in R$. Then for any $x, y \in K_\gamma$ and $0 \leq \lambda \leq 1$, we have $exp_y \lambda b\eta(x, y) \in K_\gamma$ for each $\gamma \in R$. By setting $\gamma = \max\{f(x), f(y)\}$, we get

$$f(exp_y \lambda b\eta(x, y)) \leq \gamma = \max\{f(x), f(y)\},$$

which shows that f is a quasi roughly geodesic B-invex function. This completes the proof. ■

Proposition 5.2. *Let $f : K \to R$ be a quasi roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ with roughness degree r at $y \in K$. Let $\varphi : R \to R$ be a positively homogenous nondecreasing function. Then the composite function $\varphi \circ f$ is a quasi roughly geodesic B-invex function with respect to η at y with roughness degree r.*

Proof. Since f is a quasi roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ with roughness degree r at y, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_y \lambda b\eta(x, y)) \leq \max\{f(x), f(y)\}.$$

Since φ is a positively homogenous nondecreasing function, we have

$$\varphi \circ f(exp_y \lambda b\eta(x, y)) \leq \varphi \circ (\max\{f(x), f(y)\}) \leq \max\{\varphi \circ f(x), \varphi \circ f(y)\}.$$

It follows that $\varphi \circ f$ is a quasi roughly geodesic B-invex function at y. This completes the proof. ■

Theorem 5.1. *Let $g_i : K \to R$ be quasi roughly geodesic B-invex functions with respect to the same $\eta : K \times K \to TM$ with roughness degree r on K for $i = 1, 2, \dots, m$. Then the set defined by $S = \{x \in K | g_i(x) \leq 0, i = 1, 2, \dots, m\}$ is a geodesic B-invex set.*

Proof. Since $g_i : K \to R$ are quasi roughly geodesic B-invex functions, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$g_i(exp_y \lambda b\eta(x, y)) \leq \max\{g_i(x), g_i(y)\}$$

for each $x, y \in S \subset K$ and $0 \leq \lambda \leq 1$ with $d(x, y) \geq r$. Since $x, y \in S$, we have $\max\{g_i(x), g_i(y)\} \leq 0$ and so $exp_y \lambda b\eta(x, y) \in S$. Thus, S is a geodesic B-invex set. This completes the proof. ■

Theorem 5.2. *Let K be a geodesic B-invex set and $f : K \to R$ be an η-differentiable quasi roughly geodesic B-invex function with respect to $\eta : K \times K \to R$ with roughness degree r at $y \in K$, then for each $x \in K$ with $f(x) \leq f(y)$ and $d(x, y) \geq r$, there exists a function $\bar{b}(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda) : K \times K \to R_+$ such that*

$$\bar{b}(x, y)\|\eta(x, y)\|d_{\eta(x,y)}f(y) \leq 0.$$

Proof. Since f is an η-differentiable quasi roughly geodesic B-invex function at y, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_y \lambda b\eta(x, y)) \leq \max\{f(x), f(y)\} = f(y)$$

for each $x \in K$ and $0 \leq \lambda \leq 1$ such that $d(x, y) \geq r$.

On the other hand, we have

$$f(y) + d_{\eta(x,y)}f(y)\lambda b\|\eta(x, y)\| + o^2(\lambda b) = f(exp_y \lambda b\eta(x, y)),$$

which implies that

$$f(y) + d_{\eta(x,y)}f(y)\lambda b\|\eta(x, y)\| + o^2(\lambda b) \leq f(y).$$

Dividing the above inequality by $\lambda > 0$ and taking $\lambda \to 0$, we get

$$\bar{b}(x, y)\|\eta(x, y)\|d_{\eta(x,y)}f(y) \leq 0$$

for each $x \in K$ with $d(x, y) \geq r$ and $\bar{b}(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda)$. This completes the proof. ■

Definition 5.2. A function f, defined on a geodesic B-invex set K, is said to be pseudo roughly geodesic B-invex with respect to $\eta : K \times K \to TM$ at $y \in K$ with roughness r if there exist $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ and a strictly positive function $a : K \times K \to R$ such that

$$bf(x) < bf(y) \Rightarrow f(exp_y \lambda b\eta(x, y)) \leq f(y) + \lambda(\lambda - 1)a((x, y)$$

for each $x \in K$ with $d(x, y) \geq r$.

