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STATISTICAL CONVERGENCE OF GENERALIZED DIFFERENCE SEQUENCE SPACE OF FUZZY NUMBERS

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Abstract. In this paper, we introduce and study the concept of Δ^m-summable sequence of fuzzy numbers by using a modulus function and Δ^m-statistical convergence of sequences fuzzy numbers. Also we have defined Δ^m-statistical pre-Cauchy sequences of fuzzy numbers.

1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [31] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as topological spaces, similarity relations and fuzzy orderings, fuzzy mathematical programming. Matloka [18] introduced bounded and convergent sequences of fuzzy numbers. Later on Nanda [20], Nuray and Savaş [21], Mursaleen and Basarır [19], Savaş [26], Raj et al. [22, 23, 24], Tripathy and Sarma [29] and several authors studied the sequence spaces in an analogous way as Simons [27], Maddox [16], Kızmaz [13], Et and Çolak [9] and several authors studied for scalar valued sequence spaces.

The notion of statistical convergence was introduced by Fast [11] and Schoenberg [28] independently. Fast introduced the idea of statistical convergence of real or complex numbers and Schoenberg [28] studied statistical convergence as a summability method and listed some of the properties of statistical convergence. From the point of view of sequence spaces, this concept has been generalized and developed by Fridy [12], *Šalát* [25], Connor [6], Connor et al. [7], Et and Nuray [10] and many others.

The existing literature on statistical convergence appears to have been restricted to real or complex sequences, but Nuray and Savaş [21] extended the idea to apply to

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sequences of fuzzy numbers. Later on, Aytar and Pehlivan [2], Bilgin [5], Çolak et al. [8], Kwon [14], Tripathy and Baruah [30] and many authors extended the idea of statistical convergence to the sequences of fuzzy numbers.

In this paper, we introduce and study the concept of strongly Δ^m-summable sequence of fuzzy numbers by using a modulus function and Δ^m-statistical convergence of sequences of fuzzy numbers. Also we have discussed Δ^m-statistical pre-Cauchy sequences of fuzzy numbers.

2. Definitions and Preliminaries

Definition 2.1. A fuzzy number is a fuzzy set on the real axis, i.e. a mapping $X : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions:

(i) X is normal, i.e. there exists an $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$.

(ii) X is fuzzy convex, i.e. $X(\lambda s + (1 - \lambda)t) \geq \min\{X(s), X(t)\}$ for all $s, t \in \mathbb{R}$ and for all $\lambda \in [0, 1]$.

(iii) X is upper semi-continuous.

(iv) The set $[X]^\alpha = \overline{\{t \in \mathbb{R} : X(t) > 0\}}$ is compact, where $\overline{\{t \in \mathbb{R} : X(t) > 0\}}$ denotes the closure of the set $\{t \in \mathbb{R} : X(t) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers by $L(\mathbb{R})$.

Definition 2.2. The set $L(\mathbb{R})$ forms a linear space under addition and scalar multiplication in terms of $\alpha-$ level sets as defined below:

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha \text{ and } [\lambda X]^\alpha = \lambda[X]^\alpha \text{ for each } 0 \leq \alpha \leq 1.$$

where X^α is given as

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha & \text{if } \alpha \in (0, 1] \\ t : X(t) > 0 & \text{if } \alpha = 0 \end{cases}$$

For each $\alpha \in [0, 1]$, the set X^α is a closed, bounded and nonempty interval of $\mathbb{R}$.

Let D denote the set of all closed and bounded intervals $A = [a_1, a_2]$ on the real line $\mathbb{R}$. For $A, B \in D$, (D, d) is a complete metric space where the metric d is defined as

$$d(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$$

for any $A = [a_1, a_2]$ and $B = [b_1, b_2]$.

It is easy to verify that $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ be defined by

$$\overline{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

is a metric on $L(\mathbb{R})$.

Definition 2.3. A metric $\overline{d}$ on $L(\mathbb{R})$ is said to be translation invariant if $\overline{d}(X + Z, Y + Z) = \overline{d}(X, Y)$ for all $X, Y, Z \in L(\mathbb{R})$.

Definition 2.4. Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ is said to be $\Delta-$bounded if the set $\{\Delta X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded.

Definition 2.5. Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ is said to be $\Delta-$convergent to the fuzzy number X_0, written as $\lim_{k\to\infty} \Delta X_k = X_0$, if for every $\epsilon > 0$ there exists a positive integer k_0 such that $\overline{d}(\Delta X_k, X_0) < \epsilon$ for all $k > k_0$.

Definition 2.6. Let $X = (X_k)$ be a sequence of fuzzy numbers. Then the sequence $X = (X_k)$ is said to be Δ^m-convergent to the fuzzy number X_0, written as $\lim_{k\to\infty} \Delta^m X_k = X_0$, if for every $\epsilon > 0$ there exists a positive integer k_0 such that $\overline{d}(\Delta^m X_k, X_0) < \epsilon$ for all $k > k_0$.

