LOW-DIMENSIONAL COHOMOLOGY OF LIE SUPERALGEBRA $\mathbf{A}(1,0)$ WITH COEFFICIENTS IN WITT OR SPECIAL SUPERALGEBRAS

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Abstract. Over a field of characteristic $p > 2$, using a direct sum decomposition of submodules and the weight space decomposition for the Witt superalgebra viewed as $\mathbf{A}(1,0)$-module, we compute the low-dimensional cohomology groups of the special linear Lie superalgebra $\mathbf{A}(1,0)$ with coefficients in the Witt superalgebra. We also compute the low-dimensional cohomology groups of $\mathbf{A}(1,0)$ with coefficients in the Special superalgebra.

0. INTRODUCTION

In 1997, Zhang [17] constructed four series of modular graded Lie superalgebras of Cartan type, which are analogous to the finite-dimensional modular Lie algebras of Cartan type [10] or the four series of infinite-dimensional Lie superalgebras of Cartan type defined by even differential forms over a field of characteristic zero [4]. Later, the finite-dimensional modular Lie superalgebras of Cartan type defined by odd differential forms were also constructed and studied (see [3, 7, 8], for example). Now one can find many results on the structure, representation and classification of modular Lie superalgebras, for example, see [1, 6, 9, 12, 13, 14, 15, 16] and the references therein. We should mention that the complete classification problem is still open for finite-dimensional simple modular Lie superalgebras.

In the present article, over a field of prime characteristic, we mainly compute the low-dimensional cohomology groups of the special linear Lie superalgebra $\mathbf{A}(1,0)$ with coefficients in the restricted Witt superalgebra W or Special superalgebra S viewed as $\mathbf{A}(1,0)$-modules in the natural fashion. We also give an example to illustrate that

certain classical results for the Lie superalgebra $\mathbf{A}(1,0)$ in characteristic zero do not hold in characteristic p. Generally speaking, for a graded simple Lie (super)algebra, the classical Lie (super)algebra contained in the null plays an important role in characterizing the structure of the Lie (super)algebra under consideration. We expect that this work is useful for further study of the graded modular Lie superalgebras such as characterizing the maximal graded subalgebras of the simple graded modular Lie superalgebras of Cartan type, as in the Lie algebra case [5].

Let us formulate the outline of the present paper. For our purpose we describe firstly the structure of $\mathbf{A}(1,0)$-module W and compute the weight space decomposition of W relative to the standard CSA $\mathfrak{h}$ of $\mathbf{A}(1,0)$. Then the work under consideration is reduced to computing the cohomology groups with coefficients in certain submodules and computing the so-called weight derivations from $\mathfrak{g}$ to these submodules, that is, the derivations preserving the $\mathfrak{h}^*$-gradings. Finally, since the Special superalgebra S contains $\mathbf{A}(1,0)$ as subalgebra and W contains S as $\mathbf{A}(1,0)$-submodule, we use the results obtained for W to compute the low-dimensional cohomology groups of $\mathbf{A}(1,0)$ with coefficients in the Special superalgebra S.

Let us indicate certain differences for the cohomology of $A(1,0)$ in characteristics 0 and p. Over a field of characteristic zero, the first cohomology group of $A(1,0)$ with coefficients in a finite-dimensional simple module is trivial or of dimension 1 [11]. In the characteristic p case, however, this does not hold (see Remark 5.8).

Throughout we work over a field $\mathbb{F}$ of characteristic $p > 2$. All the vectors are assumed to be finite-dimensional. Write $\mathbb{Z}$ for the set of integers and $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ the two-element field. The symbol $|x|$ implies that x is a $\mathbb{Z}_2$-homogeneous element in a $\mathbb{Z}_2$-graded vector space and meanwhile it denotes the parity of x. Write $\langle v_1, \ldots, v_k \rangle$ for the subspace spanned by $v_1, \ldots, v_k$ in a vector space.

1. Preliminaries and Main Results

We recall certain basics relative to Witt superalgebras and Special superalgebras. Fix two positive integers $m, n > 1$. Let $\mathcal{O}(m)$ be the divided power algebra with a standard basis

$$\{x^{(\alpha)} := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m} \mid \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m, \ 0 \leq \alpha_i \leq p - 1\}$$

and multiplication $x^{(\alpha)} x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}$, where $\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^m \binom{\alpha_i+\beta_i}{\alpha_i}$. Let $\Lambda(n)$ be the exterior algebra of n variables $x_{m+1}, \ldots, x_{m+n}$, which has a standard basis

$$\{x^u := x_{i_1} x_{i_2} \cdots x_{i_k} \mid m + 1 \leq i_1 < \cdots < i_k \leq m + n\},$$

where $u = \langle i_1, i_2, \ldots, i_k \rangle$ is a strictly increasing sequence of k integers between $m+1$ and $m+n$. The tensor product $\mathcal{O}(m,n) := \mathcal{O}(m) \otimes \Lambda(n)$ is an associative superalgebra in the usual fashion. We abbreviate $g \otimes f$ to gf for $g \in \mathcal{O}(m)$ and $f \in \Lambda(n)$. Let

$\partial_1, \ldots, \partial_{m+n}$ be the special superderivations of the superalgebra $\mathcal{O}(m,n)$ such that $\partial_i(x_j) = \delta_{ij}$. We write $i \in u$ if $\partial_i(x^u) \neq 0$. The finite-dimensional Witt superalgebra, denoted by $W(m,n)$, is a Lie superalgebra spanned by all $f\partial_i$, where $f \in \mathcal{O}(m,n)$ and $i = 1, \ldots, m+n$. Let $\operatorname{div} : W(m,n) \longrightarrow \mathcal{O}(m,n)$ be the divergence such that

$$\operatorname{div}(f\partial_i) = (-1)^{|f||\partial_i|}\partial_i(f) \quad \text{for } f \in \mathcal{O}(m,n).$$

Put $\bar{S}(m,n) := \langle w \in W(m,n) \mid \operatorname{div}(w) = 0 \rangle$. Then $\bar{S}(m,n)$ is a subalgebra of $W(m,n)$. Its derived algebra $S(m,n) := [\bar{S}(m,n), \bar{S}(m,n)]$ is a simple Lie superalgebra, called the Special superalgebra. In the sequel, write W, S and $\mathcal{O}$ for $W(m,n)$, $S(m,n)$ and $\mathcal{O}(m,n)$, respectively.

By definition, a $\mathbb{Z}_2$-homogeneous linear mapping φ from a Lie superalgebra L to an L-module M is called a derivation provided that

$$\varphi([x,y]) = (-1)^{|\varphi||x|}x \cdot \varphi(y) - (-1)^{|y|(|\varphi|+|x|)}y \cdot \varphi(x) \quad \text{for all } x, y \in L.$$

A derivation φ from L to M is said to be inner if there exists a fixed $m \in M$ such that $\varphi(x) = (-1)^{|x||m|}x \cdot m$ for all $x \in L$. If such an element m does not exist, φ is called an outer derivation. Denote by $\operatorname{Der}(L,M)$ and $\operatorname{Ider}(L,M)$ the derivation space and the inner derivation space, respectively. In general, $\operatorname{Der}(L,M)$ and $\operatorname{Ider}(L,M)$ are L-submodules of $\operatorname{Hom}_{\mathbb{F}}(L,M)$. The zero-dimensional cohomology group of L with coefficients in M is the maximal trivial L-submodule of M:

$$H^0(L,M) := \{m \in M \mid L \cdot m = 0\}.$$

The first cohomology group (space) of L with coefficients in M is the quotient module:

$$H^1(L,M) := \operatorname{Der}(L,M)/\operatorname{Ider}(L,M).$$

For short we write $\mathfrak{g}$ for $\mathbf{A}(1,0)$. Fix a standard basis of $\mathfrak{g}$:

$$\mathfrak{B} := \{h_1 := e_{11} + e_{33}, \; h_2 := e_{22} + e_{33}, \; e_{12}, \; e_{21}, \; e_{13}, \; e_{31}, \; e_{23}, \; e_{32}\}.$$

We identify $\mathfrak{g}$ with the subalgebra of W with a fixed ordered basis

$$\{x_1\partial_1 + x_{m+1}\partial_{m+1}, x_2\partial_2 + x_{m+1}\partial_{m+1}, x_1\partial_2, x_2\partial_1, x_1\partial_{m+1}, x_{m+1}\partial_1, x_2\partial_{m+1}, x_{m+1}\partial_2\}$$

under the canonical isomorphism given by ($1 \leq i \neq j \leq 2$):

$$h_i \longmapsto x_i\partial_i + x_{m+1}\partial_{m+1}, \; e_{ij} \longmapsto x_i\partial_j, \; e_{i3} \longmapsto x_i\partial_{m+1}, \; e_{3i} \longmapsto x_{m+1}\partial_i.$$

View W and S as $\mathfrak{g}$-modules by means of the adjoint representation. This paper aims to compute $H^i(\mathfrak{g}, X)$ for $i = 0, 1$ and $X = W, S$. In particular, we obtain the following dimension formulas:

$$\dim H^i(\mathfrak{g}, W) = \begin{cases} (2m + 2n - 5)2^{n-1}p^{m-2} & \text{if } i = 0 \\ (3m + 3n - 8)2^n p^{m-2} & \text{if } i = 1; \end{cases}$$

$$\dim H^i(\mathfrak{g}, \; S) = \begin{cases} (2m + 2n - 7)2^{n-1}p^{m-2} - 2m + 5 & \text{if } i = 0 \\ 3(2m + 2n - 7)2^{n-1}p^{m-2} - 6m + 17 & \text{if } i = 1. \end{cases}$$