Proposition 5.3. *$f : K \to R$ is a roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ with roughness r on a geodesic B-invex set K. If $b(x, y, \lambda) > 0$, f is a pseudo roughly geodesic B-invex function with respect to the same η on K.*

Proof. For all $x, y \in K$ with $bf(x) < bf(y)$ and $d(x, y) \geq r$ and $\lambda \in [0, 1]$. Since f is a roughly geodesic B-invex function and $b(x, y, \lambda) > 0$

$$f(exp_y \lambda b\eta(x, y)) \leq f(y) + \lambda b(f(x) - f(y)).$$

The above inequality can be rewritten as

$$\begin{aligned} f(exp_y \lambda b\eta(x, y)) &\leq f(y) + \lambda(1 - \lambda)b(f(x) - f(y)) \\ &= f(y) + \lambda(\lambda - 1)b(f(y) - f(x)) \\ &= f(y) + \lambda(\lambda - 1)a(x, y), \end{aligned}$$

where $a(x, y) = b(f(y) - f(x)) > 0$. Hence f is a pseudo roughly geodesic B-invex function on K. This completes the proof. ∎

Proposition 5.4. *Let $f : K \to R$ be a pseudo roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ with roughness r on a geodesic B-invex set K. If $b(x, y, \lambda) > 0$, then f is a quasi roughly geodesic B-invex function with respect to the same η on K.*

Proof. Let $f(x) \leq f(y)$. Since f is a pseudo roughly B-invex function, there exist $b(x, y, \lambda) > 0$ and $a(x, y) > 0$ such that

$$f(exp_y \lambda b\eta(x, y)) \leq f(y) + \lambda(\lambda - 1)a(x, y)$$

for all $x, y \in K$, $0 \leq \lambda \leq 1$ and $d(x, y) \geq r$. Since $a(x, y) > 0$ and $0 \leq \lambda \leq 1$,

$$f(y) + \lambda(\lambda - 1)a(x, y) \leq f(y).$$

From above inequality, we have

$$f(exp_y \lambda b\eta(x, y)) \leq f(y) = \max\{f(x), f(y)\},$$

which implies that f is a quasi roughly geodesic B-invex function on K. This completes the proof. ∎

Remark 5.1. It is easy to see that the following relationship hold:

roughly geodesic B-invex function $\Rightarrow$ pseudo roughly geodesic B-invex function
$\Rightarrow$ quasi roughly geodesic B-invex function.

Theorem 5.3. *Let K be a geodesic B-invex set. If $f : K \to R$ is an η-differentiable pseudo roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ with roughness r at $y \in K$, then there exists a function $\bar{b}(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda)$ such that*

$$\bar{b}(x, y)\|\eta(x, y)\|d_{\eta(x,y)}f(y) < 0$$

for each $x \in K$ with $d(x, y) \geq r$.

Proof. Since f is a pseudo roughly geodesic B-invex function at y, there exist $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ and strictly positive function $a : K \times K \to R$ such that

$$bf(x) \leq bf(y) \Rightarrow f(exp_y \lambda b\eta(x)) \leq f(y) + \lambda(\lambda - 1)a(x, y)$$

for each $x \in K$ and $0 \leq \lambda \leq 1$ with $d(x, y) \geq r$. Since f is η-differentiable, we have

$$f(y)+d_{\eta(x,y)}f(y)\lambda b\|\eta(x, y)\|+o^2(\lambda b) = f(exp_y \lambda b\eta(x, y)) \leq f(y)+\lambda(\lambda-1)a(x, y).$$

Dividing the above inequality by $\lambda > 0$ and taking $\lambda \to 0$, we get

$$\bar{b}(x, y)\|\eta(x, y)\|d_{\eta(x,y)}f(y) \leq -a(x, y) < 0$$

for each $x \in K$ and $d(x, y) \geq r$ with $\bar{b}(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda)$. This completes the proof. ■

Corollary 5.1. *Let $K \subset M$ be a geodesic B-invex set and $f : K \to R$ be a η-differentiable pseudo roughly geodesic B-invex function with respect to $\eta : K \times K \to TM$ with roughness degree r at $y \in K$. Then there exists a function $\bar{b}(x, y) = \lim_{\lambda \downarrow 0} b(x, y, \lambda)$ such that*

$$\bar{b}(x, y)\|\eta(x, y)\|d_{\eta(x,y)}f(y) \geq 0 \Rightarrow \bar{b}f(x) \geq \bar{b}f(y)$$

for each $x \in K$ with $d(x, y) \geq r$.