Definition 2.7. A metric $\overline{d}$ on $L(\mathbb{R})$ is said to be translation invariant if $\overline{d}(X + Z, Y + Z) = \overline{d}(X, Y)$ for all $X, Y, Z \in L(\mathbb{R})$.

Lemma 2.1. (Basarır and Mursaleen [3]). *If $\overline{d}$ is a translation invariant metric on $L(\mathbb{R})$, then*

(*i*) $\overline{d}(X + Y, \overline{0}) \leq \overline{d}(X, \overline{0}) + \overline{d}(Y, \overline{0})$

(*ii*) $\overline{d}(\lambda X, \overline{0}) \leq |\lambda|\ \overline{d}(X, \overline{0}),\ |\lambda| > 1$

Lemma 2.2. (Maddox [15]). *Let a_k, b_k for all k be sequences of complex numbers and (p_k) be a bounded sequence of positive real numbers, then*

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k})$$

and

$$|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$$

where $C = \max(1, 2^{H-1}), H = \sup p_k$ and λ is any complex number.

3. Main Results

Let $(E_k, \overline{d_k})$ be a sequence of fuzzy linear metric spaces under the translation invariant metrices $\overline{d_k}$'s such that $E_{k+1} \subseteq E_k$ for each $k \in \mathbb{N}$ where $X_k = ((X_{k,s})_{s=1}^{\infty}) \in E_k$ for each $k \in \mathbb{N}$. We define $W(E) = \{X = (X_k) : X_k \in E_k$ for each $k \in \mathbb{N}\}$. It is easy to verify that the space $W(E)$ is a linear space of fuzzy numbers under coordinatewise addition and scalar multiplication. For $X = (X_k) \in W(E)$ and $\lambda = (\lambda_k)$ a sequence of real numbers, we define $\lambda X = (\lambda_k X_k)$.

Let f be a modulus function and $p = (p_k)$ is a bounded sequence of strictly positive real numbers. Then we define the following sequence space

$$w^F(\Delta^m, f, p) = \Big\{X = (X_k) \in W(E) : \frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} \to 0 \text{ as } n \to \infty\Big\}$$

where

$$\Delta^m X_{k,s} = \sum_{i=0}^{m}(-1)^i \begin{pmatrix} m \\ i \end{pmatrix} X_{k+i,s}.$$

Theorem 3.1. *Let (p_k) be a bounded sequence of positive real numbers. Then $w^F(\Delta^m, f, p)$ is a linear space over $\mathbb{R}$.*

Proof. Using Lemma 2.1, Lemma 2.2, the subadditivity property of modulus function f and the result $f(\lambda x) \le (1+[|\lambda|])f(x)$, it is easy to show that $w^F(\Delta^m, f, p)$ is a linear space over the real field $\mathbb{R}$.

Theorem 3.2. *Let $(E_k, \overline{d_k})$ be a sequence of complete metric spaces and (p_k) be a bounded sequence of positive real numbers such that* $\inf p_k > 0$. *Then the sequence space $w^F(\Delta^m, f, p)$ is a complete metric space with respect to the metric*

$$g(X,Y)=\sum_{i=1}^{m} f(\sup_k \overline{d_k}(X_{k,i}, Y_{k,i}))+\sup_n \Big(\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, \Delta^m Y_{k,s}))\Big)^{p_k}\Big)^{\frac{1}{M}}$$

Proof. Let $(X^{(u)})$ be a Cauchy sequence in $w^F(\Delta^m, f, p)$, where $X^{(u)} = \Big((X^{(u)}_{k,s})_{s=1}^{\infty}\Big)_{k=1}^{\infty} \in w^F(\Delta^m, f, p)$ for each $u \in \mathbb{N}$. Then

$$g(X^{(u)}, X^{(v)}) \to 0 \text{ as } u, v \to \infty.$$

i.e.

$$\sum_{i=1}^{m} f(\sup_k \overline{d_k}(X^{(u)}_{k,i}, X^{(v)}_{k,i})) + \sup_n \Big(\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X^{(u)}_{k,s}, \Delta^m X^{(v)}_{k,s}))\Big)^{p_k}\Big)^{\frac{1}{M}} \to 0 \text{ as } u, v \to \infty.$$

Which implies

$$\sum_{i=1}^{m} f(\sup_k \overline{d_k}(X^{(u)}_{k,i}, X^{(v)}_{k,i})) \to 0 \text{ as } u, v \to \infty. \tag{3.1}$$

and

$$\sup_n \Big(\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u)}, \Delta^m X_{k,s}^{(v)}))\Big)^{p_k}\Big)^{\frac{1}{M}} \to 0 \text{ as } u,v\to\infty. \tag{3.2}$$