2. Reduction

We shall adopt two kinds of reductions for computing the low-dimensional cohomology groups. The first one is based on a basic fact: If L is a Lie superalgebra and M an L-module with a direct sum decomposition of L-submodules, $M = \oplus_{i=1}^{n} M_i$, then

$$(2.1) \qquad H^k(L, M) = \oplus_{i=1}^{n} H^k(L, M_i).$$

Thus, let us first decompose W into a direct sum of certain $\mathfrak{g}$-submodules. View $\mathcal{O}$ as $\mathfrak{g}$-module by the canonical embedding $\mathfrak{g} \hookrightarrow W$. Put $\mathbf{J} := \{1, 2, m+1\}$. Then $\mathcal{O}$ has a direct sum decomposition of submodules:

$$(2.2) \qquad \mathcal{O} = \widehat{\mathcal{O}} \oplus \widetilde{\mathcal{O}},$$

where

$$(2.3) \qquad \begin{aligned} \widehat{\mathcal{O}} &= \langle x^{(\alpha)} x^u \mid \partial_j(x^{(\alpha)} x^u) = 0 \quad \text{for all } j \in \mathbf{J} \rangle, \\ \widetilde{\mathcal{O}} &= \langle x^{(\alpha)} x^u \mid \partial_j(x^{(\alpha)} x^u) \neq 0 \quad \text{for some } j \in \mathbf{J} \rangle. \end{aligned}$$

It is evident that $\widehat{\mathcal{O}}$ is a trivial $\mathfrak{g}$-submodule of $\mathcal{O}$. Note that W is a free $\mathcal{O}$-module with basis $\{\partial_1, \ldots, \partial_{m+n}\}$. Then W has a direct sum decomposition of $\mathfrak{g}$-submodules:

$$W = \sum_{i \notin \mathbf{J}} \widehat{\mathcal{O}} \partial_i \oplus \sum_{i \notin \mathbf{J}} \widetilde{\mathcal{O}} \partial_i \oplus \sum_{j \in \mathbf{J}} \widetilde{\mathcal{O}} \partial_j \oplus \sum_{j \in \mathbf{J}} \widehat{\mathcal{O}} \partial_j.$$

Clearly,

$$(2.4) \quad \mathcal{T} := \sum_{i \notin \mathbf{J}} \widehat{\mathcal{O}} \partial_i = \langle x^{(\alpha)} x^u \partial_i \mid \partial_j(x^{(\alpha)} x^u) = 0 \quad \text{for all } j \in \mathbf{J}, \ i \notin \mathbf{J} \rangle$$

is a trivial $\mathfrak{g}$-submodule of W. As $\mathfrak{g}$-modules, one may easily check that

$$(2.5) \ W \simeq \mathcal{T} \oplus (\widetilde{\mathcal{O}} \otimes \langle \partial_i \mid i \notin \mathbf{J} \rangle) \oplus (\widetilde{\mathcal{O}} \otimes \langle \partial_1, \partial_2, \partial_{m+1} \rangle) \oplus (\widehat{\mathcal{O}} \otimes \langle \partial_1, \partial_2, \partial_{m+1} \rangle).$$

Obviously, as $\mathfrak{g}$-module, $\langle \partial_i \mid i \notin \mathbf{J} \rangle$ is trivial and of dimension

$$t := m + n - 3.$$

As $\mathfrak{g}$-submodule of W, $\langle x_1, x_2, x_{m+1} \rangle$ is isomorphic to the 3-dimensional standard $\mathbf{A}(1, 0)$-module $\mathfrak{sl}(2|1)$ with the standard basis

$$v_1 := (1, 0, 0)^{\mathrm{t}}, \ v_2 := (0, 1, 0)^{\mathrm{t}}, \ v_3 := (0, 0, 1)^{\mathrm{t}},$$

under the linear mapping given by

$$x_1 \longmapsto v_1, \ x_2 \longmapsto v_2, \ x_{m+1} \longmapsto v_3.$$

We have a standard $\mathfrak{g}$-module isomorphism:

$$\langle \partial_1, \partial_2, \partial_{m+1} \rangle \simeq \mathfrak{sl}(2|1)^*.$$

As usual, for an L-module M write $\mathrm{S}^k M$ for the kth super-symmetric power of M. Now consider $\mathfrak{g}$-module $\mathfrak{sl}(2|1)$. Let I be the ideal generated by v_1^p and v_2^p in the super-symmetry algebra $\mathrm{S}\mathfrak{sl}(2|1) = \oplus_{k \geq 0} \mathrm{S}^k \mathfrak{sl}(2|1)$. For $k \geq 1$, define the following $\mathfrak{g}$-modules in the usual way:

$$\mathcal{V}_k := \mathrm{S}^k \mathfrak{sl}(2|1) + I/I,$$
$$\mathcal{V} := \oplus_{k=1}^{2p-1} \mathcal{V}_k \overset{\mathrm{space}}{\simeq} \oplus_{k=1}^{2p-1} \mathrm{S}^k \mathfrak{sl}(2|1).$$

Then, as $\mathfrak{g}$-modules,

$$\mathcal{V} \overset{\sim}{\longrightarrow} \mathcal{O}^*(2,1) \hookrightarrow \mathcal{O}(m,n),$$

where $\mathcal{O}^*(2,1)$ is the unique maximal ideal of $\mathcal{O}(2,1)$ without 1. Thus, identifying $\mathcal{V}$ with $\oplus_{k=1}^{2p-1} \mathrm{S}^k \mathfrak{sl}(2|1)$ as vector spaces, we take the convention that, in $\mathcal{V}$,

$$(2.6) \qquad\qquad v_i^k = 0 \quad \text{whenever } k \geq p \quad \text{for } i = 1, 2.$$

Then one easily finds a canonical $\mathfrak{g}$-module isomorphism:

$$(2.7) \qquad\qquad \widetilde{\mathcal{O}} \simeq \widehat{\mathcal{O}} \otimes \mathcal{V}.$$

For an L-module M we also write

$$kM := M \oplus \cdots \oplus M \quad (k \text{ copies}).$$

If V and U are $\mathfrak{g}$-modules and V is trivial, then

$$(2.8) \qquad\qquad V \otimes U \simeq kU, \quad \text{where } k = \dim V.$$

From (2.7) and (2.8), one may easily find the following $\mathfrak{g}$-module isomorphisms:

$$\widetilde{\mathcal{O}} \simeq \widehat{\mathcal{O}} \otimes \mathcal{V} \simeq s\mathcal{V},$$

where

$$s := \dim \widehat{\mathcal{O}} = 2^{n-1} p^{m-2}.$$

Similarly, from (2.5) we have

$$(2.9) \qquad\qquad W \simeq \mathcal{T} \oplus s(t\mathcal{V} \oplus (\mathcal{V} \otimes \mathfrak{sl}(2|1)^*) \oplus \mathfrak{sl}(2|1)^*).$$

Note that the trivial $\mathfrak{g}$-module $\mathcal{T}$ is of dimension:

$$(2.10) \qquad \dim \mathcal{T} = ts = (m + n - 3)2^{n-1}p^{m-2}.$$

The module action of $\mathfrak{g}$ on $\mathcal{V}$ is given in the table below [Notice (2.6)]:

Table 2.1: Module action of $\mathbf{A}(1,0)$ on $\mathcal{V}$

	$v = v_1^{k_1} v_2^{k_2}$	$v = v_1^{k_1} v_2^{k_2} v_3$
h_1	$k_1 v$	$(k_1 + 1)v$
h_2	$k_2 v$	$(k_2 + 1)v$
e_{12}	$k_2 v_1^{k_1+1} v_2^{k_2-1}$	$k_2 v_1^{k_1+1} v_2^{k_2-1} v_3$
e_{21}	$k_1 v_1^{k_1-1} v_2^{k_2+1}$	$k_1 v_1^{k_1-1} v_2^{k_2+1} v_3$
e_{13}	0	$v_1^{k_1+1} v_2^{k_2}$
e_{31}	$k_1 v_1^{k_1-1} v_2^{k_2} v_3$	0
e_{23}	0	$v_1^{k_1} v_2^{k_2+1}$
e_{32}	$k_2 v_1^{k_1} v_2^{k_2-1} v_3$	0

Note that for a simple Lie superalgebra L and a trivial L-module M,

$$(2.11) \qquad \mathrm{Der}(L, M) = 0.$$

Remark 2.1. In view of (2.1), (2.9) and (2.11), It is enough to compute the low-dimensional cohomology groups of $\mathfrak{g}$ with coefficients in the submodules $\mathcal{V}$, $\mathcal{V} \otimes \mathfrak{sl}(2|1)^*$ and $\mathfrak{sl}(2|1)^*$ respectively, since $\mathcal{T}$ is trivial.

In order to explain the second kind of reductions for computing the low-dimensional cohomology groups, we introduce the following definition.

Definition 2.2. Let L be a Lie superalgebra and M an L-module. Relative to a CSA H of L, L and M have weight space decompositions $L = \oplus_{\alpha \in H^*} L_\alpha$ and $M = \oplus_{\alpha \in H^*} M_\alpha$, respectively. A derivation φ from L to M is called a weight derivation (or H^*-derivation) relative to H if $\varphi(L_\alpha) \subseteq M_\alpha$ for all $\alpha \in H^*$.

Let us state a standard fact on weight derivations, which is a super-version of the Lie algebra case [2, Theorem 1.1].

Lemma 2.3. *Each derivation is equal to a weight derivation modulo an inner derivation.*

In view of Lemma 2.3, to compute the 1-dimensional cohomology groups of $\mathfrak{g}$ to W, it is sufficient to compute the weight derivations relative to a CSA of $\mathfrak{g}$. Fix a standard CSA of $\mathfrak{g}$:

$$\mathfrak{h} = \langle h_i = e_{ii} + e_{33} \mid i = 1, 2. \rangle$$

Relative to $\mathfrak{h}$, the root space decomposition of $\mathfrak{g}$ is

(2.12) $$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_2} \oplus \mathfrak{g}_{\varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_1} \oplus \mathfrak{g}_{\varepsilon_1}.$$

All the root subspaces are 1-dimensional and the root-vectors are listed below:

Table 2.2: Roots and root-vectors for $\mathfrak{g}$

roots	$\varepsilon_1 - \varepsilon_2$	$-\varepsilon_1 + \varepsilon_2$	$-\varepsilon_2$	ε_2	$-\varepsilon_1$	ε_1
root-vectors	e_{12}	e_{21}	e_{13}	e_{31}	e_{23}	e_{32}

In view of the remarks below Lemma 2.3, for a weight derivation φ from $\mathfrak{g}$ to $\mathfrak{g}$-module M, we have

$$\varphi(\mathfrak{g}_\alpha) \subseteq M_\alpha \quad \text{for all } \alpha \in \mathfrak{h}^* = \langle \theta, \pm\varepsilon_1, \pm\varepsilon_2, \pm(\varepsilon_1 - \varepsilon_2) \rangle.$$

Thus, for a $\mathfrak{g}$-module M, we are only concerned with the weight subspaces of M corresponding to the weights of $\mathfrak{g}$. So, for later use, we list the weight-vectors corresponding to the weights in $\mathfrak{h}^*$ for $\mathfrak{g}$-module $\mathcal{V}$ and $\mathcal{V} \otimes \mathfrak{sl}(2|1)^*$, respectively. For convenience, we write $v \otimes v^*$ as vv^* for $v \in \mathcal{V}$ and $v^* \in \mathfrak{sl}(2|1)^*$. Here one should keep in mind that $p \equiv 0$ in the ground field $\mathbb{F}$.