6. Optimality Criteria

Let $\eta : K \times K \to TM$ be a mapping, $f : K \to R$ and $g_i : K \to R (i = 1, 2, \ldots, m)$ be η-differentiable roughly geodesic B-invex functions with respect to same $\eta : K \times K \to TM$ with roughness degree r on K. A roughly geodesic B-invex programming problem is formulated as follows

$$(P) \qquad \begin{cases} \min f(x) \\ x \in K = \{x \in M | g_i(x) \leq 0, i = 1, 2, \ldots, m\}. \end{cases}$$

Theorem 6.1. *Let $\eta : K \times K \to TM$ be a mapping and $f : K \to R$ be a quasi roughly geodesic B-invex function with respect to η with roughness degree r on a geodesic B-invex set K. If $x^0 \in K$ is a local solution of problem (P), then it is a global of problem (P).*

Proof. Let $x^0 \in K$ be a nonglobal minimum of the problem (P) on K. Then there exists at least one element $y^0 \in K$ such that $f(y^0) < f(x^0)$. Since f is quasi roughly geodesic B-invex, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_{x^0}\lambda b\eta(x^0, y^0)) \leq \max\{f(x^0), f(y^0)\} = f(x^0)$$

for each $0 \leq \lambda \leq 1$ with $d(x^0, y^0) \geq r$. Therefore, for any small $\varepsilon \in (0, 1)$,

$$\lambda = \frac{\varepsilon}{b\|\eta(x^0, y^0)\|}$$

and so

$$f(exp_{x^0}\frac{\eta(x^0, y^0)}{\|\eta(x^0, y^0)\|}\varepsilon) \leq f(x^0),$$

which contradicts the local optimality of x^0 for problem (P). Hence, x^0 is a global minimum of problem (P) on K. This completes the proof. ■

Remark 6.1. Theorem 6.1 generalizes Theorem 5.1 of [4] from Euclidean spaces to Hadamard manifolds.

Corollary 6.1. *Let $\eta : M \times M \to TM$ be a mapping and $f : K \to R$ be a roughly geodesic B-invex function with respect to η with roughness degree r on a geodesic B-invex set $K \subset M$. By Remark 5.1, if $x^0 \in K$ is a local solution of problem (P), then it is a global of problem (P).*

Theorem 6.2. *Let $\eta : M \times M \to TM$ be a mapping and $f : K \to R$ be a strictly quasi roughly geodesic B-invex function with respect to η with roughness degree r on a geodesic B-invex set $K \subset M$. Then the global optimal solution of problem (P) is unique.*

Proof. Let x_1, x_2 be two different global optimal solutions of problem (P). Then $f(x_1) = f(x_2)$. Since f is a strictly quasi roughly geodesic B-invex function with respect to η with roughness degree r, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_{x_2}\lambda b\eta(x_1, x_2)) < \max\{f(x_1), f(x_2)\} = f(x_1)$$

for $0 < \lambda < 1$ with $d(x_1, x_2) \geq r$. By the geodesic B-invexity of K, we have

$$exp_{x_2}\lambda b\eta(x_1, x_2) \in K,$$

which contradicts the optimality of x_1 for problem (P). Hence the global optimal solution of problem (P) is unique. This completes the proof. ■

Theorem 6.3. *Let $\eta : M \times M \to TM$ be a mapping, $f : K \to R$ be a quasi roughly geodesic B-invex function with respect to η with roughness degree r on a geodesic B-invex set $K \subset M$. Then the set of optimal solution of problem (P) is geodesic B-invex with the same η.*

Proof. Let $x^* \in K$ be an optimal solution of problem (P) and let $\beta = f(x^*)$. Let X be the set of optimal solutions for problem (P) which is written

$$X = \{x \in K | f(x) \leq \beta\}.$$

Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then $f(x_1) \leq \beta$ and $f(x_2) \leq \beta$. Since f is a quasi roughly geodesic B-invex function, there exists $b(x, y, \lambda) : K \times K \times [0, 1] \to R_+$ such that

$$f(exp_{x_2} \lambda b \eta(x_1, x_2)) \leq \max\{f(x_1), f(x_2)\} \leq \beta$$

for $0 \leq \lambda \leq 1$ with $d(x_1, x_2) \geq r$. From above inequality, we know that $exp_{x_2} \lambda b \eta(x_1, x_2) \in X$ and so X is a geodesic B-invex set. This completes the proof. ■

Remark 6.2. Theorem 6.3 generalizes Proposition 4.6 of [2] from geodesic η-invexity to roughly geodesic B-invexity.

