TAIWANESE JOURNAL OF MATHEMATICS
Vol. 17, No. 6, pp. 1869-1885, December 2013
DOI: 10.11650/tjm.17.2013.1751
This paper is available online at http://journal.taiwanmathsoc.org.tw

STABLE POISSON CONVERGENCE FOR INTEGER-VALUED RANDOM VARIABLES

Tsung-Lin Cheng* and Shun-Yi Yang

Abstract. In this paper, we obtain some stable Poisson Convergence Theorems for arrays of integer-valued dependent random variables. We prove that the limiting distribution is a mixture of Poisson distribution when the conditional second moments on a given σ-algebra of the sequence converge to some positive random variable. Moreover, we apply the main results to the indicator functions of rowise interchangeable events and obtain some interesting stable Poisson convergence theorems.

1. Introduction

It is likely that the greatest accomplishment of modern probability theory is the unified elegant theory of limits for sums of independent or stationary random variables. The former studies the limiting theorems when dependence structure is relaxed to comprising only independent random variables, while the later consider dependent but time-invariant distributed random variables. The mathematical theory of martingales may be regarded as an extension of the independence theory which has been firstly studied by Bernstein (1927) and Lévy (1935, 1937). Lévy introduced the conditional variance for martingales

$$V_n^2 = \sum_{j=1}^{n} E(X_j^2|\mathcal{F}_{j-1}),$$

where $(S_n, \mathcal{F}_n, n \geq 1)$ is a zero-mean, square integrable martingale and $X_n = S_n - S_{n-1}$ is the martingale difference. Doob (1953), Billingsley (1961), and Ibragimov (1963) established the Central Limit Theorem for martingales with stationary and ergodic differences. For such martingales the conditional variance is asymptotically constant; namely,

$$s_n^{-2} V_n^2 \xrightarrow{p} 1, \tag{1}$$

Received March 6, 2013, accepted April 27, 2013.
Communicated by Yuh-Jia Lee.
2010 *Mathematics Subject Classification*: 60F05.
Key words and phrases: Stable Poisson Convergence.
*Corresponding author.

where $s_n^2 = E(V_n^2)$. Further extensions have been made by Rosén (1967a,b), Dvoretzky (1969, 1971, 1972) and Brown (1971), among others. Especially, as indicated by Brown (1971), the crucial point for martingale convergence is the condition (1) but stationarity or ergodicity. In McLeish (1974), an elegant proof about the martingale central limit theorem and invariance principles were given. The convergence of normalized martingales to more general distributions were investigated by Brown and Eagleson (1971), Eagleson (1976), Adler et al.(1978), among others. Since its first commencement in Rényi(1963), the concept of stable convergence has been largely extended and applied to many general setting and problems concerning asymptotic behaviors of martingale arrays or stationary arrays. The stable convergence can be defined as follows. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and (Y_n) be a sequence of random variables. Let $Y_n \xrightarrow{d} Y$. If for every $B \in \mathcal{G}$, there is a countable, dense set of points x, such that

$$\lim_{n\to\infty} P(\{Y_n \le x\} \cap B) = Q(x, B)$$

exists, then we say that Y_n converges stably in $\mathcal{G}$ to Y and denote it by $Y_n \xrightarrow{stably} F_Y$(or Y) on $\mathcal{G}$, or

$$Y_n \xrightarrow{(s,\mathcal{G})} Y.$$

The mapping $Q(x, B)$ in the above definition is a distribution function when we fix $B \in \mathcal{G}$, and it is a probability measure on $\mathcal{G}$ when we fix the real number x. If $\mathcal{G} = \{\emptyset, \Omega\}$, then $\xrightarrow{(s,\mathcal{G})}$ is just $\xrightarrow{d}$. If $\mathcal{G} = \mathcal{F}$, then $\xrightarrow{(s,\mathcal{G})}$ is the usual $Y_n \xrightarrow{stably} F_Y$ (we drop the $\mathcal{F}$), denoted by $\xrightarrow{s}$. From the proof of Lemma 1 in Cheng and Chow (2002), we know that for almost every x in the value set of Y_n, $Q(x, \cdot)$ is a measure absolutely continuous with respect to P. Unlike convergence in distribution, stable convergence in distribution is a property of sequences of random variables rather than that of their corresponding distribution functions. For more details about stable convergence we refer the readers to Aldous (1978).

The idea behind Rényi's stable convergence is to generalize the renowned Central Limit Theorems to a mixture of normal distributions. The notion of a mixture of a well known distribution with a random parameter stems from Bayesian analysis. In Bayesian structure, a parameter emerging in the probability density function is assumed to be nonconstant, more generally, a random variable. The distribution of the random parameter is called the "prior". Imagine a sequence of dependent random variables which converges "in distribution" to a well known distribution, say, to a normal distribution when conditioned on the event that the random parameter is fixed at some constant value. We will try to look for the posterior utilizing the observations at hand.

Classical Poisson limit theorems assume the random variables to be i.i.d., integer-valued or just be independent but not identically distributed. For an infinite exchangeable sequence of random variables, conditioned on the tail events, the sequence will behave like an i.i.d. sequence (Chow and Teicher 1997). However, this property doesn't hold for an array of finitely exchangeable random variables. Martingale methods provide a unified approach to both situations. The martingale method was suggested by Loynes (1969) in the context of U-statistics and developed by Eagleson (1979, 1982) and Weber (1980) for exchangeable variables. A Poisson convergence theorem follows from the results for infinitely divisible laws developed by Brown and Eagleson (1971). See also Freedman (1974).

Theorem A. (Brown and Eagleson 1971, Freedman 1974). *Let* $\{A_{n,k}, \mathcal{F}_{n,k};\ 1 \le k \le n,\ n \ge 1\}$ *be an array of dependent events on the probability space* $(\Omega, \mathcal{F}, P)$ *(*$\mathcal{F}_{n,k} \subset \mathcal{F}_{n,k+1} \subset \mathcal{F}$*,* $A_{n,k}$ *is* $\mathcal{F}_{n,k}$*-measurable) and* $\mathcal{F}_{n,0}$ *be a sub-*σ*-algebra of* $\mathcal{F}_{n,1}$*. If* $\lambda > 0$ *is a constant and for* $n \to \infty$*,*

$$\sum_{k=1}^{n} P(A_{n,k} \mid \mathcal{F}_{n,k-1}) \xrightarrow{p} \lambda,$$

$$\max_{1 \le k \le n} P(A_{n,k} \mid \mathcal{F}_{n,k-1}) \xrightarrow{p} 0,$$

then $S_n = \sum_{k=1}^{n} I_{A_{n,k}} \xrightarrow{d} Poisson(\lambda)$.