Table 2.3: Weight-vectors and their $\mathbb{Z}$-degrees for $\mathfrak{g}$-module $\mathcal{V}$

weights	θ	$\varepsilon_1 - \varepsilon_2$	$-\varepsilon_1 + \varepsilon_2$	$-\varepsilon_2$	ε_2	$-\varepsilon_1$	ε_1
weight-vectors	$v_1^{p-1}v_2^{p-1}v_3$	$v_2^{p-2}v_3$ $v_1 v_2^{p-1}$	$v_1^{p-2}v_3$ $v_1^{p-1}v_2$	v_2^{p-1} $v_1^{p-1}v_2^{p-2}v_3$	v_2 $v_1^{p-1}v_3$	v_1^{p-1} $v_1^{p-2}v_2^{p-1}v_3$	v_1 $v_2^{p-1}v_3$
$\mathbb{Z}$-degrees	$2p-1$	$p-1$ p	$p-1$ p	$p-1$ $2p-2$	1 p	$p-1$ $2p-2$	1 p

Table 2.4: Weight-vectors and their $\mathbb{Z}$-degrees for $\mathfrak{g}$-module $\mathcal{V} \otimes \mathfrak{sl}(2|1)^*$

θ	$\varepsilon_1 - \varepsilon_2$	$-\varepsilon_1 + \varepsilon_2$	$-\varepsilon_2$	ε_2	$-\varepsilon_1$	ε_1
$v_1 v_1^*$ $v_2 v_2^*$ $v_3 v_3^*$ $v_1 v_2 v_3^*$ $v_1^{p-1}v_3 v_2^*$ $v_2^{p-1}v_3 v_1^*$	$v_1 v_2^*$ $v_1^2 v_3^*$ $v_1 v_2^{p-1}v_3 v_3^*$ $v_1^2 v_2^{p-1}v_1^*$ $v_1 v_2^{p-2}v_3 v_1^*$ $v_2^{p-1}v_3 v_2^*$	$v_2 v_1^*$ $v_2^2 v_3^*$ $v_1^{p-1}v_2^2 v_2^*$ $v_1^{p-1}v_2 v_3 v_3^*$ $v_1^{p-1}v_3 v_1^*$ $v_1^{p-2}v_2 v_3 v_2^*$	$v_1 v_3^*$ $v_2^{p-2}v_3 v_1^*$ $v_1 v_2^{p-1}v_1^*$ $v_2^{p-1}v_3 v_3^*$ $v_1^{p-1}v_2^{p-1}v_3 v_2^*$	$v_3 v_1^*$ $v_1 v_2 v_1^*$ $v_2^2 v_2^*$ $v_2 v_3 v_3^*$ $v_1 v_2^2 v_3^*$ $v_1^{p-1}v_2 v_3 v_2^*$	$v_2 v_3^*$ $v_1^{p-2}v_3 v_2^*$ $v_1^{p-1}v_2 v_2^*$ $v_1^{p-1}v_3 v_3^*$ $v_1^{p-1}v_2^{p-1}v_3 v_1^*$	$v_3 v_2^*$ $v_1^2 v_1^*$ $v_1 v_2 v_2^*$ $v_1 v_3 v_3^*$ $v_1^2 v_2 v_3^*$ $v_1 v_2^{p-1}v_3 v_1^*$
0 1 0 $p-1$ 0 $p-1$	0 p 1 $p-1$ p $p-1$	0 p 1 $p-1$ p $p-1$	0 $p-1$ $p-2$ $p-1$ $2p-2$	0 1 1 2 1 p	0 $p-1$ $p-2$ $p-1$ $2p-2$	0 1 1 2 1 p

3. Zero-dimensional Cohomology $H^0(\mathfrak{g}, W)$

Theorem 3.1. *The zero-dimensional cohomology group of $\mathfrak{g}$ with coefficients in W is as follows:*

$$H^0(\mathfrak{g}, W) = \mathcal{T} \oplus ts\langle v_1^{p-1}v_2^{p-1}v_3 \rangle \oplus s\langle v_1 v_1^* + v_2 v_2^* + v_3 v_3^* \rangle,$$

where

$$t = m + n - 3 \quad and \quad s = 2^{n-1}p^{m-2}.$$

In particular,

$$\dim H^0(\mathfrak{g}, W) = (2m + 2n - 5)2^{n-1}p^{m-2}.$$

    Proof.   In view of Remark 2.1, we compute the zero-dimensional cohomology groups of $\mathfrak{g}$ with coefficients in $\mathfrak{sl}(2|1)^*$, $\mathcal{V}$ and $\mathcal{V} \otimes \mathfrak{sl}(2|1)^*$, respectively. Firstly, let us compute $H^0(\mathfrak{g}, \mathfrak{sl}(2|1)^*)$. Relative to the standard CSA $\mathfrak{h}$, we have

(3.1)
$$\mathfrak{sl}(2|1)^* = \mathfrak{sl}(2|1)^*_{-\varepsilon_1} \oplus \mathfrak{sl}(2|1)^*_{-\varepsilon_2} \oplus \mathfrak{sl}(2|1)^*_{-\varepsilon_1-\varepsilon_2},$$
$$\mathfrak{sl}(2|1)^*_{-\varepsilon_1} = \langle v_1^* \rangle, \ \mathfrak{sl}(2|1)^*_{-\varepsilon_2} = \langle v_2^* \rangle, \ \mathfrak{sl}(2|1)^*_{-\varepsilon_1-\varepsilon_2} = \langle v_3^* \rangle.$$

In particular, $\mathfrak{sl}(2|1)^*_\theta = 0$. It follows that

(3.2)
$$H^0(\mathfrak{g}, \mathfrak{sl}(2|1)^*) \subseteq H^0(\mathfrak{h}, \mathfrak{sl}(2|1)^*) \subseteq \mathfrak{sl}(2|1)^*_\theta = 0.$$

Secondly, we compute $H^0(\mathfrak{g}, \mathcal{V})$. Table (2.3) shows that $\mathcal{V}_\theta = \langle v_1^{p-1}v_2^{p-1}v_3 \rangle$. Correspondingly,

(3.3)
$$H^0(\mathfrak{g}, \mathcal{V}) \subseteq H^0(\mathfrak{h}, \mathcal{V}) \subseteq \mathcal{V}_\theta = \langle v_1^{p-1}v_2^{p-1}v_3 \rangle.$$

On the other hand, keeping in mind the convention (2.6), one sees from Table 2.1 that $v_1^{p-1}v_2^{p-1}v_3 \in H^0(\mathfrak{g}, \mathcal{V})$ and therefore,

(3.4)
$$H^0(\mathfrak{g}, \mathcal{V}) = \langle v_1^{p-1}v_2^{p-1}v_3 \rangle.$$

Thirdly, let us show that

(3.5)
$$H^0(\mathfrak{g}, \mathcal{V} \otimes \mathfrak{sl}(2|1)^*) = \langle v_1 v_1^* + v_2 v_2^* + v_3 v_3^* \rangle.$$

Table 2.4 shows that

$$(\mathcal{V} \otimes \mathfrak{sl}(2|1)^*)_\theta = \langle v_1 v_1^*, \ v_2 v_2^*, \ v_3 v_3^*, \ v_1 v_2 v_3^*, \ v_1^{p-1}v_3 v_2^*, \ v_2^{p-1}v_3 v_1^* \rangle.$$

Then for any $v \in H^0(\mathfrak{g}, \mathcal{V} \otimes \mathfrak{sl}(2|1)^*) \subseteq (\mathcal{V} \otimes \mathfrak{sl}(2|1)^*)_\theta$, one may assume that

$$v = \sum_{i=1}^{3} a_i v_i v_i^* + a_4 v_1 v_2 v_3^* + a_5 v_1^{p-1}v_3 v_2^* + a_6 v_2^{p-1}v_3 v_1^*, \quad \text{where } a_i \in \mathbb{F}.$$

In the below, we use Table 2.1 without notice. Since $e_{12} \cdot v = 0$, $e_{21} \cdot v = 0$ and $e_{32} \cdot v = 0$, one gets

$$a_1 = a_2 = a_3, \quad a_4 = a_5 = a_6 = 0.$$

Then $v = a \sum_{i=1}^{3} v_i v_i^*$. On the other hand, one sees that $v \in H^0(\mathfrak{g}, \mathcal{V} \otimes \mathfrak{sl}(2|1)^*)$, proving (3.5). From (3.2)–(3.5), (2.1) and (2.9), we have immediately:

$$H^0(\mathfrak{g}, W) = \mathcal{T} \oplus ts\langle v_1^{p-1} v_2^{p-1} v_3\rangle \oplus s\langle v_1 v_1^* + v_2 v_2^* + v_3 v_3^*\rangle.$$

where $t = m + n - 3$, $s = \dim\widehat{\mathcal{O}} = 2^{n-1}p^{m-2}$. From (2.10), $\dim\mathcal{T} = ts$. Then

$$\dim H^0(\mathfrak{g}, W) = (2t+1)s = (2m + 2n - 5)2^{n-1}p^{m-2}.$$

The proof is complete.   ∎

4. First Cohomology $H^1(\mathfrak{g}, W)$

Before computing the first cohomology groups of $\mathfrak{g}$ with coefficients in W, we first introduce eight outer derivations. By (2.9), we identify W with $\mathcal{T} \oplus s(t\mathcal{V} \oplus (\mathcal{V} \otimes \mathfrak{sl}(2|1)^*) \oplus \mathfrak{sl}(2|1)^*)$. Consider the linear mappings from $\mathfrak{g}$ to W given by

$$
\begin{aligned}
&\varphi_1 : e_{13} \longmapsto v_2^*, \quad e_{23} \longmapsto -v_1^*; \\
&\varphi_2 : e_{31} \longmapsto v_2, \quad e_{32} \longmapsto -v_1; \\
&\varphi_3 : e_{12} \longmapsto v_2^{p-2}v_3, \quad e_{13} \longmapsto v_2^{p-1}; \\
&\varphi_4 : e_{21} \longmapsto v_1^{p-2}v_3, \quad e_{23} \longmapsto v_1^{p-1}; \\
(4.1) \quad &\varphi_5 : e_{12} \longmapsto v_1 v_2^{p-1}, \quad e_{32} \longmapsto v_2^{p-1}v_3; \\
&\varphi_6 : e_{21} \longmapsto v_1^{p-1}v_2, \quad e_{31} \longmapsto v_1^{p-1}v_3; \\
&\varphi_7 : e_{13} \longmapsto v_1^{p-1}v_2^{p-2}v_3, \quad e_{23} \longmapsto -v_1^{p-2}v_2^{p-1}v_3; \\
&\varphi_8 : e_{13} \longmapsto v_1^{p-1}v_2^{p-1}v_3 v_2^*, \quad e_{23} \longmapsto -v_1^{p-1}v_2^{p-1}v_3 v_1^*.
\end{aligned}
$$