Corollary 6.2. *Let $\eta : M \times M \to TM$ be a mapping and $f : K \to R$ be a roughly geodesic B-invex function with respect to η with roughness degree r on a geodesic B-invex set $K \subset M$. Then the set of optimal solution of problem (P) is geodesic B-invex with the same η.*

Theorem 6.4. *Let f be a roughly geodesic B_0-invex function with respect to η with roughness degree r_0 at $y \in K$ and g_i be roughly geodesic B-invex functions with respect to the same η with roughness degree r at $y \in K$ for $i = 1, 2, \ldots, m$. Assume that there exists $u = \{u_1, u_2, \ldots, u_m\} \in R^m$ with $u_i \geq 0$ such that (y, u) satisfies the following conditions*

$$\left.\begin{aligned} d_{\eta(x,y)} f(y) + d_{\eta(x,y)}\Big(\sum_{i=1}^{m} u_i g_i(y)\Big) &= 0, \\ \sum_{i=1}^{m} u_i g_i(y) &= 0, \\ g_i(y) &\leq 0 \end{aligned}\right\} \tag{6.1}$$

for all $i = 1, 2, \ldots, m$ and $x \in K$. If $b_0(x, y) > 0$ and $b(x, y) > 0$ for any $x \in K$, then y is an optimal solution for problem (P).

Proof. Since f is a roughly geodesic B_0-invex function, by Theorem 4.4,

$$\|\eta(x, y)\| d_{\eta(x,y)} f(y) \leq b_0(x, y)(f(x) - f(y)) \tag{6.2}$$

for each $x \in K$ with $d(x, y) \geq r_0$. Similarly, we can get

$$\|\eta(x,y)\| d_{\eta(x,y)} g_i(y) \leq b(x,y)(g_i(x) - g_i(y)) \tag{6.3}$$

+for each $x \in K$ with $i = 1, 2, \ldots, m$ and $d(x, y) \geq r$. Adding (6.2) and (6.3) by using (6.1), we have

$$\begin{aligned} b_0(x,y)(f(x) - f(y)) &\geq \|\eta(x,y)\| d_{\eta(x,y)} f(y) \\ &= -\|\eta(x,y)\| d_{\eta(x,y)} \big(\sum_{i=1}^{m} u_i g_i(y) \big) \\ &= -\sum_{i=1}^{m} u_i \|\eta(x,y)\| d_{\eta(x,y)} g_i(y) \\ &\geq b(x,y) \sum_{i=1}^{m} \big(u_i g_i(y) - u_i g_i(x) \big) \\ &= -\sum_{i=1}^{m} u_i b(x,y) g_i(x) \geq 0 \end{aligned}$$

for any $x \in K$ with $d(x, y) \geq \max\{r, r_0\}$. It follows that $f(y) \leq f(x)$. This completes the proof. ■

Remark 6.3. Theorem 6.4 generalizes Theorem 3.1 of [8] from Euclidean spaces to Hadamard manifolds.

Remark 6.4. In Theorem 6.4, since $u_i \geq 0$, $g_i(y) \leq 0$ and $\sum_{i=1}^{m} u_i g_i(y) = 0$, we know that, for each $i = 1, 2, \ldots, m$,

$$u_i g_i(y) = 0. \tag{6.4}$$

Let $I = \{i | g_i(y) = 0\}$ and $J = \{i | g_i(y) < 0\}$. Then $I \cup J = \{1, 2, \ldots, m\}$ and (6.4) gives that $u_i = 0$ for $i \in J$. From the proof of Theorem 6.4, it is easy to see that the roughly geodesic B-invexity of g_i $(i = 1, 2, \ldots, m)$ at y can be replaced by the roughly geodesic B-invexity of $g_I = g_i$ $(i \in I)$ at y.

Theorem 6.5. *Suppose that there exists a feasible point y for problem (P) and let I be as defined in Remark 6.4. Let f be a pseudo roughly geodesic B_0-invex function with respect to η with roughness degree r_0 at $y \in K$ and g_I be quasi roughly geodesic B-invex functions with respect to same η with roughness degree r at $y \in K$. Assume that there exists $u = \{u_i\} \in R^m$ such that (y, u) satisfies the condition (6.1) of Theorem 6.4. If $b(x, y) > 0$ and $b_0(x, y) > 0$ for any $x \in K$, then y is an optimal solution for problem (P).*

Proof. It can be proved as in Remark 6.4 that

$$\begin{cases} u_i = 0, \ i \in J, \\ u_i > 0, \ i \in I. \end{cases} \tag{6.5}$$