From equation (3.1),

$$f(\sup_k \overline{d_k}(X_{k,i}^{(u)}, X_{k,i}^{(v)})) \to 0 \text{ as } u,v\to\infty \text{ for each } i=1,2,\dots,m.$$

But f is a modulus function, so we have $\sup_k \overline{d_k}(X_{k,i}^{(u)}, X_{k,i}^{(v)}) \to 0$ as $u,v\to\infty$ for each $i=1,2,\dots,m$. i.e.

$$(X_{k,i}^{(u)}) \text{ is a Cauchy sequence in } E_i \text{ for each } i=1,2,\dots,m. \tag{3.3}$$

Again, from equation (3.2), since f is a modulus function, we have

$$\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u)}, \Delta^m X_{k,s}^{(v)}) \to 0 \text{ as } u,v\to\infty \text{ and for each } s=1,2,\dots,n.$$

i.e.

$$(\Delta^m X_{k,s}^{(u)}) \text{ is a Cauchy sequence in } E_k \text{ for each } k\in\mathbb{N}. \tag{3.4}$$

Now $(X_{k,i}^{(u)})$ is a Cauchy sequence in E_i, for each $i=1,2,\dots,m$ and E_i is complete so let $X_{k,i}^{(u)} \to X_{k,i}$ in E_i as $u\to\infty, i=1,2,\dots,m$. Further $(\Delta^m X_{k,s}^{(u)})$ is a Cauchy sequence in E_k for each k. Since E_k is complete for each k, so sequence $(\Delta^m X_{k,s}^{(u)})$ is convergent for each k.

Keeping u fixed and letting $v\to\infty$ in equation (3.1) and equation (3.2), we get

$$\sum_{i=1}^{m} f(\sup_k \overline{d_k}(X_{k,i}^{(u)}, X_{k,i})) \to 0 \text{ as } u\to\infty.$$

and

$$\sup_n \Big(\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u)}, \Delta^m X_{k,s}))\Big)^{p_k}\Big)^{\frac{1}{M}} \to 0 \text{ as } u\to\infty. \tag{3.5}$$

i.e.

$$g(X^{(u)}, X) \to 0 \text{ as } u\to\infty.$$

Now, we have to show that $X\in w^F(\Delta^m, f, p)$.

From equation (3.5), we have $\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u)}, \Delta^m X_{k,s}))\Big)^{p_k} \to 0$ as $u\to\infty$ for all $n\in\mathbb{N}$. i.e. given $\epsilon>0$, there exists $u_0\in\mathbb{N}$ such that

$$\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u)}, \Delta^m X_{k,s}))\Big)^{p_k} < \frac{\epsilon}{3} \text{ for all } u \geq u_0 \text{ and for all } n \in \mathbb{N}.$$

Since $X^{(u)} \in w^F(\Delta^m, f, p)$, so for each u we can find $L^{(u)}$ and $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u)}, L_k^{(u)}))\Big)^{p_k} < \frac{\epsilon}{3} \text{ for all } n \geq n_0 \text{ where } L_k^{(u)} \in E_k.$$

Similarly, for $X^{(v)} \in w^F(\Delta^m, f, p)$, so for each v we can find $L^{(v)}$ and $n_1 \in \mathbb{N}$ such that

$$\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(v)}, L_k^{(v)}))\Big)^{p_k} < \frac{\epsilon}{3} \text{ for all } n \geq n_1 \text{ where } L_k^{(v)} \in E_k.$$

Consider $u,\ v \geq u_0$ and $n_2 = \max(n_0, n_1)$. Then

$$\begin{aligned}
&\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(L_k^{(u)}, L_k^{(v)}))\Big)^{p_k} \\
&\leq C\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u)}, L_k^{(u)}))\Big)^{p_k} \\
&\quad + C\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u)}, \Delta^m X_{k,s}^{(v)}))\Big)^{p_k} \\
&\quad + C\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(v)}, L_k^{(v)}))\Big)^{p_k} \\
&< \epsilon C \text{ for all } u, v \geq n_2.
\end{aligned} \tag{3.6}$$

For suitable choice of ϵ and using the fact that the modulus function is monotone, we get

$$\overline{d_k}(L_k^{(u)}, L_k^{(v)}) < \epsilon_1 \text{ for all } u, v \geq n_2.$$

i.e. $(L_k^{(u)})$ is a Cauchy sequence in E_k. But given that E_k is complete. So let $L_k^{(u)} \to L_k$ as $u \to \infty$. Using in equation (3.6), we get

$$\frac{1}{n}\sum_{s=1}^{n}\Big(f(sup_k \overline{d_k}(L_k^{(u)}, L_k))\Big)^{p_k} < \epsilon C \text{ for all } u \geq n_2.$$

Hence we get

$$\begin{aligned}\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} &\le C\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u_0)}, \Delta^m X_{k,s}))\Big)^{p_k}\\ &\quad + C\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}^{(u_0)}, L_k^{(u_0)}))\Big)^{p_k}\\ &\quad + C\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(L_k^{(u_0)}, L_k))\Big)^{p_k}\\ &< C\frac{\epsilon}{3} + C\frac{\epsilon}{3} + \epsilon C \text{ for all } n \ge n_2.\end{aligned}$$

i.e. $\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} < \frac{2\epsilon C}{3} + \epsilon C$ for all $n \ge n_2$.
Which implies $X \in w^F(\Delta^m, f, p)$ and hence the sequence space $w^F(\Delta^m, f, p)$ is a complete metric space.