Here we take the convention that, for each $k = 1, 2, \ldots, 8$, φ_k vanishes on the standard basis elements of $\mathfrak{g}$ which do not appear. For example, $\varphi_1(\mathfrak{B} \setminus \{e_{13}, e_{23}\}) = 0$.

Lemma 4.1. *Each φ_k is both an outer derivation and a weight derivation for $k = 1, 2, \ldots$.*

Proof. First, we check that φ_k is a derivation for $k = 1, 2, \ldots, 8$. Observe that each φ_k vanishes on six basis elements in $\mathfrak{B}$ and the roots corresponding to the remaining two basis elements are indecomposable. In view of this observation, one may simplify computations in checking that φ_k is a derivation. For example, let us check that φ_3 is an odd derivation. To do that, it is sufficient to check that

$$(4.2) \qquad \varphi_3([x,y]) = (-1)^{|x|} x \cdot \varphi_3(y) - (-1)^{|y|(|x|+\bar{1})} y \cdot \varphi_3(x),$$

for $x = e_{12}$ or $x = e_{13}$ and $y \in \mathfrak{B}$. Firstly, let $x = e_{12}$ and $y = h_i$, $i = 1, 2$. In this case, (4.2) holds, since the left hand side of (4.2) is

$$\varphi_3([e_{12}, h_i]) = (-1)^i \varphi_3(e_{12}) = (-1)^i v_2^{p-2} v_3$$

and the right hand side is

$$-h_i \cdot \varphi_3(e_{12}) = -h_i \cdot v_2^{p-2} v_3 = (-1)^i v_2^{p-2} v_3.$$

Secondly, letting $x = e_{12}$ and $y = e_{ij}$ with $i \neq j$, one computes the left hand side of (4.2):

$$\varphi_3([e_{12}, e_{ij}]) = \varphi_3(\delta_{i,2} e_{1j} - \delta_{j,1} e_{i2}) = \delta_{i,2} \delta_{j,3} v_2^{p-1}$$

and the right hand side:

$$\delta_{i,1} e_{12} \cdot \varphi_3(e_{1j}) - (-1)^{|e_{ij}|} e_{ij} \cdot v_2^{p-2} v_3 = \delta_{i,2} \delta_{j,3} v_2^{p-1}.$$

Analogously, one may check (4.2) in the remaining cases $x = e_{13}$ and $y \in \mathfrak{B}$. Thus φ_3 is a derivation. In the same manner, one may check that each φ_k is a derivation, $k = 1, \ldots, 8$.

Let us show that φ_k is outer. Suppose conversely φ_k is an inner derivation given by $w_k \in W$, for $k = 1, \ldots, 8$. By (4.1), $\mathfrak{h} \cdot w_k = \varphi_k(\mathfrak{h}) = 0$ and consequently,

(4.3) $w_k \in H^0(\mathfrak{h}, W)$ for $k = 1, \ldots, 8$.

For $i = 2, \ldots, 7$, the definition (4.1) implies that $\varphi_i(\mathfrak{g}) \subseteq \mathcal{V}$. Thus one may assume that $w_i \in \mathcal{V}$. Then by (4.3), we have

$$w_i \in H^0(\mathfrak{h}, \mathcal{V})   \text{ for } i = 2, \ldots, 7.$$

It follows from (3.3) and (3.4) that

$$w_i \in H^0(\mathfrak{h}, \mathcal{V}) = H^0(\mathfrak{g}, \mathcal{V})   \text{ for } i = 2, \ldots, 7.$$

This shows that $\varphi_i = 0$ for $i = 2, \ldots, 7$. Let us consider φ_1 and φ_8. By (4.1), one sees that $\varphi_1(\mathfrak{g}) \subseteq \mathfrak{sl}(2|1)^*$ and $\varphi_8(\mathfrak{g}) \subseteq \mathcal{V} \otimes \mathfrak{sl}(2|1)^*$. Then one may assume that $w_1 \in \mathfrak{sl}(2|1)^*$ and $w_8 \in \mathcal{V} \otimes \mathfrak{sl}(2|1)^*$. By (4.3), we have $w_1 \in H^0(\mathfrak{h}, \mathfrak{sl}(2|1)^*)$ and $w_8 \in H^0(\mathfrak{h}, \mathcal{V} \otimes \mathfrak{sl}(2|1)^*)$. Thanks to (3.2), we have $w_1 = 0$. From the definition of φ_8, one sees that

$$e_{12} \cdot w_8 = e_{21} \cdot w_8 = e_{32} \cdot w_8 = 0.$$

Then, as in the proof (3.5), one gets $w_8 = v_1 v_1^* + v_2 v_2^* + v_3 v_3^*$, and then $\varphi_8(x) = x \cdot w_8 = 0$ for any $x \in \mathfrak{g}$. Summarizing, we have shown that $\varphi_k = 0$ for $k = 1, \ldots, 8$. This contradicts (4.1), proving that all φ_k are outer.

The remaining conclusion follows from Tables 2.2–2.4.   ■

In view of Remark 2.1 and Lemma 2.3, computing $H^1(\mathfrak{g}, W)$ is reduced to computing the weight derivations from $\mathfrak{g}$ to $\mathfrak{sl}(2|1)^*$, $\mathcal{V}$ and $\mathcal{V} \otimes \mathfrak{sl}(2|1)^*$, respectively.

For simplicity, for an outer derivation we write the image in the first homology group still by the outer derivation itself.

Proposition 4.2. *We have*

$$H^1(\mathfrak{g}, \mathfrak{sl}(2|1)^*) = \langle \varphi_1 \rangle.$$

In particular,

$$\dim H^1(\mathfrak{g}, \mathfrak{sl}(2|1)^*) = 1.$$

Proof. In view of Lemma 2.3, one may suppose φ is a weight derivation from $\mathfrak{g}$ to $\mathfrak{g}$-module $\mathfrak{sl}(2|1)^*$. From (2.12), (3.1) and Table 2.2, it follows that

$$\varphi(\mathfrak{B} \setminus \{e_{13}, e_{23}\}) = 0,$$
$$\varphi(e_{13}) = av_2^*, \ \varphi(e_{23}) = bv_1^* \quad \text{for some } a, b \in \mathbb{F}.$$

Clearly, $|\varphi| = \overline{1}$ and

$$av_2^* = \varphi(e_{13}) = \varphi([e_{12}, , e_{23}]) = e_{12} \cdot (bv_1^*) = -bv_2^*.$$

This forces $b = -a$ and then $\varphi = a\varphi_1$. By Lemma 4.1, φ is an outer derivation if $a \neq 0$. Now our conclusions follow from Lemma 2.3. ∎

To determine $H^1(\mathfrak{g}, \mathcal{V})$, in view of (2.1), Lemma 2.3, Tables 2.2 and 2.3, we have to compute the weight derivations φ from $\mathfrak{g}$ to $\mathcal{V}_k$ for $k = 1, p-1, p, 2p-2$ and $2p-1$.

Lemma 4.3. (1) $H^1(\mathfrak{g}, \mathcal{V}_1) = \langle \varphi_2 \rangle$, (2) $H^1(\mathfrak{g}, \mathcal{V}_{2p-2}) = \langle \varphi_7 \rangle$.

Proof. We only prove (2) while (1) can be treated analogously. Let φ be a weight derivation from $\mathfrak{g}$ to $\mathcal{V}_{2p-2}$. By Tables (2.2) and (2.3), one may assume that

$$\varphi(\mathfrak{B} \setminus \{e_{13}, e_{23}\}) = 0,$$
$$\varphi(e_{13}) = av_1^{p-1}v_2^{p-2}v_3, \ \varphi(e_{23}) = bv_1^{p-2}v_2^{p-1}v_3 \quad \text{for some } a, b \in \mathbb{F}.$$

Clearly, $|\varphi| = \overline{0}$ and

$$av_1^{p-1}v_2^{p-2}v_3 = \varphi(e_{13}) = \varphi([e_{12}, e_{23}]) = e_{12} \cdot \varphi(e_{23}) = -bv_1^{p-1}v_2^{p-2}v_3.$$

It follows that $a = -b$ and hence $\varphi = a\varphi_7$. Then (2) holds according to Lemmas 4.1 and 2.3. ∎

Lemma 4.4. (1) $H^1(\mathfrak{g}, \mathcal{V}_{p-1}) = \langle \varphi_3, \varphi_4 \rangle$, (2) $H^1(\mathfrak{g}, \mathcal{V}_p) = \langle \varphi_5, \varphi_6 \rangle$.