The functions g_I are quasi roughly geodesic B-invex functions with respect to same η with roughness degree r at $y \in K$ and

$$g_I(x) \leq 0 = g_I(y).$$

By Theorem 5.2 and $b(x, y) > 0$, we have

$$\|\eta(x, y)\| d_{\eta(x,y)} g_i(y) \leq 0$$

for each $x \in K$ and $i = 1, 2, \ldots, m$ with $d(x, y) \geq r$. Since $b_0(x, y) > 0$, it follows from Remark 6.4, (6.5) and the above inequality that

$$\begin{aligned}
0 &\geq b_0(x, y)\|\eta(x, y)\| \sum_{i \in I} u_i d_{\eta(x,y)} g_i(y) \\
&= b_0(x, y)\|\eta(x, y)\| \big(\sum_{i \in I} u_i d_{\eta(x,y)} g_i(y) + \sum_{i \in J} u_i d_{\eta(x,y)} g_i(y) \big) \\
&= b_0(x, y)\|\eta(x, y)\| \sum_{i=1}^{m} u_i d_{\eta(x,y)} g_i(y) \\
&= b_0(x, y)\|\eta(x, y)\| d_{\eta(x,y)} \big(\sum_{i=1}^{m} u_i g_i(y) \big) \\
&= -b_0(x, y)\|\eta(x, y)\| d_{\eta(x,y)} f(y).
\end{aligned}$$

By the pseudo roughly geodesic B-invexity of f at y and Corollary 5.1, we have

$$b_0(x, y) f(x) \geq b_0(x, y) f(y) \Rightarrow f(x) \geq f(y),$$

which implies that y is an optimal solution for problem (P). This completes the proof. ■

Corollary 6.3. *Suppose that there exists a feasible point y for problem (P) and let I be defined as in Remark 6.4. Let* f *be a pseudo roughly geodesic* B_0*-invex function with respect to* η *with roughness degree* r_0 *at* $y \in K$ *and* $\sum_{i \in I} u_i g_i$ *be a quasi roughly geodesic B-invex functions with respect to same* η *with roughness degree* r *at* $y \in K$*. Assume that there exists* $u = \{u_i\} \in R^m$ *such that* (y, u) *satisfies the condition (6.1) of Theorem 6.4. If* $b(x, y) > 0$ *and* $b_0(x, y) > 0$ *for any* $x \in K$*, then* y *is an optimal solution for problem (P).*

Proof. By (6.1), (6.5) and the condition of problem (P), we have

$$\sum_{i \in I} u_i g_i(x) \leq 0 = \sum_{i=1}^{m} u_i g_i(y) = \sum_{i \in I} u_i g_i(y).$$

Now the quasi roughly geodesic B-invexity of $\sum_{i\in I} u_i g_i$ at $y \in K$ implies that

$$b(x,y)\|\eta(x,y)\|d_{\eta(x,y)}\big(\sum_{i\in I} u_i g_i(y)\big) \le 0$$

for any $x \in K$ with $d(x,y) \ge r$. Thus, similar to the proof of Theorem 6.5, we can show that y is an optimal solution for problem (P). This completes the proof. ■

Corollary 6.4. *Suppose that there exists a feasible point y for problem (P) and let I be defined as in Remark 6.4. Let $f + \sum_{i\in I} u_i g_i$ be a pseudo roughly geodesic B-invex function with respect to η with roughness degree r at $y \in K$. Assume that there exists $u = \{u_i\} \in R^m$ such that (y,u) satisfies the condition (6.1) of Theorem 6.4. If $b(x,y) > 0$ for any $x \in K$, then y is an optimal solution for problem (P).*

Proof. By (6.1), (6.5) and the condition of problem (P), we have

$$\sum_{i\in I} u_i g_i(x) \le 0 = \sum_{i=1}^{m} u_i g_i(y) = \sum_{i\in I} u_i g_i(y).$$

Now the pseudo roughly geodesic B-invexity of $f + \sum_{i\in I} u_i g_i$ at $y \in K$ implies that

$$b(x,y)\|\eta(x,y)\|d_{\eta(x,y)}\big(f + \sum_{i\in I} u_i g_i\big)(y) \le 0$$

for any $x \in K$ with $d(x,y) \ge r$. Therefore, similar to the proof of Theorem 6.5, we can show that y is an optimal solution for problem (P). This completes the proof. ■

7. Duality Criteria

In this section, we consider the Mond-weir type dual and generalize its results under the geodesic roughly B-invextiy assumptions on a Hadamard manifold. Consider the following dual of problem (P) as follows

$$(D) \qquad \begin{cases} \max f(y) \\ d_{\eta(x,y)} f(y) + \displaystyle\sum_{i=1}^{m} u_i d_{\eta(x,y)} g_i(y) = 0, \\ \displaystyle\sum_{i=1}^{m} u_i g_i(y) \ge 0, \ u_i \ge 0, \end{cases}$$

for all $x \in K$, f and g_i $(i = 1, 2, \ldots, m)$ are η-differentiable functions defined on K.

