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HYPERSURFACES IN NON-FLAT PSEUDO-RIEMANNIAN SPACE FORMS SATISFYING A LINEAR CONDITION IN THE LINEARIZED OPERATOR OF A HIGHER ORDER MEAN CURVATURE

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Abstract. We study hypersurfaces either in the pseudo-Riemannian De Sitter space $\mathbb{S}^{n+1}_t \subset \mathbb{R}^{n+2}_t$ or in the pseudo-Riemannian anti De Sitter space $\mathbb{H}^{n+1}_t \subset \mathbb{R}^{n+2}_{t+1}$ whose position vector ψ satisfies the condition $L_k\psi = A\psi + b$, where L_k is the linearized operator of the $(k+1)$-th mean curvature of the hypersurface, for a fixed $k = 0, \ldots, n-1$, A is an $(n+2)\times(n+2)$ constant matrix and b is a constant vector in the corresponding pseudo-Euclidean space. For every k, we prove that when H_k is constant, the only hypersurfaces satisfying that condition are hypersurfaces with zero $(k+1)$-th mean curvature and constant k-th mean curvature, open pieces of a totally umbilical hypersurface in $\mathbb{S}^{n+1}_t$ ($\mathbb{S}^n_{t-1}(r)$, $r > 1$; $\mathbb{S}^n_t(r)$, $0 < r < 1$; $\mathbb{H}^n_{t-1}(-r)$, $r > 0$; $\mathbb{R}^n_{t-1}$), open pieces of a totally umbilical hypersurface in $\mathbb{H}^{n+1}_t$ ($\mathbb{H}^n_t(-r)$, $r > 1$; $\mathbb{H}^n_{t-1}(-r)$, $0 < r < 1$; $\mathbb{S}^n_t(r)$, $r > 0$; $\mathbb{R}^n_t$), open pieces of a standard pseudo-Riemannian product in $\mathbb{S}^{n+1}_t$ ($\mathbb{S}^m_u(r) \times \mathbb{S}^{n-m}_v(\sqrt{1-r^2})$, $\mathbb{H}^m_{u-1}(-r)\times\mathbb{S}^{n-m}_v(\sqrt{1+r^2})$, $\mathbb{S}^m_u(r)\times\mathbb{H}^{n-m}_{v-1}(-\sqrt{r^2-1})$), open pieces of a standard pseudo-Riemannian product in $\mathbb{H}^{n+1}_t$ ($\mathbb{H}^m_u(-r)\times\mathbb{S}^{n-m}_v(\sqrt{r^2-1})$, $\mathbb{S}^m_u(r)\times\mathbb{H}^{n-m}_v(-\sqrt{1+r^2})$, $\mathbb{H}^m_u(-r)\times\mathbb{H}^{n-m}_{v-1}(-\sqrt{1-r^2})$) and open pieces of a quadratic hypersurface $\{x \in \mathbb{M}^{n+1}_t(c) \mid \langle Rx, x\rangle = d\}$, where R is a self-adjoint constant matrix whose minimal polynomial is $\mu_R(z) = z^2 + az + b$, $a^2 - 4b \leq 0$, and $\mathbb{M}^{n+1}_t(c)$ stands for $\mathbb{S}^{n+1}_t \subset \mathbb{R}^{n+2}_t$ or $\mathbb{H}^{n+1}_t \subset \mathbb{R}^{n+2}_{t+1}$.

1. INTRODUCTION

The Laplacian operator Δ of a hypersurface M^n immersed into $\mathbb{R}^{n+1}$ can be seen as the first one of a sequence of operators $\{L_0 = \Delta, L_1, \ldots, L_{n-1}\}$, where L_k stands

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for the linearized operator of the first variation of the $(k+1)$-th mean curvature, arising from normal variations of the hypersurface (see, for instance, [21]). These operators are defined by $L_k(f) = \mathrm{tr}(P_k \circ \nabla^2 f)$, for a smooth function f on M, where P_k denotes the k-th Newton transformation associated to the second fundamental form of the hypersurface, and $\nabla^2 f$ denotes the self-adjoint linear operator metrically equivalent to the hessian of f.