Proof. We only prove (1) while (2) can be treated analogously. Let φ be a weight derivation from $\mathfrak{g}$ to $\mathcal{V}_{p-1}$. By Tables 2.2 and 2.3, one may assume that

$$\varphi(\mathfrak{B} \setminus \{e_{12}, e_{21}, e_{13}, e_{23}\}) = 0,$$
$$\varphi(e_{12}) = a_{12}v_2^{p-2}v_3, \ \varphi(e_{21}) = a_{21}v_1^{p-2}v_3,$$
$$\varphi(e_{13}) = a_{13}v_2^{p-1}, \quad \varphi(e_{23}) = a_{23}v_1^{p-1} \quad \text{for some } a_{ij} \in \mathbb{F}.$$

Clearly, $|\varphi| = \bar{1}$ and

$$a_{13}v_2^{p-1} = \varphi(e_{13}) = \varphi([e_{12}, e_{23}]) = e_{12} \cdot \varphi(e_{23}) + e_{23} \cdot \varphi(e_{12}) = a_{12}v_2^{p-1}.$$

It follows that $a_{12} = a_{13}$. Analogously,

$$a_{21}v_1^{p-2}v_3 = \varphi(e_{21}) = \varphi([e_{23}, e_{31}]) = -e_{23} \cdot \varphi(e_{31}) - e_{31} \cdot \varphi(e_{23}) = a_{23}v_1^{p-2}v_3.$$

This forces $a_{23} = a_{21}$. Let $a = a_{12} = a_{13}$ and $b = a_{23} = a_{21}$. By (4.1), $\varphi = a\varphi_3 + b\varphi_4$. ∎

Lemma 4.5. $H^1(\mathfrak{g}, \mathcal{V}_{2p-1}) = 0$.

Proof. Let φ be a weight derivation from $\mathfrak{g}$ to $\mathcal{V}_{2p-1}$. By Tables 2.2 and 2.3,

$$\varphi(\mathfrak{B} \setminus \{h_1, h_2\}) = 0$$

and then

$$\varphi(h_i) = \varphi([e_{i3}, e_{3i}]) = 0 \quad \text{for } i = 1, 2.$$

Thus $\varphi = 0$. By Lemma 2.3, the conclusion holds. ∎

Proposition 4.6. *We have*

$$H^1(\mathfrak{g}, \mathcal{V}) = \langle \varphi_2, \ldots, \varphi_7 \rangle.$$

In particular,

$$\dim H^1(\mathfrak{g}, \mathcal{V}) = 6.$$

Proof. By Lemmas 4.3–4.5, the first conclusion holds. For the dimension formula, it suffices to show that $\varphi_2, \ldots, \varphi_7 \in \mathrm{Der}(\mathfrak{g}, \mathcal{V})$ are linearly independent modulo the inner derivation space $\mathrm{Ider}(\mathfrak{g}, \mathcal{V})$. By the definitions (4.1) and Lemma 4.1, $\varphi_2, \ldots, \varphi_7$ are weight derivations and their $\mathbb{Z}$-degrees are

$$\deg(\varphi_2) = 1, \ \deg(\varphi_3) = \deg(\varphi_4) = p-1, \ \deg(\varphi_5) = \deg(\varphi_6) = p, \ \deg(\varphi_7) = 2p-2.$$

Thus it suffices to show that $\{\varphi_3, \varphi_4\}$ and $\{\varphi_5, \varphi_6\}$ are linearly independent modulo the inner derivation space $\mathrm{Ider}(\mathfrak{g}, \mathcal{V})$, respectively. Indeed, this follows from the general fact that $\mathrm{Der}(\mathfrak{g}, \mathcal{V})$ are $\mathbb{Z} \times \mathfrak{h}^*$-graded and an observation from Table 2.3:

$$\mathcal{V}_{p-1} \cap \mathcal{V}_\theta = 0 = \mathcal{V}_p \cap \mathcal{V}_\theta,$$

since $\varphi_3, \ldots, \varphi_6 \in \mathrm{Der}(\mathfrak{g}, \mathcal{V})$ are linearly independent. ∎

In the below we compute $H^1(\mathfrak{g}, \mathcal{V} \otimes \mathfrak{sl}(2|1)^*)$. As before, we only need to compute the weight derivations from $\mathfrak{g}$ to $\mathcal{V}_k \otimes \mathfrak{sl}(2|1)^*$ for $k = 1, 2, 3, p-1, \ p, p+1$ and $2p-1$.

Lemma 4.7. $H^1(\mathfrak{g}, \mathcal{V}_k \otimes \mathfrak{sl}(2|1)^*) = 0$ *for* $k = 1, 2, 3, p-1, p, p+1$.

Proof. (1) Let φ be a weight derivation from $\mathfrak{g}$ to $\mathcal{V}_1 \otimes \mathfrak{sl}(2|1)^*$. According to Tables 2.2 and 2.4, suppose

$$\varphi(h_i) = \sum_{j=1}^{3} a_{ij} v_j v_j^* \quad \text{for } i = 1, 2,$$

$$\varphi(e_{ij}) = b_{ij} v_i v_j^* \quad \text{for } 1 \leq i \neq j \leq 3.$$

Note that $|\varphi| = \bar{0}$. We have

$$(4.4) \quad \sum_{j=1}^{3} a_{ij} v_j v_j^* = \varphi(h_i) = \varphi([e_{i3}, e_{3i}]) = (b_{i3} + b_{3i}) v_i v_i^* + (b_{i3} - b_{3i}) v_3 v_3^*.$$

It follows that

$$(4.5) \qquad\qquad a_{12} = a_{21} = 0.$$

Thus

$$\varphi(h_1 - h_2) = a_{11} v_1 v_1^* - a_{22} v_2 v_2^* + (a_{13} - a_{23}) v_3 v_3^*.$$

On the other hand

$$(4.6) \quad \varphi(h_1 - h_2) = \varphi([e_{12}, e_{21}]) = (b_{12} + b_{21}) v_1 v_1^* - (b_{12} + b_{21}) v_2 v_2^*.$$

Then

$$(4.7) \qquad\qquad a_{11} = a_{22}, \quad a_{13} = a_{23}.$$

Noting that $b_{12} v_1 v_2^* = \varphi(e_{12}) = \varphi([h_1, e_{12}]) = (b_{12} + a_{11}) v_1 v_2^*$, one gets

$$(4.8) \qquad\qquad a_{11} = 0.$$

Similarly, since $0 = \varphi(0) = \varphi([h_1, e_{13}]) = -a_{13} v_1 v_3^*$, one gets

$$(4.9) \qquad\qquad a_{13} = 0.$$

From (4.5) and (4.7)–(4.9), we have

$$(4.10) \qquad\qquad a_{ij} = 0 \quad \text{for } i = 1, 2 \text{ and } j = 1, 2, 3.$$

Applying (4.10) to (4.4), one gets

$$(4.11) \qquad\qquad b_{13} = b_{31} = b_{23} = b_{32} = 0.$$

Since
$$b_{13}v_1v_3^* = \varphi(e_{13}) = \varphi([e_{12}, e_{23}]) = (b_{32} + b_{12})v_1v_3^*,$$

it follows from (4.6) that

(4.12) $$b_{12} = b_{21} = 0.$$

According to (4.10)–(4.12), we have $\varphi = 0$. By Lemma 2.3, $H^1(\mathfrak{g}, \mathcal{V}_1 \otimes \mathfrak{sl}(2|1)^*) = 0$.

(2) Let φ be a weight derivation from $\mathfrak{g}$ to $\mathcal{V}_2 \otimes \mathfrak{sl}(2|1)^*$. According to Tables (2.2) and (2.4), one may assume that

$$\varphi(e_{i3}) = 0, \quad \varphi(h_i) = a_i v_1 v_2 v_3^*, \quad \varphi(e_{ij}) = a_{ij} v_i^2 v_3^*,$$
$$\varphi(e_{3i}) = a_{3i} v_j^2 v_j^* + b_{3i} v_i v_j v_i^* + c_{3i} v_j v_3 v_3^* \quad \text{for } i, j = 1, 2 \text{ and } i \neq j.$$

Note that $|\varphi| = \overline{1}$ and

$$(-1)^{i-1}\varphi(e_{12}) = \varphi([h_i, e_{12}]) = h_i \cdot \varphi(e_{12}) - e_{12} \cdot \varphi(h_i) = (-1)^{i-1}\varphi(e_{12}) - a_i v_1^2 v_3^*$$

It follows that $a_i = 0$, $i = 1, 2$. That is

(4.13) $$\varphi(\mathfrak{h}) = 0.$$

Applying (4.13) to the equations

$$\varphi(h_1) = \varphi([e_{13}, e_{31}]) = (a_{31} - c_{31})v_1 v_2 v_3^*$$

and

$$\varphi(h_2) = \varphi([e_{23}, e_{32}]) = (b_{32} - c_{32})v_1 v_2 v_3^*,$$

one has

(4.14) $$a_{31} = c_{31}, \quad b_{32} = c_{32}.$$

Noting that

$$0 = \varphi([e_{31}, e_{31}]) = -2e_{31} \cdot \varphi(e_{31}) = -2(a_{31} + c_{31})v_2 v_3 v_1^*$$

and

$$0 = \varphi([e_{32}, e_{32}]) = -2e_{32} \cdot \varphi(e_{32}) = -2(b_{32} + c_{32})v_1 v_3 v_2^*,$$

one has

(4.15) $$a_{31} = -c_{31}, \quad b_{32} = -c_{32}.$$

Combining (4.14) and (4.15), one gets

(4.16) $$a_{31} = c_{31} = 0, \quad b_{32} = c_{32} = 0.$$

By (4.16), a simple computation shows

$$0 = \varphi([e_{31}, e_{32}]) = -2a_{32}v_1v_3v_1^* - 2b_{31}v_2v_3v_2^*.$$

It follows that

(4.17) $$a_{32} = 0, \quad b_{31} = 0.$$

According to (4.16) and (4.17), we have

(4.18) $$\varphi(e_{31}) = \varphi(e_{32}) = 0.$$

Noting that

$$0 = \varphi(e_{32}) = \varphi([e_{31}, e_{12}]) = -2a_{12}v_1v_3v_3^* - a_{12}v_1^2v_1^*$$

and

$$0 = \varphi(e_{31}) = \varphi([e_{32}, e_{21}]) = -2a_{21}v_2v_3v_3^* - a_{21}v_2^2v_2^*,$$

we have $a_{12} = 0$, $a_{21} = 0$. Therefore,

(4.19) $$\varphi(e_{12}) = \varphi(e_{21}) = 0.$$

From (4.13), (4.18) and (4.19), we have $\varphi = 0$. By Lemma 2.3, the conclusion holds.