The conditions in Theorem A have also been used by Kaplan(1977), Brown(1978) and Silverman and Brown(1978). However, so far as our knowledge goes, there were no literature discussing stable Poisson convergence. Therefore, we are going to try to derive a stable Poisson convergence theorem (SPCT, for abbreviation) with the limiting distribution of the type of a Poisson mixture, namely, with the intensity parameter λ being a nonnegative random variable. In Cheng and Chow (2002), they proved an auxiliary lemma and obtained some interesting theorems on stable convergence to normal mixture. Under a mild (but not trivial) modification, we may obtain some theorems on stable Poisson convergence.

This paper is organized as below: In Section 2, we consider λ to be a random variable and generalize the Poisson convergence theorem to comprises the stable Poisson convergence theorems by exploiting the conditional characteristic function introduced by Brown (1971), Hall and Heyde (1980), and Cheng and Chow (2002). In Section 3, we apply our main results to arrays of row-exchangeable events. Throughout this paper, all equalities and inequalities between random variables are in the sense of "with probability one" and I_A denotes the indicator function of the set. All kinds of convergence, in distribution, in probability and in L_p, are denoted respectively by $\xrightarrow{d}$, $\xrightarrow{p}$, and $\xrightarrow{L_p}$. The fact that $X_n \xrightarrow{p} 0$ in probability is abbreviated by $X_n = o_p(1)$.

2. Main Results

The following result is an auxiliary lemma for proving stable convergence.

Lemma 2.1. (Cheng and Chow, 2002). *$Y_n \overset{(s,\mathcal{G})}{\longrightarrow} Y$ iff there exists a r.v. Y' on $(\Omega \times I, \mathcal{G} \times \mathcal{B}, P \times u)$, where $I = (0, 1)$, $\mathcal{B}$ is the class of all Borel sets in I and u is the Lebesque measure on $\mathcal{B}$, with the same distribution as Y such that for all real t, $E(e^{itY_n} I_A) \to E(e^{itY'} I_{A\times I})$, for all $A \in \mathcal{G}$, and $E(e^{itY'} I_{A\times I})$ is continuous at $t = 0$ for all $A \in \mathcal{G}$.*

Let $\{A_{n,k}, \mathcal{F}_{n,k}; 1 \le k \le m_n, n \ge 1\}$ be an array of events on the probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_{n,0}$ be a sub-σ-algebra of $\mathcal{F}_{n,1}$. Set $\widetilde{X_{n,k}} = I_{A_{n,k}}$, $\widetilde{S_n} = \sum_{k=1}^{m_n} \widetilde{X_{n,k}}$, and $f_n(t) = \prod_{k=1}^{m_n} \mathrm{E}_{n,k-1}(e^{it\widetilde{X_{n,k}}})$, where $\mathrm{E}_{n,k-1}(X) = E(X|\mathcal{F}_{n,k-1})$.

Theorem 2.1. *Suppose there exists a sub-σ-algebra $\mathcal{G} \subseteqq \bigcap_{n=1}^{\infty} \mathcal{F}_{n,0}$ and a positive $\mathcal{G}$-measurable random variable λ such that for $n \to \infty$,*

$$\sum_{k=1}^{m_n} P(A_{n,k} \,|\, \mathcal{F}_{n,k-1}) \quad \overset{p}{\to} \lambda, \tag{2}$$

$$\max_{1\le k\le m_n} P(A_{n,k} \,|\, \mathcal{F}_{n,k-1}) \quad \overset{p}{\to} 0, \tag{3}$$

$$E(e^{it\widetilde{S_n}} - f_n(t) \,|\, \mathcal{G}) \quad \overset{p}{\to} 0, \tag{4}$$

then $\widetilde{S_n} \overset{(s,\mathcal{G})}{\longrightarrow} Z$, where the random variable Z has characteristic function $E(exp\{\lambda(e^{it} - 1)\})$ and for any $A \in \mathcal{G}$, $E(e^{it\widetilde{S_n}} I_A) \to E(exp\{\lambda(e^{it} - 1)\} I_A)$.

Proof. For clarity and convenience, we define $\mathrm{E}_{n,k-1}(X) = \mathrm{E}(X \,|\, \mathcal{F}_{n,k-1})$. First, we want to show that $f_n(t) \overset{p}{\to} exp\{\lambda(e^{it} - 1)\}$. According to the inequality $|e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!}| \le \min\{\frac{2|x|^n}{n!}, \frac{|x|^{n+1}}{(n+1)!}\}$, we have $|e^{ix} - 1 - ix| \le \min\{2|x|, \frac{|x|^2}{2}\}$. We might set $B(x) \equiv 2 \wedge \frac{|x|}{2}$ and consider a function, say A(x), with $|A(x)| \le B(x)$ such that $e^{ix} = 1 + ix + x \cdot A(x)$. It follows that

$$\begin{aligned} \mathrm{E}_{n,k-1}(e^{it\widetilde{X_{n,k}}}) &= 1 + i\mathrm{E}_{n,k-1}(t\widetilde{X_{n,k}}) + t\mathrm{E}_{n,k-1}(\widetilde{X_{n,k}} \cdot A(t\widetilde{X_{n,k}})) \\ &= 1 + it\mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}) + tA(t)\mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}). \end{aligned}$$

Fix t, define $\delta_{n,k} \equiv it\mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}) + tA(t)\mathrm{E}_{n,k-1}(\widetilde{X_{n,k}})$.
Since $|A(t)| \le 2$, we have

$$|\delta_{n,k}| = |it\mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}) + tA(t)\mathrm{E}_{n,k-1}(\widetilde{X_{n,k}})| \le 3|t| \cdot \mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}).$$

By (3),

$$\max_{1\le k\le m_n} |\delta_{n,k}| \le 3|t| \cdot \max_{1\le k\le m_n} \mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}) \overset{p}{\to} 0. \tag{5}$$

Since for any z∈ C with $|z| \le 1$,

$$\log(1+z) - z = \int_0^z (\frac{1}{1+w} - 1)dw = \int_0^z \frac{-w}{1+w}dw.$$

Hence,

$$|\log(1+z) - z| \le \int_0^z |\frac{-w}{1+w}|dw \le \frac{1}{1-|z|} \cdot \int_0^z |w|dw \le \frac{1}{1-|z|} \cdot \frac{|z|^2}{2},$$

and

$$\begin{aligned}\log f_n(t) &= \sum_{k=1}^{m_n} \log \mathrm{E}_{n,k-1}(e^{it\widetilde{X_{n,k}}}) = \sum_{k=1}^{m_n} \log(1+\delta_{n,k}) \\ &= \sum_{k=1}^{m_n} \delta_{n,k} + R_n,\end{aligned}$$

where

$$|R_n| \le \frac{1}{2} \sum_{k=1}^{m_n} \frac{|\delta_{n,k}|^2}{1-|\delta_{n,k}|},$$

and $\max_{1\le k\le m_n} |\delta_{n,k}| < 1$.