From this point of view, and inspired by Garay's extension of Takahashi theorem and its subsequent generalizations and extensions ([24, 6, 10, 8, 12, 1, 2, 3]), Alías and Gürbüz initiated in [4] the study of hypersurfaces in Euclidean space satisfying the general condition $L_k\psi = A\psi + b$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant matrix and $b \in \mathbb{R}^{n+1}$ is a constant vector. Recently, we have completely extended to the Lorentz-Minkowski space the previous classification theorem obtained by Alías and Gürbüz. In particular, we proved in [15] that the only hypersurfaces immersed in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ satisfying the condition $L_k\psi = A\psi + b$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant matrix and $b \in \mathbb{L}^{n+1}$ is a constant vector, are open pieces of hypersurfaces with zero $(k+1)$-th mean curvature, or open pieces of totally umbilical hypersurfaces $\mathbb{S}^n_1(r)$ or $\mathbb{H}^n(-r)$, or open pieces of generalized cylinders $\mathbb{S}^m_1(r) \times \mathbb{R}^{n-m}$, $\mathbb{H}^m(-r) \times \mathbb{R}^{n-m}$, with $k+1 \le m \le n-1$, or $\mathbb{L}^m \times \mathbb{S}^{n-m}(r)$, with $k+1 \le n-m \le n-1$.

In [5], and as a natural continuation of the study started in [4], Alías and Kashani consider the study of hypersurfaces M^n immersed either into the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ or into the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ whose position vector ψ satisfies the condition $L_k\psi = A\psi + b$, for some constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some constant vector $b \in \mathbb{R}^{n+2}_q$, $q = 0, 1$. They obtain classification results in two cases: when A is self-adjoint and $b = 0$, and when the k-th mean curvature H_k is constant and b is a non-zero constant vector. When the ambient space is a Lorentzian space form $\mathbb{S}^{n+1}_1$ or $\mathbb{H}^{n+1}_1$, the shape operator of the hypersurface needs not be diagonalizable, condition which plays a chief role in the Riemannian case. In this case, the shape operator of the hypersurface can be expressed, in an appropriate frame, in one of four types. In [16] we have extended, to the Lorentzian case, the results obtained in [5].

However, when the ambient space is a general pseudo-Riemannian space form $\mathbb{S}^{n+1}_t \subset \mathbb{R}^{n+2}_t$ or $\mathbb{H}^{n+1}_t \subset \mathbb{R}^{n+2}_{t+1}$, the shape operator of the hypersurface can be much more complicated than in the Riemannian or Lorentzian cases, and then the reasoning followed in [5] and [16] is not applicable in the general case. In this paper, we extend to arbitrary pseudo-Riemannian space forms $\mathbb{S}^{n+1}_t$ or $\mathbb{H}^{n+1}_t$ the results obtained in [5] and [16].

Our approach in this paper is completely different to that given in above papers. First, we do not assume that A is a self-adjoint matrix, but we only assume that the k-th mean curvature of the hypersurface is constant. Secondly, the techniques developed in [4, 5, 15, 16] are not applicable in the general case, so that we have needed to follow a different way. The new and more general proof is based on the complexification of the shape operator of the hypersurface (see sections 2 and 5 for details).

For the sake of simplifying the notation and unifying the statements of our main results, let us denote by $\mathbb{M}^{n+1}_t(c)$ either the pseudo-Riemannian De Sitter space $\mathbb{S}^{n+1}_t \subset \mathbb{R}^{n+2}_t$ if $c=1$, or the pseudo-Riemannian anti De Sitter space $\mathbb{H}^{n+1}_t \subset \mathbb{R}^{n+2}_{t+1}$ if $c=-1$. In this paper, we are able to give the following classification result.

Theorem 1. *Let $\psi : M^n_s \to \mathbb{M}^{n+1}_t(c) \subset \mathbb{R}^{n+2}_q$ be an orientable hypersurface immersed into the pseudo-Riemannian space form $\mathbb{M}^{n+1}_t(c)$, and let L_k be the linearized operator of the $(k+1)$-th mean curvature of M^n_s, for some fixed $k=0,1,\dots,n-1$. Assume that H_k is constant. Then the immersion satisfies the condition $L_k\psi = A\psi + b$, for some constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some constant vector $b \in \mathbb{R}^{n+2}_q$, if and only if it is one of the following hypersurfaces:*