Theorem 7.1. *Suppose that there exist a feasible solution (y,u) for the problem (D) and $X \subset K$ is feasible region for (P). Let f be a roughly geodesic B_0-invex function with respect to $\eta : M \times M \to TM$ with roughness degree r_0 at $y \in K$ and g_i be roughly geodesic B-invex functions with respect to the same η with roughness degree r at y. If $b(x,y) > 0$ and $b_0(x,y) > 0$ for all $x \in X$, then y is an optimal solution for problem (P).*

Proof. Since f is a roughly geodesic B_0-invex function and $b_0(x,y) > 0$, we have

$$\|\eta(x,y)\| d_{\eta(x,y)} f(y) \le b_0(x,y)(f(x) - f(y))$$

for any $x \in X$ with $d(x,y) \ge r_0$. Similarly, for all $i = 1, 2, \ldots, m$ and $x \in X$, we have

$$\|\eta(x,y)\| d_{\eta(x,y)} g_i(y) \le b(x,y)(g_i(x) - g_i(y))$$

Since $u_i \ge 0$,

$$\begin{aligned} &\|\eta(x,y)\| u_i d_{\eta(x,y)} g_i(y) \le u_i b(x,y)(g_i(x) - g_i(y)) \\ \Rightarrow &\|\eta(x,y)\| \sum_{i=1}^{m} u_i d_{\eta(x,y)} g_i(y) \le \sum_{i=1}^{m} u_i b(x,y) g_i(x) - \sum_{i=1}^{m} u_i b(x,y) g_i(y). \end{aligned}$$

By the conditions of (D) and the above inequality, we get

$$\begin{aligned} b_0(x,y)(f(x) - f(y)) &\ge \|\eta(x,y)\| d_{\eta(x,y)} f(y) \\ &= \|\eta(x,y)\| \left(-\sum_{i=1}^{m} u_i d_{\eta(x,y)} g_i(y) \right) \\ &\ge \sum_{i=1}^{m} u_i b(x,y) g_i(y) - \sum_{i=1}^{m} u_i b(x,y) g_i(x) \\ &= -b(x,y) \sum_{i=1}^{m} u_i g_i(x) \\ &\ge 0, \end{aligned}$$

which implies that y is an optimal solution for problem (P). This completes the proof. ∎

Theorem 7.2. *Let x^* be an optimal solution for (P) and g_i $(i = 1, 2, \ldots, m)$ satisfy the Kuhn-Tucker constraint qualification at x^*. Then there exists $u^* \in R^m$ such that (x^*, u^*) is a feasible solution for (D) and the (P)-objective at x^* is equal to the (D)-objective at (x^*, u^*). If f is a roughly geodesic B_0-invex with respect to $\eta : K \times K \to TM$ with roughness degree r_0 on K and $g_i : K \to R$ are roughly geodesic B-invex functions with respect to the same η with roughness degree r on K. If $b(x,y) > 0$ and $b_0(x,y) > 0$ for any $x, y \in K$, then (x^*, u^*) is an optimal solution for problem (D).*

Proof. Since g_i satisfies the Kuhn-Tucker constraint qualification at x^*, there exists $u^* \in R^m$ such that the following the Kuhn-Tucker conditions hold:

$$d_{\eta(x,x^*)}f(x^*) + \sum_{i=1}^{m} u_i^* d_{\eta(x,x^*)}g_i(x^*) = 0, \tag{7.1}$$

$$\sum_{i=1}^{m} u_i^* g_i(x^*) = 0, \tag{7.2}$$

$$g_i(x^*) \le 0, \tag{7.3}$$

$$u_i^* \ge 0 \tag{7.4}$$

for any $x \in K$ and all $i = 1, 2, \dots, m$. Now (7.1), (7.2) and (7.4) yield that (x^*, u^*) is a feasible solution for problem (D). Also (7.1)(7.2)(7.3) yields that the (P)-objective at x^* is equal to the (D)-objective at (x^*, u^*).