On the set $\{w \in \Omega\ ; \max_{1\le k\le m_n} |\delta_{n,k}| < 1\}$, by (2), (3), we have

$$\begin{aligned}|R_n| &\le \frac{1}{2}\sum_{k=1}^{m_n} \frac{|\delta_{n,k}|^2}{1-|\delta_{n,k}|} \le \frac{1}{1-\max_{1\le k\le m_n}|\delta_{n,k}|} \cdot \sum_{k=1}^{m_n} |\delta_{n,k}|^2 \\ &= \left(\frac{1-\max_{1\le k\le m_n}|\delta_{n,k}|}{1-\max_{1\le k\le m_n}|\delta_{n,k}|} + \frac{\max_{1\le k\le m_n}|\delta_{n,k}|}{1-\max_{1\le k\le m_n}|\delta_{n,k}|}\right) \cdot \sum_{k=1}^{m_n} |\delta_{n,k}|^2 \\ &= (1+o_p(1)) \cdot \sum_{k=1}^{m_n} |\delta_{n,k}|^2 = 9t^2(1+o_p(1)) \cdot \sum_{k=1}^{m_n} \mathrm{E}^2_{n,k-1}(\widetilde{X_{n,k}}) \\ &\le Ct^2 \cdot \max_{1\le k\le m_n} \mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}) \cdot \sum_{k=1}^{m_n} \mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}) \xrightarrow{p} 0.\end{aligned} \tag{6}$$

Hence, by (6), for all $\varepsilon > 0$, as $n \to \infty$,

$$\begin{aligned}\mathrm{P}(|R_n| > \varepsilon) &= \mathrm{P}(|R_n| > \varepsilon, \max_{1\le k\le m_n} |\delta_{n,k}| < 1) + \mathrm{P}(|R_n| > \varepsilon, \max_{1\le k\le m_n} |\delta_{n,k}| \ge 1) \\ &\le \mathrm{P}(|R_n| > \varepsilon, \max_{1\le k\le m_n} |\delta_{n,k}| < 1) + \mathrm{P}(\max_{1\le k\le m_n} |\delta_{n,k}| \ge 1) \\ &\longrightarrow 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\log f_n(t) &= \sum_{k=1}^{m_n} \delta_{n,k} + o_p(1) \\ &= it \sum_{k=1}^{m_n} \mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}) + tA(t) \sum_{k=1}^{m_n} \mathrm{E}_{n,k-1}(\widetilde{X_{n,k}}) + o_p(1) \\ &\xrightarrow{p} it\lambda + tA(t)\lambda = \lambda(it + tA(t)) = \lambda(e^{it} - 1).\end{aligned}$$

Thus, $f_n(t) \xrightarrow{p} exp\{\lambda(e^{it} - 1)\}$, and as a result, for all $A \in \mathcal{G}$,

$$f_n(t) \cdot I_A \xrightarrow{p} exp\{\lambda(e^{it} - 1)\} \cdot I_A.$$

By uniform integrability,

$$\mathrm{E}(f_n(t) \cdot I_A) \to E[exp\{\lambda(e^{it} - 1)\} \cdot I_A].$$

By (4) and uniform integrability, for $A \in \mathcal{G}$,

$$\mathrm{E}[(e^{it\widetilde{S_n}} - f_n(t)) \cdot I_A] \to 0.$$

Consequently,

$$\mathrm{E}(e^{it\widetilde{S_n}} I_A) \to \mathrm{E}(exp\{\lambda(e^{it} - 1)\} I_A).$$

Next, put $\beta(\omega, x) = P(X \leq x|\lambda)$, where X is a mixture of a Poisson distribution with random intensity λ. Define $Y'(\omega, y)$ on $(\Omega \times I, \mathcal{G} \times \mathcal{B}, P \times u)$ as in Lemma 1 of Cheng and Chow (2002), i.e. $Y'(\omega, y) = inf\{x : -\infty < x < \infty, \beta(\omega, x) \geq y\}$. Then

$$\begin{aligned}E\Big(e^{itY'(\omega,y)} \cdot I_{A\times I}\Big) &= \int_A dP \int_0^1 e^{itY'(\omega,y)dy} \\ \int_A dP \int_{-\infty}^{\infty} e^{itx} d\beta(\omega, x) &= \int_A exp\{\lambda(e^{it} - 1)\} dP \\ = \ & E[exp\{\lambda(e^{itx} - 1)\} \cdot I_A].\end{aligned}$$

By Lemma 2.1, we complete the proof. ■

Remark 2.1. It seems that the conditions of Theorem 2.1 are stronger than Theorem A. It is because that, similar to Theorem 1 in Cheng and Chow (2002), the stable convergence is a generalization of the classical distributional convergence. When $\mathcal{G} = \{\emptyset, \Omega\}$, condition (4) is satisfied automatically (see Remark 2.2 below), and in this case, Theorem A is valid as a result. The function $f_n(t)$ plays an important role in proving the stable convergence. However, when conditioned on $\mathcal{G} \neq \{\emptyset, \Omega\}$, it might not be coincident with $E(e^{it\tilde{S}_n})$ which is crucial in proving distributional convergence. It will be interesting to study the conditions implying the coincidence of $f_n(t)$ and $E(e^{it\tilde{S}_n})$ when conditioned on $\mathcal{G} \neq \{\emptyset, \Omega\}$.

In order to obtain a more complete version of SPCT, we add a condition to the original assumptions of Theorem 2.1 to ensure the SPCT for the partial sum S_n. Let $\{X_{n,k}, \mathcal{F}_{n,k}; 1 \leq k \leq m_n\}$ be any array of nonnegative integer-valued random varibles on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_{n,0}$ be a sub-σ-algebra of $\mathcal{F}_{n,1}$. Set $S_n = \sum_{k=1}^{m_n} X_{n,k}$, $\tilde{X}_{n,k} = I_{\{X_{n,k}=1\}}$, $\tilde{S}_n = \sum_{k=1}^{m_n} \tilde{X}_{n,k}$, and $f_n(t) = \prod_{k=1}^{m_n} E_{n,k-1}(e^{it\tilde{X}_{n,k}})$.