(1) *a hypersurface having zero $(k+1)$-th mean curvature and constant k-th mean curvature.*
(2) *an open piece of one of the following totally umbilical hypersurfaces in $\mathbb{S}^{n+1}_t$: $\mathbb{S}^n_{t-1}(r)$, $r>1$; $\mathbb{S}^n_t(r)$, $0<r<1$; $\mathbb{H}^n_{t-1}(-r)$, $r>0$; $\mathbb{R}^n_{t-1}$.*
(3) *an open piece of one of the following totally umbilical hypersurfaces in $\mathbb{H}^{n+1}_t$: $\mathbb{H}^n_t(-r)$, $r>1$; $\mathbb{H}^n_{t-1}(-r)$, $0<r<1$; $\mathbb{S}^n_t(r)$, $r>0$; $\mathbb{R}^n_t$.*
(4) *an open piece of a standard pseudo-Riemannian product in $\mathbb{S}^{n+1}_t$: $\mathbb{S}^m_u(r) \times \mathbb{S}^{n-m}_v(\sqrt{1-r^2})$, $\mathbb{H}^m_{u-1}(-r) \times \mathbb{S}^{n-m}_v(\sqrt{1+r^2})$, $\mathbb{S}^m_u(r) \times \mathbb{H}^{n-m}_{v-1}(-\sqrt{r^2-1})$.*
(5) *an open piece of a standard pseudo-Riemannian product in $\mathbb{H}^{n+1}_t$: $\mathbb{H}^m_u(-r) \times \mathbb{S}^{n-m}_v(\sqrt{r^2-1})$, $\mathbb{S}^m_u(r) \times \mathbb{H}^{n-m}_v(-\sqrt{1+r^2})$, $\mathbb{H}^m_u(-r) \times \mathbb{H}^{n-m}_{v-1}(-\sqrt{1-r^2})$.*
(6) *an open piece of a quadratic hypersurface $\{x \in \mathbb{M}^{n+1}_t(c) \subset \mathbb{R}^{n+2}_q \mid \langle Rx, x\rangle = d\}$, where R is a self-adjoint constant matrix whose minimal polynomial is z^2+az+b, $a^2-4b \le 0$.*

In the case when $b=0$, the condition that the matrix A is self-adjoint implies that the k-th mean curvature H_k is constant, and then we obtain the following consequence.

Theorem 2. *Let $\psi : M^n_s \to \mathbb{M}^{n+1}_t(c) \subset \mathbb{R}^{n+2}_q$ be an orientable hypersurface immersed into the pseudo-Riemannian space form $\mathbb{M}^{n+1}_t(c)$, and let L_k be the linearized operator of the $(k+1)$-th mean curvature of M^n_s, for some fixed $k=0,1,\dots,n-1$. Then the immersion satisfies the condition $L_k\psi = A\psi$, for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$, if and only if it is one of the following hypersurfaces:*

(1) *a hypersurface having zero $(k+1)$-th mean curvature and constant k-th mean curvature;*
(2) *an open piece of a standard pseudo-Riemannian product in $\mathbb{S}^{n+1}_t$: $\mathbb{S}^m_u(r) \times \mathbb{S}^{n-m}_v(\sqrt{1-r^2})$, $\mathbb{H}^m_{u-1}(-r) \times \mathbb{S}^{n-m}_v(\sqrt{1+r^2})$, $\mathbb{S}^m_u(r) \times \mathbb{H}^{n-m}_{v-1}(-\sqrt{r^2-1})$.*

(3) *an open piece of a standard pseudo-Riemannian product in* $\mathbb{H}_t^{n+1}$*:* $\mathbb{H}_u^m(-r) \times \mathbb{S}_v^{n-m}(\sqrt{r^2-1})$, $\mathbb{S}_u^m(r) \times \mathbb{H}_v^{n-m}(-\sqrt{1+r^2})$, $\mathbb{H}_u^m(-r) \times \mathbb{H}_{v-1}^{n-m}(-\sqrt{1-r^2})$.

(4) *an open piece of a quadratic hypersurface* $\{x \in \mathbb{M}_t^{n+1}(c) \subset \mathbb{R}_q^{n+2} \mid \langle Rx, x\rangle = d\}$*, where* R *is a self-adjoint constant matrix whose minimal polynomial is* $z^2 + az + b$*,* $a^2 - 4b \leq 0$.