If (x^*, u^*) is not an optimal solution for problem (D), then there exists a feasible solution $(\bar{x}, \bar{u}) \neq (x^*, u^*)$ such that $f(\bar{x}) > f(x^*)$. Since f is a roughly geodesic B_0-invex function and $b_0(x, \bar{x}) > 0$,

$$\|\eta(x, \bar{x})\| d_{\eta(x,\bar{x})} f(\bar{x}) \le b_0(x, y)(f(x) - f(\bar{x}))$$

for any $x \in K$ with $d(x, \bar{x}) \ge r_0$. Similarly, for any $x \in K$ with $i = 1, 2, \dots, m$ and $d(x, \bar{x}) \ge r$, we have

$$\|\eta(x, \bar{x})\| d_{\eta(x,\bar{x})} g_i(\bar{x}) \le b(x, y)(g_i(x) - g_i(\bar{x})).$$

By (7.4) and the conditions of (D), we have

$$\begin{aligned} b(x,y)\left(\sum_{i=1}^{m} \bar{u}_i g_i(x) - \sum_{i=1}^{m} \bar{u}_i g_i(\bar{x})\right) &\ge \|\eta(x,\bar{x})\| \sum_{i=1}^{m} \bar{u}_i d_{\eta(x,\bar{x})} g_i(\bar{x}) \\ &= -\|\eta(x,\bar{x})\| d_{\eta(x,\bar{x})} f(\bar{x}) \\ &\ge b_0(x,y)(f(\bar{x}) - f(x)) \end{aligned}$$

for any $x \in K$ with $d(x, \bar{x}) \ge \max\{r, r_0\}$. Setting $x = x^*$, from above inequality and (7.3), we have

$$\sum_{i=1}^{m} \bar{u}_i g_i(x^*) - \sum_{i=1}^{m} \bar{u}_i g_i(\bar{x}) \ge \frac{b_0(x,y)}{b(x,y)} (f(\bar{x}) - f(x^*)) > 0$$

$$\Rightarrow \sum_{i=1}^{m} \bar{u}_i g_i(\bar{x}) < \sum_{i=1}^{m} \bar{u}_i g_i(x^*) \le 0,$$

which contradicts that $(\bar{x}, \bar{u}) \neq (x^*, u^*)$ is a feasible solution for problem (D). Therefore, (x^*, u^*) is an optimal solution for problem (D). This completes the proof. ∎

Remark 7.1. Theorem 7.2 generalizes Theorem 4.1 of [8] from Euclidean spaces to Hadamard manifolds.

Acknowledgments

The authors are grateful to the editor and the referees for their valuable comments and suggestions.

References

1. R. P. Agarwal and I. Ahmad, Akhlad Iqbal and Shahid Ali, Geodesic G-invex sets and semistrictly geodesic η-preinvex functions, *Optimization*, **61** (2012), 1169-1174.
2. I. Ahmad, Akhlad Iqbal and Shahid Ali, On properties of geodesic η-preinvex functions, *Adv. in Oper. Res.*, **2009** (2009), DOI:10.1155/2009/381831.
3. D. Azagra, J. Ferrera and F. Lopez-Mesas, Nonsmooth analysis and Hamilton-Jaccobi equations on Riemannian manifolds, *J. Func. Anal.*, **220** (2005), 304-361.
4. A. Barani and M. R. Pouryayevali, Invex sets and preinvex functions on Riemannian manifolds, *J. Math. Anal. Appl.*, **328** (2007), 767-779.
5. R. Bartolo, A. Germinario and M. Sanchez, Convextiy of domains of Riemannian manifolds, *Annals of Global Analysis and Geometry*, **21** (2002), 63-84.
6. M. S. Bazaraa and C. M. Shetty, *Nonlinear Programming-Theory and Algorithms*, Wiley, New York, 1979.
7. C. R. Bector and C. Singh, B-vex functions, *J. Optim. Theory Appl.*, **71** (1991), 237-253
8. C. R. Bector, S. K. Suneja and C. Lalitha, Generalized B-vex functions and generalized B-vex programming, *J. Optim. Theory Appl.*, **76** (1993), 561-576.
9. A. A. Borisenko, E. Gallego and A. Reventos, Relation between area and volume for λ-convex sets in Hadamard manifolds. *Diff. Geo. Appl.*, **14** (2001), 267-280.
10. I. Chavel, *Riemannian Geometry-A Modern Introduction*, Cambridge University Press, 1993.
11. T. Emam, Roughly B-invex programming problems, *Calcolo*, **48** (2011), 173-188.
12. O. P. Ferreira, Dini derivative and a characterization for Lipschitz and convex functions on Riemannian manifolds, *Nonlinear Anal.*, **68** (2008), 1517-1528.
13. O. P. Ferreira, L. R. Lucambio Perez and S. Z. Nemeth, Singularities of monotone vector fields and an extragradient-type algorithm, *J. Global Optim.*, **31** (2005), 133-151.
14. O. P. Ferreira and P. R. Oliveira, Proximal point algorithm on Riemannian manifolds, *Optimization*, **51** (2002), 257-270.
15. M. A. Hanson, On sufficiency of Kuhn-Tucker conditions, *J. Math. Anal. Appl.*, **80** (1981), 545-550