Corollary 2.1. *Let λ be a positive $\mathcal{G}$-measurable random variable, where $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. If for $n \to \infty$,*

$$(7) \qquad \sum_{k=1}^{m_n} P(X_{n,k} = 1 \,|\, \mathcal{F}_{n,k-1}) \quad \xrightarrow{p} \lambda,$$

$$(8) \qquad \max_{1 \leq k \leq m_n} P(X_{n,k} = 1 \,|\, \mathcal{F}_{n,k-1}) \quad \xrightarrow{p} 0,$$

then $f_n(t) \xrightarrow{p} exp\{\lambda(e^{it} - 1)\}$. Moreover, if $\mathcal{G} \subseteqq \bigcap_{n=1}^{\infty} \mathcal{F}_{n,0}$ and

$$(9) \qquad E(e^{it\widetilde{S_n}} - f_n(t) \,|\, \mathcal{G}) \quad \xrightarrow{p} 0,$$

$$(10) \qquad \sum_{k=1}^{m_n} P(X_{n,k} \geq 2) \quad \to 0,$$

then $S_n \xrightarrow{(s,\mathcal{G})} Z$, where where the random variable Z has characteristic function $E(exp\{\lambda(e^{it} - 1)\})$ and for $A \in \mathcal{G}$, $E(e^{itS_n} I_A) \to E(exp\{\lambda(e^{it} - 1)\} I_A)$.

Proof. Let $A_{n,k} = \{X_{n,k} = 1\}$. From (10), we have

$$\mathrm{P}(S_n \neq \widetilde{S_n}) \quad = \mathrm{P}(S_n - \widetilde{S_n} > 0) \leq \textstyle\sum_{k=1}^{m_n} \mathrm{P}(X_{n,k} \geq 2) \to 0.$$

Therefore, by Theorem 2.1, we have completed the proof. ∎

Remark 2.2. When λ is a constant, according to the Theorem 3 of Beśka(1982), we have $\widetilde{S_n} \xrightarrow{d} Poisson(\lambda)$, if $f_n(t) \xrightarrow{p} exp\{\lambda(e^{it} - 1)\}$. And by (10), $S_n \xrightarrow{d} Poisson(\lambda)$.

From Corollary 2.1 and Remark 2.2, we can obtain the following corollary concerning classical Poisson convergence theorem for arrays of independent integer-valued random variables.

Corollary 2.2. *Let $X_{n,k}$, $1 \leq k \leq m_n$, be independent nonnegative integer-valued* *random variables. If* $\lambda > 0$ *is a constant and for* $n \to \infty$*,*

$$\sum_{k=1}^{m_n} P(X_{n,k} = 1) \quad \to \lambda,$$
$$\max_{1 \le k \le m_n} P(X_{n,k} = 1) \quad \to 0,$$
$$\sum_{k=1}^{m_n} P(X_{n,k} \ge 2) \quad \to 0,$$

then

$$S_n = \sum_{k=1}^{m_n} X_{n,k} \xrightarrow{d} Poisson(\lambda).$$

The following theorem can be proven in a fashion similar to Theorem 2.1 given in Section 2.

Theorem 2.2. *Suppose there exists a sub-*σ*-algebra* $\mathcal{G} \subset \bigcap_{n=1}^{\infty} \mathcal{F}_{n,0}$ *such that* $\{A_{n,k};\ 1 \le k \le m_n,\ n \ge 1\}$ *be conditional independent given* $\mathcal{G}$ *in each row. Let* λ *be a positive* $\mathcal{G}$*-measurable random variable. If for* $n \to \infty$*,*

$$\sum_{k=1}^{m_n} P(A_{n,k} \,|\, \mathcal{G}) \quad \xrightarrow{p} \lambda, \tag{11}$$

$$\max_{1 \le k \le m_n} P(A_{n,k} \,|\, \mathcal{G}) \quad \xrightarrow{p} 0, \tag{12}$$

then $\widetilde{S_n} \xrightarrow{(s,\mathcal{G})} Z$*, where the random variable Z has characteristic function* $E(exp\{\lambda(e^{it} - 1)\})$ *and for* $A \in \mathcal{G}$*,* $E(e^{it\widetilde{S_n}} I_A) \to E(exp\{\lambda(e^{it} - 1)\} I_A)$*.*

Proof. By the same way of Theorem 2.1, we have

$$\prod_{k=1}^{m_n} \mathrm{E}(e^{it\widetilde{X_{n,k}}} \,|\, \mathcal{G}) \xrightarrow{p} exp\{\lambda(e^{it} - 1)\},$$

which implies that for any $A \in \mathcal{G}$,

$$\prod_{k=1}^{m_n} \mathrm{E}(e^{it\widetilde{X_{n,k}}} \,|\, \mathcal{G}) \cdot I_A \xrightarrow{p} exp\{\lambda(e^{it} - 1)\} \cdot I_A.$$

Due to the property of conditional independence,

$$\mathrm{E}(e^{it\widetilde{S_n}} | \,\mathcal{G}) = |\, \mathcal{G}) = \cdot \prod_{k=1}^{m_n} \mathrm{E}(e^{it\widetilde{X_{n,k}}} \,|\, \mathcal{G}).$$

Hence,

$$(e^{it\widetilde{S_n}} I_A |\, \mathcal{G}) \xrightarrow{p} exp\{\lambda(e^{it}-1)\} \cdot I_A.$$

Since $\{\mathrm{E}(e^{it\widetilde{S_n}} |\, \mathcal{G}); n \geq 1\}$ is uniformly integrable, for $A \in \mathcal{G}$

$$\mathrm{E}(e^{it\widetilde{S_n}} \cdot I_A) \to \mathrm{E}(exp\{\lambda(e^{it}-1)\} \cdot I_A). \qquad \blacksquare$$

3. Applications

Let $\{A_{n,k},\ k = 1, 2, \ldots, m_n,\ n \geq 1\}$ be an array of row-exchangeable events. Set $S_{n,j} = \sum_{k=1}^{j} I_{A_{n,k}}$, $\mathcal{F}_{n,0} = \sigma(\sum_{k=1}^{m_n} I_{A_{n,k}})$, and $\mathcal{F}_{n,j} = \sigma(I_{A_{n,1}}, I_{A_{n,2}}, \ldots, I_{A_{n,j}}, \sum_{k=j+1}^{m_n} I_{A_{n,k}})$, for $j = 1, 2, \ldots, m_n$. Let $f_n(t) = \prod_{k=1}^{n} \mathrm{E}_{n,k-1}(e^{itI_{A_{n,k}}})$. Similar to Eagleson (1979), we may obtain the following results.