(3) Let φ be a weight derivation from $\mathfrak{g}$ to $\mathcal{V}_3 \otimes \mathfrak{sl}(2|1)^*$. According to Tables (2.2) and (2.4), one may assume that

$$\varphi(\mathfrak{B} \setminus \{e_{31}, e_{32}\}) = 0,$$
$$\varphi(e_{31}) = a_{31}v_1v_2^2v_3^*, \quad \varphi(e_{32}) = a_{32}v_1^2v_2v_3^* \quad \text{for some } a_{31}, a_{32} \in \mathbb{F}.$$

Clearly, $|\varphi| = \overline{0}$ and

$$0 = \varphi([e_{31}, e_{32}]) = 2(a_{32} + a_{31})v_1v_2v_3v_3^* + a_{32}v_1^2v_2v_1^* + a_{31}v_1v_2^2v_2^*.$$

Then we have $a_{31} = a_{32} = 0$. Consequently, $\varphi = 0$ and then $H^1(\mathfrak{g}, \mathcal{V}_3 \otimes \mathfrak{sl}(2|1)^*) = 0$.

Similarly, one may check the conclusion for $k = p - 1, p, p + 1$. ∎

Lemma 4.8. $H^1(\mathfrak{g}, \mathcal{V}_{2p-1} \otimes \mathfrak{sl}(2|1)^*) = \langle \varphi_8 \rangle$.

Proof. Let φ be a weight derivation from $\mathfrak{g}$ to $\mathcal{V}_{2p-1} \otimes \mathfrak{sl}(2|1)^*$. According to Tables 2.2 and 2.4, suppose

$$\varphi(\mathfrak{B} \setminus \{e_{13}, e_{23}\}) = 0,$$
$$\varphi(e_{13}) = a_{13}v_1^{p-1}v_2^{p-1}v_3v_2^*, \quad \varphi(e_{23}) = a_{23}v_1^{p-1}v_2^{p-1}v_3v_1^* \quad \text{for some } a_{13}, a_{23} \in \mathbb{F}.$$

Note that $|\varphi| = \bar{0}$ and

$$0 = \varphi([e_{13}, e_{23}]) = (a_{13} + a_{23})v_1^{p-1}v_2^{p-1}v_3 v_3^*.$$

It follows that $a_{13} = -a_{23}$ and then

$$\varphi = a\varphi_8.$$

According to Lemma 2.3, the conclusion holds.   ∎

The following is a direct consequence of Lemmas 4.1, 4.7 and 4.8.

Proposition 4.9. *We have*

$$H^1(\mathfrak{g}, \mathcal{V} \otimes \mathfrak{sl}(2|1)^*) = \langle \varphi_8 \rangle,$$

In particular,

$$\dim H^1(\mathfrak{g}, \mathcal{V} \otimes \mathfrak{sl}(2|1)^*) = 1.$$

We are in position to prove the main result of this paper:

Theorem 4.10. *The 1-dimensional cohomology group of $\mathfrak{g}$ with coefficients in W is as follows:*

$$H^1(\mathfrak{g}, W) = 2^{n-1}p^{m-2}(\langle \varphi_1 \rangle \oplus (m+n-3)\langle \varphi_2, \ldots, \varphi_7 \rangle \oplus \langle \varphi_8 \rangle).$$

In particular,

$$\dim H^1(\mathfrak{g}, W) = (3m + 3n - 8)2^n p^{m-2}.$$

Proof.   Using the fundamental fact (2.11), we have $H^1(\mathfrak{g}, \mathcal{T}) = 0$. Then, by (2.1) and (2.9),

$$H^1(\mathfrak{g}, W) = s(H^1(\mathfrak{g}, \mathfrak{sl}(2|1)^*) \oplus tH^1(\mathfrak{g}, \mathcal{V}) \oplus H^1(\mathfrak{g}, \mathcal{V} \otimes \mathfrak{sl}(2|1)^*))$$

and our conclusions follow from Propositions 4.2, 4.5 and 4.9.   ∎

5. APPLICATION

In this section, we apply the results obtained in Section 4 to compute the low-dimensional cohomology groups of $\mathfrak{g}$ with coefficients in the Special superalgebra. Recall that

(5.1) $$\bar{S}(m,n) = S(m,n) \oplus \langle x^{(\pi - (p-1)\varepsilon_i)}x^\omega \partial_i \mid 1 \leq i \leq m \rangle,$$

and

(5.2) $$\dim S(m,n) = (m+n-1)2^n p^m - m + 1,$$

where $\pi = (p-1, \ldots, p-1) \in \mathbb{N}^m$ and $\omega = \langle m+1, \ldots, m+n \rangle$ for all $m+1 \leq j \leq m+n$. Before computing the zero-dimensional cohomology groups, we establish a technical lemma.

Lemma 5.1. *Let $d = v_1 v_1^* + v_2 v_2^* + v_3 v_3^*$ and $\mathcal{N} = (\mathcal{T} \oplus \widehat{\mathcal{O}}d) \cap S$. Then*

$$\dim \mathcal{N} = (m + n - 3)2^{n-1}p^{m-2} - m + 2.$$

Proof. Fix the standard basis of $\widehat{\mathcal{O}}$ as in (2.3): $\{f_i \mid i = 1, \ldots, s\}$, where

$$f_s = x_3^{p-1} \cdots x_m^{p-1} x_{m+2} \cdots x_{m+n}.$$

For f_i, $i \neq s$, suppose that k_i is the first one of the ordered set $\{3, \ldots, m, m + 2, \ldots, m+n\}$ such that $x_{k_i} f_i \neq 0$. Set

$$(5.3) \qquad g_i = -(-1)^{(|f_i|+1)|k_i|} c_i x_{k_i} f_i,$$

where $c_i \in \mathbb{F}$ is chosen so that the coefficient of $c_i x_{k_i} f_i$ is 1. One sees that such g_i and k_i are uniquely determined by f_i for $i = 1, \ldots, s - 1$. By (2.4),

$$(5.4) \qquad g_i \partial_{k_i} \in \mathcal{T} = \langle \widehat{\mathcal{O}} \partial_i \mid i \notin \mathbf{J} \rangle.$$

(5.3) ensures that

$$(5.5) \qquad \operatorname{div}(f_i d + g_i \partial_{k_i}) = 0.$$

By (5.1), $f_i d + g_i \partial_{k_i} \in S$. Clearly, $\{f_i d + g_i \partial_{k_i} \mid i = 1, \ldots, s - 1\}$ is linearly independent.

Next, let us show that

$$(5.6) \qquad \mathcal{N} = (\mathcal{T} \cap S) \oplus \langle f_i d + g_i \partial_{k_i} \mid i = 1, \ldots, s - 1 \rangle.$$

The inclusion "$\supseteq$" is clear. To show the converse, suppose that

$$x = y + \sum_{i=1}^{s} a_i f_i d \in \mathcal{N} = (\mathcal{T} \oplus \widehat{\mathcal{O}}d) \cap S,$$

where $y \in \mathcal{T}$ and $\sum_{i=1}^{s} a_i f_i d \in \widehat{\mathcal{O}}d$, $a_i \in \mathbb{F}$. By (5.1),

$$(5.7) \qquad \operatorname{div} y = -\sum_{i=1}^{s} a_i f_i.$$

It follows that $a_s = 0$. So one may assume that $x = y + \sum_{i=1}^{s-1} f_i d$. On the other hand,

$$x = y - \left(\sum_{i=1}^{s-1} a_i g_i \partial_{k_i} - \sum_{i=1}^{s-1} a_i g_i \partial_{k_i} \right) + \sum_{i=1}^{s} a_i f_i d$$

$$= \left(y - \sum_{i=1}^{s-1} a_i g_i \partial_{k_i} \right) + \sum_{i=1}^{s-1} a_i (f_i d + g_i \partial_{k_i}).$$

(5.4), (5.5) and (5.7) imply that $y - \sum_{i=1}^{s-1} a_i g_i \partial_{k_i} \in \mathcal{T} \cap S$. Then $x \in (\mathcal{T} \cap S) \oplus \langle f_i d + g_i \partial_{k_i} \mid i = 1, \ldots, s-1 \rangle$. Thus, (5.6) holds.

Note that as a vector space, $\mathcal{T} \cap S$ is isomorphic to $S(m-2, n-1)$. By (5.2),

$$\dim(\mathcal{T} \cap S) = \dim S(m-2, n-1) = (t-1)s - m + 3.$$

According to (5.6), we have

$$\dim \mathcal{N} = (t-1)s - m + 3 + (s-1) = (m+n-3)2^{n-1}p^{m-2} - m + 2. \qquad \blacksquare$$

Theorem 5.2. *The* 0*-dimensional cohomology group of* $\mathfrak{g}$ *with coefficients in* S *is*

$$H^0(\mathfrak{g}, S) = \mathcal{N} \oplus \dim S(m-2, n-1) \langle v_1^{p-1} v_2^{p-1} v_3 \rangle.$$

In particular,

$$\dim H^0(\mathfrak{g}, S) = (2m + 2n - 7)2^{n-1}p^{m-2} - 2m + 5.$$

Clearly, $H^0(\mathfrak{g}, S) = H^0(\mathfrak{g}, W) \cap S$. By Theorem 3.1,

$$H^0(\mathfrak{g}, W) = \mathcal{T} \oplus st\langle v \rangle \oplus s\langle d \rangle,$$

where $s = 2^{n-1}p^{m-2}$, $t = m+n-3$ and $v = v_1^{p-1} v_2^{p-1} v_3$, $d = v_1 v_1^* + v_2 v_2^* + v_3 v_3^*$. It is easy to see that for any $x \in \mathcal{T} \setminus S$, $y \in st\langle v \rangle \setminus S$ and $z \in s\langle d \rangle \setminus S$, one has

$$\mathrm{div}(x + y) \neq 0 \quad \text{and} \quad \mathrm{div}(y + z) \neq 0.$$

Then, according to (5.1),

$$x + y \notin S \quad \text{and} \quad y + z \notin S.$$

Thus,

$$(5.8) \qquad H^0(\mathfrak{g}, S) = H^0(\mathfrak{g}, W) \cap S = ((\mathcal{T} \oplus s\langle d \rangle) \cap S) \oplus (st\langle v \rangle \cap S).$$