Theorem 3.3. *Let (p_k), (t_k) be two sequences of positive real numbers and assume that for each $k \in \mathbb{N}$, $0 < p_k \le t_k$ and the sequence $(\frac{t_k}{p_k})$ be bounded. Then $w^F(\Delta^m, f, t) \subset w^F(\Delta^m, f, p)$.*

Proof. Let $X \in w^F(\Delta^m, f, t)$ which implies $\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{t_k} \to 0$ as $n \to \infty$.

Consider $\mu_k = \Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{t_k}$ and $\lambda_k = \frac{p_k}{t_k}$ be such that $0 < \lambda \le \lambda_k \le 1$. Define

$$u_k = \begin{cases} \mu_k & \text{if } \mu_k \ge 1 \\ 0 & \text{if } \mu_k < 1 \end{cases} \quad \text{and } v_k = \begin{cases} 0 & \text{if } \mu_k \ge 1 \\ \mu_k & \text{if } \mu_k < 1 \end{cases}$$

Then we have $\mu_k = u_k + v_k$ and $\mu_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$ and it follows that $u_k^{\lambda_k} \le u_k \le \mu_k$ and $v_k^{\lambda_k} \le v_k^{\lambda}$. Therefore

$$\begin{aligned}&\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k}\\ &\le \frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{t_k} + \frac{1}{n}\sum_{s=1}^{n} v_k^{\lambda}\end{aligned} \tag{3.7}$$

and the right hand side of equation (3.7) $\to 0$ as $n \to \infty$ which implies $X \in w^F(\Delta^m, f, p)$.

Theorem 3.4. *Let f and g be two modulus functions. Then we have*

(i) $w^F(\Delta^m, f, p) \cap w^F(\Delta^m, g, p) \subseteq w^F(\Delta^m, f+g, p)$.

(ii) $w^F(\Delta^m, f, p) = w^F(\Delta^m, g, p)$ *if* $0 < \inf \frac{f(x)}{g(x)} \leq \sup \frac{f(x)}{g(x)} < \infty$.

Proof. The proof is very easy. So we omit it.

4. Δ^m–Statistical Convergence

The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

The natural density of a subset K of $\mathbb{N}$ is defined by $\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in K\}|$, where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n. We shall be concerned with integer sets having natural density zero.

If $X = (X_k)$ is a sequence that satisfies a property P for almost all k except a set of natural density zero, then we say that X_k satisfies P for almost all k and we write it by a.a.k.

Definition 4.1. The sequence $X = (((X_{k,s})_{s=1}^{\infty})_k)$ of fuzzy numbers is said to be Δ^m–statistically convergent to the fuzzy number $L = (L_1, L_2, L_3, \ldots)$ where $L_k \in E_k$, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{s \leq n : \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) \geq \epsilon\}| = 0.$$

Let $S^F(\Delta^m)$ denotes the set of all Δ^m–statistically convergent sequences of fuzzy numbers.

Definition 4.2. The sequence $X = (((X_{k,s})_{s=1}^{\infty})_k)$ of fuzzy numbers is said to be Δ^m–statistically Cauchy sequence, if for any $\epsilon > 0$, there exists a positive integer s_0 (depends upon ϵ only) such that

$$\lim_{n\to\infty} \frac{1}{n}|\{s \leq n : \sup_k \overline{d_k}(\Delta^m X_{k,s}, \Delta^m X_{k,s_0}) \geq \epsilon\}| = 0.$$

Definition 4.3. The sequence $X = (((X_{k,s})_{s=1}^{\infty})_k)$ of fuzzy numbers is said to be Δ^m–statistically pre-Cauchy sequence, if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n^2}|\{(i,j) : i, j \leq n, \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j}) \geq \epsilon\}| = 0.$$

Remark 4.1. If a sequence is Δ^m–convergent, then it is Δ^m–statistically convergent. But the converse is not true. This is clear from the following example.