16. C. D. Horvath, Contractibility and generalized convexity, *J. Math. Anal. Appl.*, **156** (1991), 341-357.

17. R. N. Kaul and S. Kuar, Optimality criteria in nonlinear programming involving non convex functions, *J. Math. Anal. Appl.*, **105** (1985), 104-112.

18. W. Klingenberg, *A Course in Differential Geometry*, Springer-Verlag, 1978.

19. Y. S. Ledyaev and Q. J. Zhu, *Nonsmooth Analysis on Smooth Manifolds*, Transactions American Mathematical Society, 2007.

20. S. L. Li, C. Li, Y. C. Liou and J. C. Yao, Existence of solutions for variational inequalities on Riemannian manifolds, *Nonlinear Anal.*, **71** (2009), 5695-5706.

21. C. Li, B. S. Mordukhovich, J. H. Wang and J. C. Yao, Weak sharp minima on Riemannian manifolds, *SIAM J. Optim.*, **21** (2011), 1523-1560.

22. C. Li, G. Lopez and V. Martin-Marquez, Iterative algorithms for nonexpansive mappings on Hadamard manifolds, *Taiwanese J. Math.*, **14** (2010), 541-559.

23. C. Li, G. Lopez and V. Martin-Marquez, Monotone vector fields and the proximal point algorithm on Hadamard manifolds, *J. London Math. Soc.*, **79(2)** (2009), 663-683.

24. S. Mititelu, Generalized invexity and vector optimization on differentiable manifolds, *Differential Geometry-Dynamical Systems*, **3** (2001), 21-31.

25. T. Morsy, *A Study on Generalized Convex Mathematical Programming Problems*, Master Thesis, Faculty of Science, Suez Canal University, Egypt, 2003.

26. S. Z. Németh, Variational inequalities on Hadamard manifolds, *Nonlinear Anal.*, **52** (2003), 1491-1498.

27. H. X. Phu, Six kinds of roughly convex functions, *J. Optim. Theory Appl.*, **92** (1997), 357-375.

28. R. Pini, Convextiy along curves and invexity, *Optimization*, **29** (1994), 301-309.

29. T. Rapcsak, *Smooth Nonlinear Optimization in* R^n, Kluwer Academic Publishers, 1997.

30. T. Rapcsak, Geodesic convexity in nonlinear optimization, *J. Optim. Theory Appl.*, **69** (1991), 169-183.

31. S. K. Sunela, C. Singh and C. R. Bector, Generalization of preinvex and B-vex functions, *J. Optim. Theory Appl.*, **76(3)** (1993), 277-287

32. G. J. Tang, L. W. Zhou and N. J. Huang, The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds, *Optim. Lett.*, (2012), DOI: 10.1007/s11590-012-0459-7.

33. C. Udriste, Convex Functions and Optimization Methods on Riemannian Manifolds, in: *Mathematics and its Applications*, Vol. 297, Kluwer Academic Publishers, 1994.

34. C. Udriste, Kuhn-Tucker theorem on Riemannian manifolds, *Colloquia Math. Soc. Janos Bolyai, Topics in Diff. Geometry*, Debrecen, Hungary, 1984, pp. 1247-1259.

35. J. H. Wang, J. C. Yao and C. Li, Gauss-Newton methods for convex composite optimization on Riemannian manifolds, *J. Glob. Optim.*, **53** (2012), 5-28.

36. L. W. Zhou and N. J. Huang, Existence of solutions for vector optimizations on Hadamard manifolds, *J. Optim. Theory Appl.*, (2012), DOI: 10.1007/s10957-012-0186-3.

Li-wen Zhou and Nan-jing Huang
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
P. R. China
E-mail: nanjinghuang@hotmail.com