2. Preliminaries

In this section we will recall basic formulas and notions about hypersurfaces in pseudo-Riemannian space forms that will be used later on. Let $\mathbb{R}_q^{n+2}$ be the $(n+2)$-dimensional pseudo-Euclidean space of index $q \geq 0$, whose metric tensor $\langle,\rangle$ is given by

$$\langle,\rangle = -\sum_{i=1}^{q} \mathrm{d}x_i \otimes \mathrm{d}x_i + \sum_{i=q+1}^{n+2} \mathrm{d}x_i \otimes \mathrm{d}x_i,$$

where $x = (x_1, \ldots, x_{n+2})$ denotes the usual rectangular coordinates in $\mathbb{R}^{n+2}$. The pseudo-Riemannian De Sitter space of index t is defined by

$$\mathbb{S}_t^{n+1}(r) = \{x \in \mathbb{R}_t^{n+2} \mid \langle x, x\rangle = r^2\}, \quad r > 0,$$

and the pseudo-Riemannian anti-De Sitter space of index t is defined by

$$\mathbb{H}_t^{n+1}(-r) = \{x \in \mathbb{R}_{t+1}^{n+2} \mid \langle x, x\rangle = -r^2\}, \quad r > 0.$$

Throughout this paper, we will consider both the case of hypersurfaces immersed into pseudo-Riemannian De Sitter space $\mathbb{S}_t^{n+1} \equiv \mathbb{S}_t^{n+1}(1)$, and the case of hypersurfaces immersed into pseudo-Riemannian anti De Sitter space $\mathbb{H}_t^{n+1} \equiv \mathbb{H}_t^{n+1}(-1)$. In order to simplify our notation and computations, we will denote by $\mathbb{M}_t^{n+1}(c)$ both the De Sitter space $\mathbb{S}_t^{n+1}$ and the anti De Sitter space $\mathbb{H}_t^{n+1}$ according to $c = 1$ or $c = -1$, respectively. We will use $\mathbb{R}_q^{n+2}$ to denote the corresponding pseudo-Euclidean space where $\mathbb{M}_t^{n+1}(c)$ lives, so that $q = t$ if $c = 1$ and $q = t + 1$ if $c = -1$. Then the metric of $\mathbb{R}_q^{n+2}$ is given by

$$\langle,\rangle = -\sum_{i=1}^{t} \mathrm{d}x_i \otimes \mathrm{d}x_i + c\,\mathrm{d}x_{t+1} \otimes \mathrm{d}x_{t+1} + \sum_{i=t+2}^{n+2} \mathrm{d}x_i \otimes \mathrm{d}x_i,$$

and we can write

$$\mathbb{M}_t^{n+1}(c) = \{x \in \mathbb{R}_q^{n+2} \mid -\sum_{i=1}^{t} x_i^2 + c\,x_{t+1}^2 + \sum_{i=t+2}^{n+2} x_i^2 = c\}.$$

It is well known that $\mathbb{S}_t^{n+1} \subset \mathbb{R}_t^{n+2}$ and $\mathbb{H}_t^{n+1} \subset \mathbb{R}_{t+1}^{n+2}$ are pseudo-Riemannian totally umbilical hypersurfaces with constant sectional curvature $+1$ and -1, respectively.

Let $\psi : M_s^n \longrightarrow \mathbb{M}_t^{n+1}(c) \subset \mathbb{R}_q^{n+2}$ be an isometric immersion of a connected orientable hypersurface M_s^n of index s with Gauss map N, $\langle N, N\rangle = \varepsilon$ (where $\varepsilon = 1$ if $s = t$ or $\varepsilon = -1$ if $s = t-1$). Let ∇^0, $\overline{\nabla}$ and ∇ denote the Levi-Civita connections on $\mathbb{R}_q^{n+2}$, $\mathbb{M}_t^{n+1}(c)$ and M_s^n, respectively. Then the Gauss and Weingarten formulae are given by

$$\nabla^0_X Y = \nabla_X Y + \varepsilon \langle SX, Y\rangle N - c \langle X, Y\rangle \psi, \tag{1}$$

$$SX = -\overline{\nabla}_X N = -\nabla^0_X N, \tag{2}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$, where $S : \mathfrak{X}(M_s^n) \longrightarrow \mathfrak{X}(M_s^n)$ stands for the shape operator (or Weingarten endomorphism) of M_s^n, with respect to the chosen orientation N.