Example 4.1. Let $E_k = L(\mathbb{R})$ for each $k \in \mathbb{N},\ m = 1$ and consider the sequence X as
when $k = 10^n$

$$X_k(t) = \begin{cases} \frac{k}{k-1}(t + 2 - \frac{1}{k}) & \text{if } \frac{1-2k}{k} \leq t \leq -1 \\ \frac{k}{k+1}(\frac{1}{k} - t) & \text{if } -1 \leq t \leq \frac{1}{k} \\ 0 & \text{otherwise} \end{cases}$$

and when $k \neq 10^n$

$$X_k(t) = \begin{cases} t - 5 & \text{if } 5 \leq t \leq 6 \\ 7 - t & \text{if } 6 \leq t \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

The figure for the sequence (X_k) looks like as below:

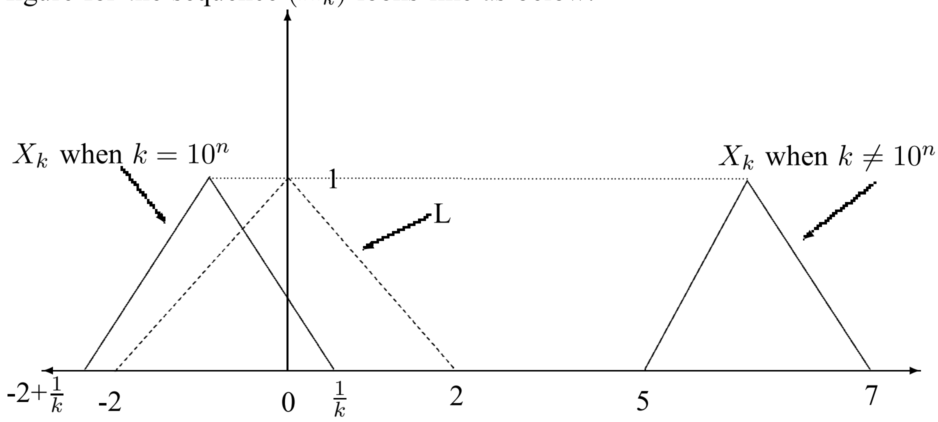

Figure 4.1

Then

$$[X_k]^\alpha = \begin{cases} [\frac{1-2k+k\alpha-\alpha}{k}, \frac{1-k\alpha-\alpha}{k}] & \text{when } k = 10^n \\ [5 + \alpha, 7 - \alpha] & \text{otherwise} \end{cases}$$

i.e.

$$[\Delta X_k]^\alpha = \begin{cases} [\frac{1-9k+2k\alpha-\alpha}{k}, \frac{1-2k\alpha-5k-\alpha}{k}] & \text{when } k = 10^n \\ [\frac{5k+2k\alpha+4+3\alpha}{k+1}, \frac{9k-2k\alpha+8-\alpha}{k+1}] & \text{when } k + 1 = 10^n \\ [-2 + 2\alpha, 2 - 2\alpha] & \text{otherwise} \end{cases}$$

which implies that $\Delta X_k \to L$ statistically, where $L = [-2 + 2\alpha, 2 - 2\alpha]$, but (ΔX_k) is not a convergent sequence.

Theorem 4.5. *Let f be any modulus function and* $0 < h = \inf p_k \leq p_k \leq \sup p_k = H < \infty$. *Then* $w^F(\Delta^m, f, p) \subsetneq S^F(\Delta^m)$.

Proof. Let $X \in w^F(\Delta^m, f, p)$ and $\epsilon > 0$ be given. Then

$$\begin{aligned}
&\frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} \\
&= \frac{1}{n}\sum_{\substack{s=1, \\ \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k)\geq\epsilon}}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} \\
&\quad + \frac{1}{n}\sum_{\substack{s=1, \\ \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k)<\epsilon}}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} \\
&\geq \frac{1}{n}\sum_{\substack{s=1, \\ \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k)\geq\epsilon}}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} \\
&\geq \min(f(\epsilon)^h, f(\epsilon)^H)\frac{1}{n}|\{s \leq n : \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) \geq \epsilon\}|
\end{aligned}$$

which implies X is Δ^m-statistically convergent sequence.

Remark 4.2. The inclusion is strict. This is clear from the following example.

Example 4.2. Let $f(x) = x,\ m = 1,\ p_k = 1$ for each $k \in \mathbb{N},\ E_k = L(\mathbb{R})$ for each $k \in \mathbb{N}$ and consider the sequence X_k as
when $k = 5^n$

$$X_k(t) = \begin{cases} k(t + \frac{1}{k}) & \text{if } \frac{-1}{k} \leq t \leq 0 \\ k(\frac{1}{k} - t) & \text{if } 0 \leq t \leq \frac{1}{k} \\ 0 & \text{otherwise} \end{cases}$$

and when $k \neq 5^n$

$$X_k(t) = \begin{cases} t - 5 & \text{if } 5 \leq t \leq 6 \\ 7 - t & \text{if } 6 \leq t \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

The figure for the sequence (X_k) looks like as below:

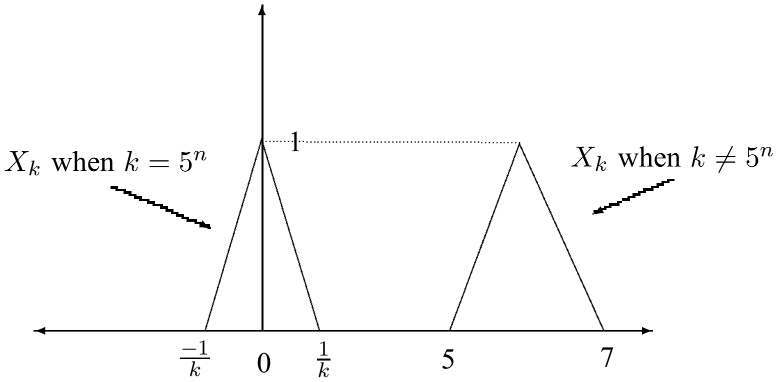

Figure 4.2

Then

$$[X_k]^\alpha = \begin{cases} [\frac{\alpha-1}{k}, \frac{1-\alpha}{k}] & \text{when } k = 5^n \\ [5+\alpha, 7-\alpha] & \text{otherwise} \end{cases}$$

i.e.

$$[\Delta X_k]^\alpha = \begin{cases} [\frac{\alpha-1-7k+\alpha k}{k}, \frac{1-5k-\alpha-\alpha k}{k}] & \text{when } k = 5^n \\ [\frac{5k+k\alpha+2\alpha+4}{k+1}, \frac{7k-k\alpha+8-2\alpha}{k+1}] & \text{when } k+1 = 5^n \\ [-2+2\alpha, 2-2\alpha] & \text{otherwise} \end{cases}$$

Then $\Delta X_k \to L$ statistically, where $L = [-2+2\alpha, 2-2\alpha]$, but $(\Delta X_k) \notin w^F(\Delta^m, f, p)$.

Theorem 4.6. *If f is a bounded modulus function, then* $S^F(\Delta^m) \subseteq w^F(\Delta^m, f, p)$.

Proof. Let $\epsilon > 0$ be given and f be any modulus function. Since f is a bounded modulus function, there exists an integer K such that $f(x) < K$ for all $x \geq 0$.
Let X is Δ^m-statistically convergent sequence. Consider

$$\begin{aligned} & \frac{1}{n}\sum_{s=1}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} \\ = & \frac{1}{n}\sum_{\substack{s=1,\\ \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k)\geq\epsilon}}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} \\ & +\frac{1}{n}\sum_{\substack{s=1,\\ \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k)<\epsilon}}^{n}\Big(f(\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k))\Big)^{p_k} \end{aligned}$$

$$\begin{aligned} &\leq \max(K^h, K^H)\frac{1}{n}|\{s \leq n : \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) \geq \epsilon\}| \\ &\quad + \max(f(\epsilon)^h, f(\epsilon)^H) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

i.e. $X \in w^F(\Delta^m, f, p)$ which implies $S^F(\Delta^m) \subseteq w^F(\Delta^m, f, p)$.

Theorem 4.7. *If the sequence X is Δ^m-statistically convergent, then X is Δ^m-statistically Cauchy.*

Proof. Let X is Δ^m-statistically convergent sequence and let $\epsilon > 0$ be given. Then we have

$$\lim_{n\to\infty} \frac{1}{n}|\{s \leq n : \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) \geq \epsilon\}| = 0.$$

i.e.

$$\sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) < \epsilon \ a.a.s.$$

In particular choose $s_1 \in \mathbb{N}$ such that

$$\sup_k \overline{d_k}(\Delta^m X_{k,s_1}, L_k) < \epsilon.$$

$$\begin{aligned} \sup_k \overline{d_k}(\Delta^m X_{k,s}, \Delta^m X_{k,s_1}) &\leq \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) + \sup_k \overline{d_k}(\Delta^m X_{k,s_1}, L_k) \\ &< \epsilon + \epsilon = 2\epsilon \ a.a.s. \end{aligned}$$

which implies X is a Δ^m-statistically Cauchy sequence.

Theorem 4.8. *If $X = (((X_{k,s})_{s=1}^{\infty})_k)$ is a sequence for which there is a Δ^m-statistically convergent sequence $Y = (((Y_{k,s})_{s=1}^{\infty})_k)$ such that $\Delta^m X_{k,s} = \Delta^m Y_{k,s}$ a.a.s. Then the sequence X is also Δ^m-statistically convergent sequence.*

Proof. Let $\Delta^m X_{k,s} = \Delta^m Y_{k,s}$ $a.a.s$ and Y is Δ^m-statistically convergent sequence. Let $\epsilon > 0$ be given. Then for each n,

$$\begin{aligned} &\{s \leq n : \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) \geq \epsilon\} \\ &\subseteq \{s \leq n : \sup_k \overline{d_k}(\Delta^m Y_{k,s}, L_k) \geq \epsilon\} \cup \{s \leq n : \Delta^m X_{k,s} \nsim \Delta^m Y_{k,s}\}. \end{aligned}$$