Recall that as $\mathfrak{g}$-module, $s\langle d \rangle \simeq \widehat{\mathcal{O}} d$. By Lemma 5.1, one gets

$$(5.9) \qquad\qquad\qquad (\mathcal{T} \oplus s\langle d \rangle) \cap S \simeq \mathcal{N}.$$

Analogously, as $\mathfrak{g}$-module, $st\langle v \rangle \simeq \langle v \widehat{\mathcal{O}} \partial_i \mid i \notin J \rangle$. Then

$$st\langle v \rangle \cap S \simeq \dim(\langle v \widehat{\mathcal{O}} \partial_j \mid i \notin J \rangle \cap S) \langle v \rangle.$$

As vector spaces, $\langle v \widehat{\mathcal{O}} \partial_i \mid i \notin J \rangle \cap S \simeq S(m-2, n-1)$. Then

$$(5.10) \qquad\qquad\qquad st\langle v \rangle \cap S \simeq \dim S(m-2, n-1) \langle v \rangle$$

In view of Lemma 5.1, from (5.8)–(5.10) and (5.2) we have

$$H^0(\mathfrak{g}, S) = \mathcal{N} \oplus \dim S(m-2, n-1)\langle v \rangle$$

and

$$\dim H^0(\mathfrak{g}, S) = (2m + 2n - 7)2^{n-1}p^{m-2} - 2m + 5. \qquad \blacksquare$$

Finally, we compute $H^1(\mathfrak{g}, S)$. Since S is a subalgebra of W, each derivation from $\mathfrak{g}$ to S may be viewed as a derivation from $\mathfrak{g}$ to W. Thus, it is sufficient to compute the derivations in $\mathrm{Der}(\mathfrak{g}, W)$ which are outer derivations from $\mathfrak{g}$ to S. For convenience, write

$$\mathcal{D}_k := \begin{cases} \langle \sigma \in \mathrm{Der}(\mathfrak{g}, W) \mid \exists f \in \widehat{\mathcal{O}} : \sigma(x) = f\varphi_k(x) & \text{for all } x \in \mathfrak{g} \rangle, & k = 1, 8; \\ \langle \sigma \in \mathrm{Der}(\mathfrak{g}, W) \mid \exists T \in \mathcal{T} : \sigma(x) = \varphi_k(x)T & \text{for all } x \in \mathfrak{g} \rangle, & k = 2, \ldots, 7. \end{cases}$$

By (4.1), we have

(5.11) $$\bar{\mathcal{D}}_1 := \mathcal{D}_1 \cap \mathrm{Der}(\mathfrak{g}, S) = \mathcal{D}_1 \quad \text{and} \quad \dim \bar{\mathcal{D}}_1 = s.$$

Lemma 5.3. *Let* $\bar{\mathcal{D}}_k := \mathcal{D}_k \cap \mathrm{Der}(\mathfrak{g}, S)$ *for* $k = 2, \ldots, 7$. *Then*

$$\dim \bar{\mathcal{D}}_k = (m + n - 4)2^{n-1}p^{m-2} - m + 3.$$

Proof. As vector spaces, $\mathcal{D}_k \simeq W(m-2, n-1)$ for $k = 2, \ldots, 7$ and $\bar{\mathcal{D}}_k \simeq S(m-2, n-1)$. By (5.2), we have

$$\dim \bar{\mathcal{D}}_k = \dim S(m-2, n-1) = (m+n-4)2^{n-1}p^{m-2} - m + 3. \qquad \blacksquare$$

Let us consider $(\mathcal{D}_7 \oplus \mathcal{D}_8) \cap \mathrm{Der}(\mathfrak{g}, S)$. Since $\dim \mathcal{D}_8 = \dim \widehat{\mathcal{O}} = s$, fix a standard basis of $\mathcal{D}_8$:

$$\{\varphi_{8,i} \in \mathcal{D}_8 \mid \varphi_{8,i}(x) = f_i\varphi_8(x), \ x \in \mathfrak{g}, \ i = 1, \ldots, s\},$$

where $\{f_i \mid i = 1, \ldots, s\}$ is the standard basis of $\widehat{\mathcal{O}}$ in (2.3) and

$$f_s = x_3^{p-1} \cdots x_m^{p-1} x_{m+2} \cdots x_{m+n}.$$

For f_i with $i \neq s$, suppose that k_i is the first element in the ordered set $\{3, \ldots m, m+2, \ldots, m+n\}$ such that $x_{k_i}f_i \neq 0$. Set

(5.12) $$g_i = -(-1)^{|f_i||k_i|+|k_i|+|f_i|}c_i x_{k_i} f_i,$$

where $c_i \in \mathbb{F}$ is chosen so that the coefficient of $c_i x_{k_i} f_i$ is 1. One sees that such g_i and k_i are uniquely determined by f_i. Let $\varphi_{7,i}$ be the derivation in $\mathcal{D}_7$ such that $\varphi_{7,i}(x) = \varphi_7(x) g_i \partial_{k_i}$. By (5.12) and (4.1),

$$(5.13) \qquad \operatorname{div}(\varphi_{8,i}(x) + \varphi_{7,i}(x)) = 0 \quad \text{for } x \in \mathfrak{g}.$$

According to (5.1), $\varphi_{8,i} + \varphi_{7,i} \in (\mathcal{D}_7 \oplus \mathcal{D}_8) \cap \operatorname{Der}(\mathfrak{g}, S)$ for $i \neq s$. Clearly, $\{\varphi_{8,i} + \varphi_{7,i} \mid i = 1, \ldots, s-1\}$ is linearly independent. Let

$$(5.14) \qquad \bar{\mathcal{D}}_8 := \langle \varphi_{8,i} + \varphi_{7,i} \mid i = 1, \ldots, s-1 \rangle.$$

Then

$$(5.15) \qquad \dim \bar{\mathcal{D}}_8 = s - 1.$$

Lemma 5.4. $(\mathcal{D}_7 \oplus \mathcal{D}_8) \cap \operatorname{Der}(\mathfrak{g}, S) = \bar{\mathcal{D}}_7 \oplus \bar{\mathcal{D}}_8$.

Proof. Suppose $\sigma_7 + \sigma_8 \in \operatorname{Der}(\mathfrak{g}, S)$ and $\sigma_7 \in \mathcal{D}_7$, $\sigma_8 \in \mathcal{D}_8$. Then for any $x \in \mathfrak{g}$,

$$(5.16) \qquad \operatorname{div}(\sigma_7(x)) = -\operatorname{div}(\sigma_8(x)).$$

Suppose $\sigma_8 = \sum_{i=1}^{s} a_i \varphi_{8,i}$ and $a_i \in \mathbb{F}$. We assert that $a_s = 0$. Otherwise, $\operatorname{div}(\sigma_7(e_{13}) + \sigma_8(e_{13})) \neq 0$ for any $\sigma_7 \in \mathcal{D}_7$. By (5.1), $\sigma_7 + \sigma_8 \notin \operatorname{Der}(\mathfrak{g}, S)$, contradicting our assumption. So one may assume that $\sigma = \sum_{i=1}^{s-1} a_i \varphi_{8,i}$. Then

$$\begin{aligned}
\sigma_7 + \sigma_8 &= \sigma_7 - \Big(\sum_{i=1}^{s-1} a_i \varphi_{7,i} - \sum_{i=1}^{s-1} a_i \varphi_{7.i} \Big) + \sum_{i=1}^{s-1} a_i \varphi_{8,i} \\
&= \Big(\sigma_7 - \sum_{i=1}^{s-1} a_i \varphi_{7,i} \Big) + \sum_{i=1}^{s-1} a_i (\varphi_{8,i} + \varphi_{7,i}).
\end{aligned}$$

(5.13) implies that for any $x \in \mathfrak{g}$,

$$(5.17) \qquad \operatorname{div}\Big(\sum_{i=1}^{s-1} a_i \varphi_{7,i}(x) \Big) = -\operatorname{div}\Big(\sum_{i=1}^{s-1} a_i \varphi_{8,i}(x) \Big) = -\operatorname{div}(\sigma_8(x)).$$

Then (5.16) and (5.17) show that

$$\operatorname{div}(\sigma_7(x)) = \operatorname{div}\Big(\sum_{i=1}^{s-1} a_i \varphi_7(x) g_i \partial_{k_i} \Big) \quad \text{for any } x \in \mathfrak{g}.$$

It follows that

$$\sigma_7 - \sum_{i=1}^{s-1} a_i \varphi_{7,i} \in \bar{\mathcal{D}}_7.$$

Thus $\sigma_7 + \sigma_8 \in \bar{\mathcal{D}}_7 + \bar{\mathcal{D}}_8$. Clearly, $\bar{\mathcal{D}}_7 + \bar{\mathcal{D}}_8 \subseteq (\mathcal{D}_7 \oplus \mathcal{D}_8) \cap \operatorname{Der}(\mathfrak{g}, S)$ and $\bar{\mathcal{D}}_7 \cap \bar{\mathcal{D}}_8 = 0$. So we have $(\mathcal{D}_7 \oplus \mathcal{D}_8) \cap \operatorname{Der}(\mathfrak{g}, S) = \bar{\mathcal{D}}_7 \oplus \bar{\mathcal{D}}_8$. ∎

Lemma 5.5. *Suppose $w \in W_\theta$. Then $\mathrm{ad}w \equiv \mathrm{ad}(fv_1v_1^*) \pmod{\mathrm{Ider}(\mathfrak{g},S)}$ for some $f \in \widehat{\mathcal{O}}$.*

Proof. By Tables 2.3 and 2.4 and, (3.1) and (3.3), one may assume that

$$w = \sum_{j \in \mathbf{J}} g_j v_j v_j^* + g_4 v_1 v_2 v_3^* + g_5 v_1^{p-1} v_3 v_2^* + g_6 v_2^{p-1} v_3 v_1^*, \quad \text{where } g_i \in \widehat{\mathcal{O}}.$$