Theorem 3.3. *Let $\mathcal{G}$ be a sub-σ-algebra of $\bigcap_{n=1}^{\infty} \mathcal{F}_{n,0}$, and λ a positive $\mathcal{G}$-measurable random variable. If for $n \to \infty$,*

$$n \cdot P(A_{n,1} \,|\, \mathcal{G}) \xrightarrow{L_1} \lambda, \tag{13}$$

$$n^2 \cdot P(A_{n,1} \,|\, \mathcal{G})^2 - n^2 \cdot P(A_{n,1} \cap A_{n,2} \,|\, \mathcal{G}) \xrightarrow{L_1} 0, \tag{14}$$

$$n \cdot P(A_{n,1} A_{n,2}^c) \to 0, \tag{15}$$

$$m_n / n \to \infty, \tag{16}$$

then $S_{n,n} \xrightarrow{(s,\mathcal{G})} Z$, where the random variable Z has characteristic function $E(exp\{\lambda(e^{it}-1)\})$ and for $A \in \mathcal{G}$, $E(e^{itS_{n,n}} I_A) \to E(exp\{\lambda(e^{it}-1)\} I_A)$.

Proof. By (13), we have

$$n\mathrm{P}(A_{n,1}) \to \mathrm{E}(\lambda) < \infty \tag{17}$$

which yields

$$\mathrm{P}(A_{n,1}) \to 0. \tag{18}$$

Similarly, we have

$$\mathrm{E}[n^2 \cdot \mathrm{P}(A_{n,1} \,|\, \mathcal{G})^2 - n^2 \cdot \mathrm{P}(A_{n,1} \cap A_{n,2} \,|\, \mathcal{G})] \to 0,$$

which implies

$$n^2 \cdot \mathrm{E}(I_{A_{n,1}} \cdot \mathrm{P}(A_{n,1} \,|\, \mathcal{G})) - n^2 \cdot \mathrm{P}(A_{n,1} \cap A_{n,2}) \to 0. \tag{19}$$

Fix $n, j, \quad 1 \leq j \leq m_n$.

By exchangeability, for any $B \in \mathcal{F}_{n,j-1}$, and each $k, j \le k \le m_n$,

$$\mathrm{E}(I_{A_{n,j}} \cdot I_B) = \mathrm{E}(I_{A_{n,k}} \cdot I_B). \tag{20}$$

Hence, for any $B \in \mathcal{F}_{n,j-1}$,

$$\mathrm{E}(I_{A_{n,j}} \cdot I_B) = \mathrm{E}(\frac{\sum_{k=j}^{m_n} I_{A_{n,k}}}{m_n - j + 1} \cdot I_B).$$

And $\frac{\sum_{k=j}^{m_n} I_{A_{n,k}}}{m_n-j+1}$ is $\mathcal{F}_{n,j-1}$-measurable, we have

$$\mathrm{E}(I_{A_{n,j}} \,|\, \mathcal{F}_{n,j-1}) = \frac{\sum_{k=j}^{m_n} I_{A_{n,k}}}{m_n - j + 1}.$$

Similarly, for each t with $j \le t \le m_n$,

$$\mathrm{P}(A_{n,t} \,|\, \mathcal{F}_{n,j-1}) = \frac{\sum_{k=j}^{m_n} I_{A_{n,k}}}{m_n - j + 1}. \tag{21}$$

In particular, for each j with $1 \le j \le m_n$,

$$\mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) = \mathrm{P}(A_{n,m_n} \,|\, \mathcal{F}_{n,j-1}). \tag{22}$$

Moreover, for each t with $1 \le t \le m_n$,

$$\begin{aligned} \mathrm{P}(A_{n,1} \,|\, \mathcal{G}) &= \mathrm{E}(\mathrm{P}(A_{n,j} \,|\, \mathcal{G}) \,|\, \mathcal{F}_{n,0}) = \mathrm{E}(\mathrm{P}(A_{n,1} \,|\, \mathcal{F}_{n,0}) \,|\, \mathcal{G}) \\ &= E(\mathrm{P}(A_{n,t} \,|\, \mathcal{F}_{n,0}) \,|\, \mathcal{G}) = \mathrm{P}(A_{n,t} \,|\, \mathcal{G}). \end{aligned} \tag{23}$$

Now, we want to claim that $\max_{1 \le j \le n} \mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) \xrightarrow{p} 0$.
Since for each $n \ge 1$, $\{\mathrm{P}(A_{n,m_n} \,|\, \mathcal{F}_{n,j-1}), \mathcal{F}_{n,j-1}, 1 \le j \le n\}$ is a martingale.
Hence, by (21), (22) and Doob's inequality, for any $\varepsilon > 0$,

$$\begin{aligned} \mathrm{P}(\max_{1 \le j \le n} \mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) > \varepsilon) &= \mathrm{P}(\max_{1 \le j \le n} \mathrm{P}(A_{n,m_n} \,|\, \mathcal{F}_{n,j-1}) > \varepsilon) \\ &\le \frac{\mathrm{E}[\mathrm{P}(A_{n,m_n} \,|\, \mathcal{F}_{n,n-1})]}{\varepsilon} \\ &= \frac{\mathrm{P}(A_{n,m_n})}{\varepsilon} = \frac{\mathrm{P}(A_{n,1})}{\varepsilon} \to 0 \end{aligned}$$

Next, claim that $\sum_{j=1}^{n} \mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) \xrightarrow{p} \lambda$. By (13), we only need to show that $\sum_{j=1}^{n} \mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) - n \cdot \mathrm{P}(A_{n,1} \,|\, \mathcal{G}) \xrightarrow{p} 0$.

Since for each j with $1 \leq j \leq n$, by (13), (14), (15) and Jensen's inequality, we have

$$\begin{aligned}
&(\mathrm{E}|\mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) - \mathrm{P}(A_{n,1} \,|\, \mathcal{G})|)^2 = (\mathrm{E}|\mathrm{P}(A_{n,m_n} \,|\, \mathcal{F}_{n,j-1}) - \mathrm{P}(A_{n,m_n} \,|\, \mathcal{G})|)^2 \\
\leq\ & \mathrm{E}|\mathrm{P}(A_{n,m_n} \,|\, \mathcal{F}_{n,j-1}) - \mathrm{P}(A_{n,m_n} \,|\, \mathcal{G})|^2 \\
=\ & \mathrm{E}(I_{A_{n,m_n}} \cdot \mathrm{P}(A_{n,m_n} \,|\, \mathcal{F}_{n,j-1})) - \mathrm{E}(I_{A_{n,m_n}} \cdot \mathrm{P}(A_{n,m_n} \,|\, \mathcal{G})) \\
=\ & \mathrm{E}(I_{A_{n,m_n}} \cdot \frac{\sum_{k=j}^{m_n} I_{A_{n,k}}}{m_n - j + 1}) - \mathrm{E}(I_{A_{n,m_n}} \cdot \mathrm{P}(A_{n,m_n} \,|\, \mathcal{G})) \\
\leq\ & \frac{\mathrm{P}(A_{n,1})}{m_n - j + 1} + \mathrm{P}(A_{n,1} \cap A_{n,2}) - \mathrm{E}(I_{A_{n,m_n}} \cdot \mathrm{P}(A_{n,m_n} \,|\, \mathcal{G})),
\end{aligned}$$