Since Y is Δ^m-statistically convergent sequence, which implies the set $\{s \leq n : \sup_k \overline{d_k}(\Delta^m Y_{k,s}, L_k) \geq \epsilon\}$ contains a fixed number of elements say $s_0 = s_0(\epsilon)$. Then,

$$\begin{aligned} &\frac{1}{n}|\{s \leq n : \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) \geq \epsilon\}| \\ \leq\ &\frac{s_0}{n} + \frac{1}{n}|\{s \leq n : \Delta^m X_{k,s} \nsim \Delta^m Y_{k,s}\}| \\ &\to 0 \text{ as } n \to \infty \text{ (because } \Delta^m X_{k,s} = \Delta^m Y_{k,s} \ a.a.s.) \end{aligned}$$

which implies X is a Δ^m-statistically convergent sequence.

Theorem 4.9. *If X be a sequence of fuzzy numbers such that X is Δ^m-statistically convergent sequence. Then X is Δ^m-statistically bounded sequence.*

Proof. Let X is Δ^m-statistically convergent sequence. Then given $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n}|\{s \leq n : \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) \geq \epsilon\}| = 0.$$

Since L is a fuzzy number, so we have $\sup_k \overline{d_k}(L_k, \overline{0}) < T$(say). Then, we have

$$\begin{aligned} \sup_k \overline{d_k}(\Delta^m X_{k,s}, \overline{0}) &\leq \sup_k \overline{d_k}(\Delta^m X_{k,s}, L_k) + \sup_k \overline{d_k}(L_k, \overline{0}) \\ &\leq \epsilon + T \ a.a.k. \end{aligned}$$

which implies X is Δ^m-statistically bounded sequence.

Remark 4.3. In general the converse is not true. This is clear from the following example.

Example 4.3. Let $f(x) = x,\ m = 1,\ p_k = 1$ for each $k \in \mathbb{N},\ E_k = L(\mathbb{R})$ for each $k \in \mathbb{N}$ and consider the sequence X_k as
when $k = 10^n$

$$X_k(t) = \begin{cases} kt + 1 & \text{if } \frac{-1}{k} \leq t \leq 0 \\ 1 - kt & \text{if } 0 \leq t \leq \frac{1}{k} \\ 0 & \text{otherwise} \end{cases}$$

when $k \neq 10^n$ and k is odd

$$X_k(t) = \begin{cases} t + 7 & \text{if } -7 \leq t \leq -6 \\ -t - 5 & \text{if } -6 \leq t \leq -5 \\ 0 & \text{otherwise} \end{cases}$$

and when $k \neq 10^n$ and k is even

$$X_k(t) = \begin{cases} t - 5 & \text{if } 5 \leq t \leq 6 \\ 7 - t & \text{if } 6 \leq t \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

The figure for the sequence (X_k) looks like as below:

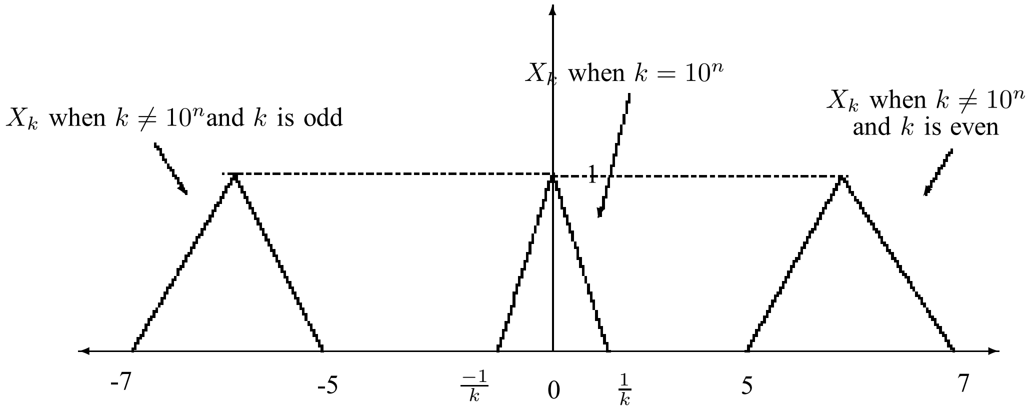

Figure 4.3

Then

$$[X_k]^\alpha = \begin{cases} [\frac{\alpha-1}{k}, \frac{1-\alpha}{k}] & \text{when } k = 10^n \\ [-7+\alpha, -5-\alpha] & \text{when } k \neq 10^n \text{and k is odd} \\ [5+\alpha, 7-\alpha] & \text{when } k \neq 10^n \text{and k is even} \end{cases}$$

i.e.

$$[\Delta X_k]^\alpha = \begin{cases} [\frac{\alpha-1+\alpha k+5k}{k}, \frac{1-\alpha+7k-\alpha k}{k}] & \text{when } k = 10^n \\ [\frac{-7k+k\alpha+2\alpha-8}{k+1}, \frac{-5k-k\alpha-4-2\alpha}{k+1}] & \text{when } k+1 = 10^n \\ [-14+2\alpha, -10-2\alpha] & \text{when } k \neq 10^n \text{and k is odd} \\ [10+2\alpha, 14-2\alpha] & \text{when } k \neq 10^n \text{and k is even} \end{cases}$$

which implies X is Δ^m-statistically bounded sequence, but not Δ^m-statistically convergent sequence.