A simple computation and (5.1) show that $\mathrm{ad}w(\mathfrak{g}) \subseteq S$. Then $\mathrm{ad}w \in \mathrm{Der}(\mathfrak{g},S)$. Note that

$$g_4 v_1 v_2 v_3^* + g_5 v_1^{p-1} v_3 v_2^* + g_6 v_2^{p-1} v_3 v_1^* \in S$$

and

$$\sum_{j \in \mathbf{J}} g_j v_j v_j^* \equiv g_1 v_1 v_1^* \pmod{S}.$$

It follows that $\mathrm{ad}w \equiv \mathrm{ad}(fv_1v_1^*) \pmod{\mathrm{Ider}(\mathfrak{g},S)}$ for some $f \in \widehat{\mathcal{O}}$. $\blacksquare$

Set

(5.18) $$\mathcal{D}_9 := \langle \mathrm{ad}(fv_1v_1^*) \mid f \in \widehat{\mathcal{O}} \rangle.$$

Then $\bar{\mathcal{D}}_9 := \mathcal{D}_9 \cap \mathrm{Der}(\mathfrak{g},S) = \mathcal{D}_9$ and

(5.19) $$\dim \bar{\mathcal{D}}_9 = s = 2^{n-1} p^{m-2}.$$

Lemma 5.6. *Let $\sigma_k \in \mathcal{D}_k$ for $k = 1, \ldots, 9$. Then*

$$\sum_{k=1}^{9} \sigma_k \in \mathrm{Der}(\mathfrak{g},S) \Longleftrightarrow \sigma_k \in \bar{\mathcal{D}}_k \text{ for } k \neq 7,8 \text{ and } \sigma_7 + \sigma_8 \in \bar{\mathcal{D}}_7 \oplus \bar{\mathcal{D}}_8.$$

Proof. It suffices to show the implication "$\Longrightarrow$". Since $\bar{\mathcal{D}}_1 = \mathcal{D}_1$ and $\bar{\mathcal{D}}_9 = \mathcal{D}_9$, one may suppose that $\sigma = \sum_{k=2}^{8} \sigma_k \in \mathrm{Der}(\mathfrak{g},S)$. (4.1) shows that for any $x \in \mathfrak{g}$, $\varphi_k(x) \in \mathcal{O}(2,1)$, $k = 2, \ldots, 7$. Since $\sigma_k \in \mathcal{D}_k$, one may suppose that

$$\sigma_k(x) = \varphi_k(x)q_k, \quad \text{where } q_k \in \mathcal{T}, \quad k = 2, \ldots, 7.$$

Assume that $\sigma_2 \notin \bar{\mathcal{D}}_2$. Then there exists some $x \in \mathfrak{B}$ such that $\sigma_2(x) \notin S$. By (4.1), $x = e_{31}$ or e_{32}. If $x = e_{31}$, again by (4.1), $\sigma_k(e_{31}) = 0$ for $k = 3,4,5,7,8$ and $\sigma_2(e_{31}) = v_2 q_2$, $\sigma_6(e_{31}) = v_1^{p-1} v_3 q_6$. Thus,

(5.20) $$\sigma(e_{31}) = \sigma_2(e_{31}) + \sigma_6(e_{31}) = v_2 q_2 + v_1^{p-1} v_3 q_6 \in S.$$

The assumption that $\sigma_2(e_{31}) \notin S$ forces $\sigma_6(e_{31}) \notin S$. Then

$$\mathrm{div}\sigma_2(e_{31}) = \mathrm{div}(v_2 q_2) = v_2 \mathrm{div}q_2 \neq 0,$$
$$\mathrm{div}\sigma_6(e_{31}) = \mathrm{div}(v_1^{p-1} v_3 q_6) = \pm v_1^{p-1} v_3 \mathrm{div}q_6 \neq 0.$$

It follows that $\mathrm{div} q_2 \neq 0$ and $\mathrm{div} q_6 \neq 0$. Then $\mathrm{div}(\sigma_2(e_{31}) + \sigma_6(e_{31})) \neq 0$. This contradicts (5.20).

If $x = e_{32}$, a similar discussion yields that $\sigma_2 \in \bar{\mathcal{D}}_2$. Next, we want to show that $\sigma_3 \in \bar{\mathcal{D}}_3$. Assume that $\sigma_3 \notin \bar{\mathcal{D}}_3$. According to (4.1), $\sigma_3(x) \notin S$ for $x = e_{12}$ or $x = e_{13}$. If $x = e_{13}$, then $\sigma_k(e_{13}) = 0$ for $k = 4, 5, 6$ and

$$\sigma_3(e_{13}) = v_2^{p-1} q_3,$$

$$\sigma_7(e_{13}) = \pm v_1^{p-1} v_2^{p-2} v_3 q_7,$$

$$\sigma_8(e_{13}) = f v_1^{p-1} v_2^{p-1} v_3 \partial_2, \ f \in \widehat{\mathcal{O}}.$$

Thus,

(5.21) $\sigma(e_{13}) = \sigma_3(e_{13}) + (\sigma_7 + \sigma_8)(e_{13}) \in S.$

The assumption that $\sigma_3(e_{13}) \notin S$ forces $(\sigma_7 + \sigma_8)(e_{13}) \notin S$. Then

$$\mathrm{div}\sigma_3(e_{13}) = v_2^{p-1} \mathrm{div} q_3 \neq 0,$$
$$\mathrm{div}((\sigma_7 + \sigma_8)(e_{13})) = v_1^{p-1} v_2^{p-2} v_3 (\pm \mathrm{div} q_7 + f) \neq 0.$$

It follows that

$$\mathrm{div}(\sigma_3(e_{13}) + (\sigma_7 + \sigma_8)(e_{13})) \neq 0.$$

This contradicts (5.21). If $x = e_{12}$, the discussion is similar. So we have $\sigma_3 \in \bar{\mathcal{D}}_3$.

Analogously, one may prove that $\sigma_k \in \bar{\mathcal{D}}_k$ for $k = 4, 5, 6$. Then $\sigma_7 + \sigma_8 \in \mathrm{Der}(\mathfrak{g}, S)$, since $\sum_{k=2}^{6} \sigma_k \in \mathrm{Der}(\mathfrak{g}, S)$. By Lemma 5.4, $\sigma_7 + \sigma_8 \in \bar{\mathcal{D}}_7 \oplus \bar{\mathcal{D}}_8$. The proof is complete. ∎

Theorem 5.7. *The* 1-*dimensional cohomology group of* $\mathfrak{g}$ *with coefficients in* S *is*

$$H^1(\mathfrak{g}, S) = \oplus_{k=1}^{9} \bar{\mathcal{D}}_k.$$

In particular,

$$\dim H^1(\mathfrak{g}, S) = 3(2m + 2n - 7)2^{n-1} p^{m-2} - 6m + 17.$$

Proof. Let σ be an outer derivation from $\mathfrak{g}$ to S. By Theorem 4.10,

$$\mathrm{Der}(\mathfrak{g}, W) = \oplus_{k=1}^{8} \mathcal{D}_k \oplus \mathrm{Ider}(\mathfrak{g}, W).$$

Then one may assume that

$$\sigma = \sum_{k=1}^{8} \sigma_k + \mathrm{ad} w \in \mathrm{Der}(\mathfrak{g}, S), \quad \text{where } \sigma_k \in \mathcal{D}_k, \ w \in W.$$

Since σ_k are weight derivations, by Lemma 2.3, one may assume that $w \in W_\theta$. According to Lemma 5.5 and (5.18), one may assume further that $\sigma = \sum_{k=1}^{9} \sigma_k$, where $\sigma_k \in \mathcal{D}_k$. Then by Lemma 5.6, one gets $\sigma \in \oplus_{k=1}^{9} \bar{\mathcal{D}}_k$.

Next, we are going to show that any nonzero $\sigma \in \oplus_{k=1}^{9} \bar{\mathcal{D}}_k$ is an outer derivation from $\mathfrak{g}$ to S. By (5.18), assume that

$$\sigma = \sum_{k=1}^{8} \sigma_k + \mathrm{ad}(f v_1 v_1^*), \quad \text{where } \sigma_k \in \bar{\mathcal{D}}_k, \ a_i \in \mathbb{F}, \ f \in \hat{\mathcal{O}}.$$

If σ is inner, there exists some $z \in S$ such that

$$\sum_{k=1}^{8} \sigma_k + \mathrm{ad}(f v_1 v_1^*) = \mathrm{ad}z.$$

Recall that nonzero $\phi_k \in \mathcal{D}_k$, $k = 1, \ldots, 8$, are linearly independent modulo $\mathrm{Ider}(\mathfrak{g}, W)$. Consequently, $\sigma_k = 0$, $k = 1, \ldots, 8$, and then $\mathrm{ad}(f v_1 v_1^*) = \mathrm{ad}z$. Note that $\mathrm{ad}(f v_1 v_1^*)$ is an outer derivation from $\mathfrak{g}$ to S when $f \neq 0$. Since $z \in S$, one has $f = 0$. Thus $\sigma = 0$, contradicting our assumption. So σ is outer.

Up to now we have shown that $H^1(\mathfrak{g}, S) = \oplus_{k=1}^{9} \bar{\mathcal{D}}_k$. Then, by Lemma 5.3 and, (5.11), (5.15) and (5.19), we have

$$\begin{aligned}
\mathrm{dim}H^1(\mathfrak{g}, S) &= \sum_{k=1}^{9} \mathrm{dim}\bar{\mathcal{D}}_k \\
&= s + 6((m+n-4)2^{n-1}p^{m-2} - m + 3) + (s-1) + s \\
&= 3(2m+2n-7)2^{n-1}p^{m-2} - 6m + 17. \qquad \blacksquare
\end{aligned}$$

Finally, we explain that, as in Lie algebra case, some classical conclusions in characteristic zero do not hold in characteristic $p > 0$:

Remark 5.8. Over a field $\mathbb{F}$ of characteristic zero, the first cohomology group of $\mathrm{A}(1,0)$ with coefficients in a finite-dimensional simple module is trivial or of dimension 1 (see [11]). However, in the case of characteristic $p > 2$, from Lemma 4.4 and Proposition 4.9 one sees that

$$\mathrm{dim}H^1(\mathrm{A}(1,0), \mathcal{V}_{p-1}) = \mathrm{dim}H^1(\mathrm{A}(1,0), \mathcal{V}_p) = 2,$$

while both $\mathcal{V}_{p-1}$ and $\mathcal{V}_p$ as $\mathrm{A}(1,0)$-modules are finite-dimensional and simple.

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