which implies

$$\begin{aligned}
&\mathrm{E}|\mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) - \mathrm{P}(A_{n,1} \,|\, \mathcal{G})| \\
\leq\ & [\tfrac{\mathrm{P}(A_{n,1})}{m_n - j + 1}]^{\frac{1}{2}} + [\mathrm{P}(A_{n,1} \cap A_{n,2}) - \mathrm{E}(I_{A_{n,m_n}} \cdot \mathrm{P}(A_{n,m_n} \,|\, \mathcal{G}))]^{\frac{1}{2}}.
\end{aligned}$$

Moreover, by (13)-(16), we have

$$\begin{aligned}
&\sum_{j=1}^{n} \mathrm{E}|\mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) - \mathrm{P}(A_{n,1} \,|\, \mathcal{G})| \leq [n \cdot \mathrm{P}(A_{n,1})]^{\frac{1}{2}} \cdot [\frac{n}{m_n - n}]^{\frac{1}{2}} \\
&+[n^2 \cdot \mathrm{P}(A_{n,1} \cap A_{n,2}) - n^2 \cdot \mathrm{E}(I_{A_{n,m_n}} \cdot \mathrm{P}(A_{n,m_n} \,|\, \mathcal{G}))]^{\frac{1}{2}} \\
&\longrightarrow \mathrm{E}(\lambda) \cdot 0 + 0 = 0.
\end{aligned}$$

Hence, for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathrm{P}(|\sum_{j=1}^{n} \mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) - n \cdot \mathrm{P}(A_{n,1} \,|\, \mathcal{G})| > \varepsilon) \\
\leq\ & \frac{1}{\varepsilon} \cdot \mathrm{E}|\sum_{j=1}^{n} [\mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) - \mathrm{P}(A_{n,1} \,|\, \mathcal{G})]| \\
\leq\ & \frac{1}{\varepsilon} \cdot \sum_{j=1}^{n} \mathrm{E}|\mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) - \mathrm{P}(A_{n,1} \,|\, \mathcal{G})| \longrightarrow 0.
\end{aligned}$$

Therefore, $\sum_{j=1}^{n} \mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) \xrightarrow{p} \lambda$.

Next, since for any fixed $n, 1 \leq k \leq n$,

$$
\begin{aligned}
&|I_{A_{n,k}} - \mathrm{E}(I_{A_{n,k}} \,|\, \mathcal{F}_{n,k-1})| \\
&= I_{A_{n,k}} - \frac{1}{m_n - k + 1} \sum_{j=k}^{m_n} I_{A_{n,j}} \cdot I_{A_{n,k}} + \frac{1}{m_n - k + 1} \sum_{j=k}^{m_n} I_{A_{n,j}} \cdot I_{A^c_{n,k}} \\
&= \frac{1}{m_n - k + 1} \sum_{j=k+1}^{m_n} (I_{A_{n,k}} - I_{A_{n,j}} \cdot I_{A_{n,k}} + I_{A_{n,j}} \cdot I_{A^c_{n,k}}) \\
&= \frac{1}{m_n - k + 1} [\sum_{j=k+1}^{m_n} (I_{A_{n,k}} \cdot I_{A^c_{n,j}} + I_{A_{n,j}} \cdot I_{A^c_{n,k}})],
\end{aligned}
$$

we have for $n \to \infty$, by (15),

$$
\begin{aligned}
&\sum_{k=1}^{n} \mathrm{E}|I_{A_{n,k}} - \mathrm{E}(I_{A_{n,k}} \,|\, \mathcal{F}_{n,k-1})| \\
&= 2 \cdot \sum_{k=1}^{n} \frac{m_n - k}{m_n - k + 1} \cdot \mathrm{P}(A_{n,1} A^c_{n,2}) \le 2 \cdot \sum_{k=1}^{n} \mathrm{P}(A_{n,1} A^c_{n,2}) = 2n \cdot \mathrm{P}(A_{n,1} A^c_{n,2}) \to 0.
\end{aligned}
$$

Note that for any fixed $n, 1 \le k \le n$,

$$
\begin{aligned}
&|I_{A^c_{n,k}} - \mathrm{E}_{n,k-1}(I_{A^c_{n,k}})| = |1 - I_{A_{n,k}} - 1 + \mathrm{E}_{n,k-1}(I_{A_{n,k}})| \\
&= |\mathrm{E}_{n,k-1}(I_{A_{n,k}}) - I_{A_{n,k}}| = |I_{A_{n,k}} - \mathrm{E}_{n,k-1}(I_{A_{n,k}})|.
\end{aligned}
$$

Therefore we have for $n \to \infty$,

$$
\begin{aligned}
&\mathrm{E}(\sum_{k=1}^{n} |e^{itI_{A_{n,k}}} - \mathrm{E}_{n,k-1}(e^{itI_{A_{n,k}}})|) \\
&= \sum_{k=1}^{n} \mathrm{E}|e^{it} \cdot I_{A_{n,k}} + I_{A^c_{n,k}} - e^{it} \cdot \mathrm{E}_{n,k-1}(I_{A_{n,k}}) - \mathrm{E}_{n,k-1}(I_{A^c_{n,k}})| \\
&\le \sum_{k=1}^{n} \mathrm{E}|I_{A_{n,k}} - \mathrm{E}_{n,k-1}(I_{A_{n,k}})| + \sum_{k=1}^{n} \mathrm{E}|I_{A^c_{n,k}} - \mathrm{E}_{n,k-1}(I_{A^c_{n,k}})| \to 0.
\end{aligned}
$$

So, for $n \to \infty$,

$$
\begin{aligned}
\mathrm{P}\{|\mathrm{E}(e^{itS_{n,n}} - f_n(t) \,|\, \mathcal{G})| > \varepsilon\} &\le \mathrm{P}\{\mathrm{E}(|e^{itS_{n,n}} - f_n(t)| \,|\, \mathcal{G}) > \varepsilon\} \\
&\le \frac{1}{\varepsilon} \cdot \mathrm{E}(|e^{itS_{n,n}} - f_n(t)|) \\
&\le \frac{1}{\varepsilon} \cdot \mathrm{E}(\sum_{k=1}^{n} |e^{itI_{A_{n,k}}} - \mathrm{E}_{n,k-1}(e^{itI_{A_{n,k}}})|) \to 0.
\end{aligned}
$$

By Theorem 2.1, we have completed the proof. ∎

Let $\{A_{n,k}\}_{k\geq 1}$ be an array of rowise exchangeable events. Set $S_{n,j} = \sum_{k=1}^{j} I_{A_{n,k}}$, $\mathcal{G}_{n,j} = \sigma(\sum_{k=1}^{j} I_{A_{n,k}}, I_{A_{n,j+1}}, I_{A_{n,j+2}}, \ldots), j \geq 1$, and $\mathcal{G}_{n,\infty} = \bigcap_{j=1}^{\infty} \mathcal{G}_{n,j}$, $\mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_{n,\infty}$. Similar to Weber(1980), we may obtain the following results.