Remark 4.4. A sequence X is Δ^m-statistically pre-Cauchy sequence, but not Δ^m-statistically convergent sequence.

Example 4.4. Let $f(x) = x,\ p_k = 1$ for each $k \in \mathbb{N},\ E_k = L(\mathbb{R})$ for each $k \in \mathbb{N}$ and consider the sequence X_k as
when k is odd

$$X_k(t) = \begin{cases} t+7 & \text{if } -7 \leq t \leq -6 \\ -t-5 & \text{if } -6 \leq t \leq -5 \\ 0 & \text{otherwise} \end{cases}$$

and when k is even

$$X_k(t) = \begin{cases} t-5 & \text{if } 5 \leq t \leq 6 \\ 7-t & \text{if } 6 \leq t \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$[X_k]^\alpha = \begin{cases} [-7+\alpha, -\alpha-5] & \text{when } k \text{ is odd} \\ [5+\alpha, 7-\alpha] & \text{when } k \text{ is even} \end{cases}$$

i.e.

$$[\Delta^m X_k]^\alpha = \begin{cases} [2^m(-7+\alpha), 2^m(-\alpha-5)] & \text{when } k \text{ is odd} \\ [2^m(5+\alpha), 2^m(7-\alpha)] & \text{when } k \text{ is even} \end{cases}$$

which implies the sequence X is Δ^m-statistically pre-Cauchy sequence, but not Δ^m-statistically convergent sequence.

Theorem 4.10. *Let X be a sequence of fuzzy numbers such that $(\Delta^m X_k)$ is bounded. Then X is said to be Δ^m-statistically pre-Cauchy if and only if $\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j\leq n} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) = 0$, for any bounded modulus function f.*

Proof. Let $\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j\leq n} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) = 0.$

Given $\epsilon > 0$ and for any $n \in \mathbb{N}$, we have

$$\begin{aligned} & \frac{1}{n^2} \sum_{i,j\leq n} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) \\ = & \frac{1}{n^2} \sum_{\substack{i,j\leq n \\ \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j}) < \epsilon}} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) \\ & + \frac{1}{n^2} \sum_{\substack{i,j\leq n \\ \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j}) \geq \epsilon}} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) \\ \geq & \frac{1}{n^2} \sum_{\substack{i,j\leq n \\ \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j}) \geq \epsilon}} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) \\ \geq & f(\epsilon) \frac{1}{n^2} |\{(i,j) : i,j \leq n, \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j}) \geq \epsilon\}| \end{aligned}$$

and thus X is Δ^m-statistically pre-Cauchy sequence.

Conversely, let X is Δ^m-statistically pre-Cauchy sequence and $\epsilon > 0$ be given. Choose $\delta > 0$ such that $f(\delta) < \frac{\epsilon}{2}$. Since f is a bounded modulus function, so there exist an integer B such that $f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) < B$. Now for each $n \in \mathbb{N}$, consider

$$\begin{aligned}
& \frac{1}{n^2}\sum_{i,j\leq n} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) \\
= & \frac{1}{n^2}\sum_{\substack{i,j\leq n \\ \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})<\delta}} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) \\
& +\frac{1}{n^2}\sum_{\substack{i,j\leq n \\ \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})\geq\delta}} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) \\
\leq & f(\delta) + B\frac{1}{n^2}|\{(i,j) : i,j \leq n, \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j}) \geq \delta\}| \\
\leq & \frac{\epsilon}{2} + B\frac{1}{n^2}|\{(i,j) : i,j \leq n, \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j}) \geq \delta\}|
\end{aligned}$$

Since let X is Δ^m-statistically pre-Cauchy sequence, so we have

$$\frac{1}{n^2}|\{(i,j) : i,j \leq n, \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j}) \geq \delta\}| \to 0 \text{ as } n \to \infty.$$

i.e. there exist $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n^2}|\{(i,j) : i,j \leq n, \sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j}) \geq \delta\}| < \frac{\epsilon}{2B} \text{ for all } n \geq n_0.$$

i.e.

$$\frac{1}{n^2}\sum_{i,j\leq n} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) \leq \epsilon \text{ for all } n \geq n_0.$$

Hence we have $\lim_{n\to\infty} \frac{1}{n^2}\sum_{i,j\leq n} f(\sup_k \overline{d_k}(\Delta^m X_{k,i}, \Delta^m X_{k,j})) = 0$.

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