Theorem 3.4. *If there exists a $\mathcal{G}$-measurable integrable random variable λ such that for $n \to \infty$,*

$$n \cdot P(A_{n,1} \,|\, \mathcal{G}_{n,\infty}) \xrightarrow{p} \lambda, \tag{24}$$

$$n \cdot P(A_{n,1} A_{n,2}^c) \to 0, \tag{25}$$

then $S_{n,n} \xrightarrow{(s,\mathcal{G})} Z$, where the random variable Z has characteristic function $E(exp\{\lambda(e^{it} - 1)\})$ and for $A \in \mathcal{G}$, $E(e^{itS_{n,n}} I_A) \to E(exp\{\lambda(e^{it} - 1)\} I_A)$.

Proof. Fix n, j. By exchangeability, for any $B \in \mathcal{G}_{n,j}$, and each $k, 1 \leq k \leq j$,

$$\mathrm{E}(I_{A_{n,1}} \cdot I_B) = \mathrm{E}(I_{A_{n,k}} \cdot I_B).$$

Hence, for any $B \in \mathcal{G}_{n,j}$,

$$\mathrm{E}(I_{A_{n,1}} \cdot I_B) = \mathrm{E}(\frac{\sum_{k=1}^{j} I_{A_{n,k}}}{j} \cdot I_B).$$

And $\frac{\sum_{k=1}^{j} I_{A_{n,k}}}{j}$ is $\mathcal{G}_{n,j}$-measurable, we have

$$\mathrm{P}(A_{n,1} \,|\, \mathcal{G}_{n,j}) = \frac{\sum_{k=1}^{j} I_{A_{n,k}}}{j}. \tag{26}$$

For each n, since $\mathcal{G}_{n,j} \supseteq \mathcal{G}_{n,j+1}$, then $\{\mathrm{P}(A_{n,1} \,|\, \mathcal{G}_{n,j}), \mathcal{G}_{n,j}, n \geq 1\}$ is a reversed martingale, and by the reversed martingale convergence theorem , as $j \to \infty$,

$$\mathrm{P}(A_{n,1} \,|\, \mathcal{G}_{n,j}) \xrightarrow{\mathcal{L}_2} \mathrm{P}(A_{n,1} \,|\, \mathcal{G}_{n,\infty}). \tag{27}$$

Next, for some $m \in \mathrm{N}$, we consider σ-field

$$\mathcal{F}_{n,j-1} \equiv \sigma(I_{A_{n,1}}, I_{A_{n,2}}, \ldots, I_{A_{n,j-1}}, \sum_{k=j}^{m} I_{A_{n,k}}),$$

then for fixed n and j, by(26) and (27), as $m \longrightarrow \infty$,

$$\begin{aligned}
\mathrm{P}(A_{n,j} \,|\, \mathcal{F}_{n,j-1}) &= \frac{\sum_{k=j}^{m} I_{A_{n,k}}}{m - j + 1} \\
&= \frac{(\sum_{k=1}^{m} I_{A_{n,k}} - \sum_{k=1}^{j-1} I_{A_{n,k}})}{m - j + 1} \\
&= \frac{m}{m - j + 1} \cdot \mathrm{P}(A_{n,1} \,|\, \mathcal{G}_{n,m}) - \frac{j - 1}{m - j + 1} \cdot \mathrm{P}(A_{n,1} \,|\, \mathcal{G}_{n,j-1}) \\
&\xrightarrow{\mathcal{L}_1} \mathrm{P}(A_{n,1} \,|\, \mathcal{G}_{n,\infty}),
\end{aligned}$$

which in turn implies for each n, as $m \longrightarrow \infty$,

$$\sum_{j=1}^{n} \mathrm{P}(A_{n,j} \mid \mathcal{F}_{n,j-1}) \xrightarrow{\mathcal{L}_1} n \cdot \mathrm{P}(A_{n,1} \mid \mathcal{G}_{n,\infty}).$$

Thus, for each n, there exists $m_n \in \mathrm{N}$ such that for any $m \geq m_n \geq n$, we have

$$\mathrm{E}(|\sum_{j=1}^{n} \mathrm{P}(A_{n,j} \mid \mathcal{F}_{n,j-1}) - n \cdot \mathrm{P}(A_{n,1} \mid \mathcal{G}_{n,\infty})|) < \frac{1}{n}.$$

Hence, for each n, we can choose σ-fields $\mathcal{F}^*_{n,j-1} \equiv \sigma(I_{A_{n,1}}, I_{A_{n,2}}, \ldots, I_{A_{n,j-1}}, \sum_{k=j}^{m_n} I_{A_{n,k}})$, $1 \leq j \leq n$, such that

$$\sum_{j=1}^{n} \mathrm{P}(A_{n,j} \mid \mathcal{F}^*_{n,j-1}) - n \cdot \mathrm{P}(A_{n,1} \mid \mathcal{G}_{n,\infty}) \xrightarrow{\mathcal{L}_1} 0.$$

So, by (24),

$$\sum_{j=1}^{n} \mathrm{P}(A_{n,j} \mid \mathcal{F}^*_{n,j-1}) \xrightarrow{p} \lambda.$$

And, in the same way as Theorem 3.1, we have $\max_{1 \leq j \leq n} \mathrm{P}(A_{n,j} \mid \mathcal{F}^*_{n,j-1}) \xrightarrow{p} 0$.

Taking $\mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{F}^*_{n,0}$ and $f_n(t) = \prod_{k=1}^{n} \mathrm{E}(e^{itI_{A_{n,k}}} \mid \mathcal{F}^*_{n,k-1})$,
we also have

$$\mathrm{E}(e^{itS_{n,n}} - f_n(t) \mid \mathcal{G}) \xrightarrow{p} 0.$$

Then, by Theorem 2.1, we complete the proof. ∎

References

1. R. J. Adler and D. J. Scott, Martingale central limit theorems without negligibility conditions, *Corrigendum. Bull. Austral. Math. Soc.*, **18** (1978), 311-319.

2. D. J. Aldous and G. K. Eagleson, On mixing and stability of limit theorems, *Ann. Probab.*, **6** (1978), 325-331.

3. D. J. Aldous, *Exchangeability and Related Topics*, Lecture Notes in Mathematics, Springer-Verlag, 1983.

4. A. D. Barbour and L. Holst, Some applications of the Stein-Chen method for proving Poisson convergence, *Adv. Appl. Prob.*, **21** (1989), 74-90.

5. S. Bernstein, Sur l'extension du théoréme limite du calcul des probabilitiés aux sommes de quantités dépendantes, *Math. Ann.*, **85** (1927), 1-59.

6. M. Beška, A. Klopotowski and L. Slomišski, Limit Theorems for Random Sums of Dependent d-Dimensional Random Vectors, *Z. Wahrscheinlichkeitsth*, **61** (1982), 43-57.

7. P. Billingsley, The Lindeberg-Lévy theorem for martingales, *Proc. Amer. Math. Soc.*, **12** (1961b), 788-792.

8. B. Brown, A martingale approach to the Poisson convergence of simple point processes, *Ann. Prob.*, **6** (1978), 615-628.

9. B. Brown and G. K. Eagleson, Martingale convergence to infinitely divisible laws with finite variance, *Trans. Amer. Math. Soc.*, **162** (1971), 449-453.

10. B. Brown, Martingale central limit theorems, *Ann. Math. Statist.*, **42** (1971), 59-66.

11. L. H. Y. Chen, On the convergence of Poisson binomial to Poisson distributions, *Ann. Prob.*, **2** (1974), 178-180.

12. L. H. Y. Chen, Poisson approximation for dependent trials, *Ann. Prob.*, **3** (1975), 534-545.

13. T. L. Cheng and Y. S. Chow, On stable convergence in the central limit theorem, *Statist. Probab. Lett.*, **57** (2002), 307-313.

14. Y. S. Chow and H. Teicher, *Probability Theory*, 3rd Edition, Springer, Berlin, 1997.

15. J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.

16. R. Durrent, *Probability: Theory and Examples*, Wadsworth and Brooks/Cole, Belmont, CA, 1991.

17. A. Dvoretzky, Central limit theorems for dependent random variables and some applications, *Abstract. Ann. Math. Statist*, **40** (1969), 1871.

18. A. Dvoretzky, The central limit problem for dependent random variables. in: *Actes du Congres International deś Mathematiciens, Nice*, 565-570, Gauthier-Villars, Paris, 1971.

19. A. Dvoretzky, *Asymptotic normality for sums of dependent random variables*, Proc. Sixth Berkeley Symp. Math. Statist, Probability, Univ. of California Pess, 1972, pp. 513-535.

20. G. K. Eagleson, Martingale convergence to the Poisson distribution, *Casopis Pest. Mat.*, **101** (1976), 271-277.

21. G. K. Eagleson, A Poisson Limit Theorem for Weakly Exchangeable Events, *J. Appl. Prob.*, **16** (1979), 794-802.

22. G. K. Eagleson, Weak limit theorems for exchangeable random variables, in: *Exchangeability in Probability and Statistics*, G. Koch and F. Spizzichino, eds., North-Holland, Amsterdam-New York, 1982, pp. 251-268.

23. D. Freedman, The Poisson approximation for dependent events, *Ann. Prob.*, **2** (1974), 256-269.

24. P. Hall and C. C. Heyde, *Martingale Limit Theory and Its Application*, Academic Press, New York, 1980.

25. I. S. Helland, Central limit theorems for martingales with discrete or continuous time, *Scand. J. Statist*, **9** (1982), 79-94.

26. I. A. Ibragimov, A central limit theorem for a class of dependent random variables, *Theory Probab. Appl.*, **8** (1963), 83-89.

27. N. Kaplan, Two applications of a Poisson approximation for dependent events, *Ann. Prob.*, **5** (1977), 787-794.

28. R. G. Laha and V. K. Rohagi, *Probability Theory*, 1970.

29. P. Lévy, Propriétés asymptotiques des sommes de variables aléatoires enchainées, *Bull. Sci. Math.*, **59**(ser.2) (1935a), 84-96, 109-128.

30. P. Lévy, Propriétés asymptotiques des sommes de variables aléatoires independantes ou enchainées, *J. Math. Pures Appl.*, bf 14(ser.9) (1935b), 347-402.

31. P. Lévy, *Théorie de l'adition des Variables Aléatoires*, Gauthier-Villars, Paris, 1937.

32. R. M. Loynes, The central limit theorem for backwards martingales, *Z. Wahrsch. Verw. Gebiete.*, **13** (1969), 1-8.

33. D. L. McLeish, Dependent central limit theorem and invariance principles, *Ann. Probab.*, **2** (1974), 620-628.

34. A. Rényi, On stable sequences of events, *Sankhya Ser. A*, **25** (1963), 293-302.

35. A. Rényi, *Foundations of Probability*, Holden-Day, San Francisco, 1970.

36. B. Rosén, On the central limit theorem for sums of dependent random variables, *Z. Wahrsch. Verw. Gebiete.*, **7** (1967a), 48-52.

37. Rosén, B. On asymptotic normality of sums of dependent random variables, *Z. Wahrsch. Verw. Gebiete*, **7** (1967b), 95-102.

38. A. N. Shiryayev, *Probability*, Springer, Berlin, New York, 1984.

39. A. G. Sholomitskii, On the necessary conditions of normal convergence for martingales, *Theory Probab. Appl.*, **43** (1998), 434-448.

40. A. G. Sholomitskii, On the necessary conditions of poisson convergence for martingales, *Theory Probab. Appl.*, **49** (2005), 735-737.

41. B. W. Silverman and T. C. Brown, Short distances, flat triangles and Poisson limits, *J. Appl. Prob.*, **15** (1978), 815-825.

42. N. C. Weber, A Martingale Approach to Central Limit Theorems for Exchangeable Random Variables, *J. Appl. Prob.*, **17** (1980), 662-673.

43. J. Wesollowski, Poisson Process via Martingale and Related Characteristics, *J. Appl. Prob.*, **36** (1999), 919-926.

Tsung-Lin Cheng and Shun-Yi Yang
Department of Mathematics
National Changhua University of Education
Changhua 500, Taiwan
E-mail: tlcheng@cc.ncue.edu.tw