It is well-known [20, pp. 261–262] that a linear self-adjoint endomorphism B on a vector space V can be expressed as a direct sum of subspaces V_ℓ that are mutually orthogonal (hence non-degenerate) and B-invariant, and each $B_\ell = B|_{V_\ell}$ has a matrix of form either

$$\text{I.}\quad \begin{pmatrix} \kappa & & & & \mathbf{0} \\ 1 & \kappa & & & \\ & \ddots & \ddots & & \\ & & 1 & \kappa & \\ \mathbf{0} & & & 1 & \kappa \end{pmatrix}$$

relative to a basis $\{E_1, \dots, E_p\}$ ($p \geq 1$) such that

$$\langle E_i, E_j\rangle = \begin{cases} \epsilon = \pm 1 & \text{if } i+j = p+1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

or

$$\text{II.}\quad \begin{pmatrix} \alpha & \beta & & & & & \mathbf{0} \\ -\beta & \alpha & & & & & \\ 1 & 0 & \alpha & \beta & & & \\ 0 & 1 & -\beta & \alpha & & & \\ & & \ddots & & \ddots & & \\ & & & 1 & 0 & \alpha & \beta \\ \mathbf{0} & & & 0 & 1 & -\beta & \alpha \end{pmatrix} \qquad (\beta \neq 0)$$

relative to a basis $\{E_1, \dots, E_q\}$ ($q \geq 2$ and even) such that

$$\langle E_i, E_j\rangle = \begin{cases} 1 & \text{if } i, j \text{ are odd and } i+j = q \\ -1 & \text{if } i, j \text{ are even and } i+j = q+2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Here p, ϵ and q depend on V_ℓ. A matrix of type I is called a Jordan block corresponding to the (real) eigenvalue κ, whereas a matrix of type II is said to be a Jordan block corresponding to the (complex) eigenvalue $\alpha + i\beta$.

Jordan blocks of type II can be transformed in matrices of form I by a complexification process, see [22]. If V is a real vector space, then the set $V^{\mathbb{C}} = V \times V$ of ordered pairs, with component addition

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)$$

and scalar multiplication over $\mathbb{C}$ defined by

$$(\alpha + i\beta)(u, v) = (\alpha u - \beta v, \beta u + \alpha v),$$

for $\alpha, \beta \in \mathbb{R}$, is a complex vector space, called the complexification of V. The set $V^{\mathbb{C}}$ can be described as $V^{\mathbb{C}} = \{u + iv \mid u, v \in V\}$ and then the addition and scalar multiplication operations resemble the usual for complex numbers:

$$\begin{aligned}(u_1 + iv_1) + (u_2 + iv_2) &= (u_1 + u_2) + i(v_1 + v_2),\\ (\alpha + i\beta)(u + iv) &= (\alpha u - \beta v) + i(\beta u + \alpha v).\end{aligned}$$

An interesting map from V to $V^{\mathbb{C}}$ is the complexification map cpx : $V \to V^{\mathbb{C}}$ defined by cpx$(v) = v + i0$. It is easy to see that cpx is an injective linear transformation, and in this way we can say that $V^{\mathbb{C}}$ contains an embedded copy of V. If $\mathcal{B} = \{v_j \mid j \in I\}$ is a basis of V over $\mathbb{R}$ then the complexification of $\mathcal{B}$, cpx$(\mathcal{B}) = \{v_j + i0 \mid v_j \in \mathcal{B}\}$, is a basis for $V^{\mathbb{C}}$ over $\mathbb{C}$. Hence, $\dim_{\mathbb{C}}(V^{\mathbb{C}}) = \dim_{\mathbb{R}}(V)$.

A linear operator τ on a real vector space V can be extended to a linear operator $\tau^{\mathbb{C}}$ on the complexification $V^{\mathbb{C}}$ by defining

$$\tau^{\mathbb{C}}(u + iv) = \tau(u) + i\tau(v).$$

The following properties of this complexification can be easily obtained. If τ, σ are linear operators on V, then

(1) $(a\tau)^{\mathbb{C}} = a\tau^{\mathbb{C}}, \quad a \in \mathbb{R}$.
(2) $(\tau + \sigma)^{\mathbb{C}} = \tau^{\mathbb{C}} + \sigma^{\mathbb{C}}$.
(3) $(\tau\sigma)^{\mathbb{C}} = \tau^{\mathbb{C}}\sigma^{\mathbb{C}}$.
(4) $[\tau(v)]^{\mathbb{C}} = \tau^{\mathbb{C}}(v^{\mathbb{C}})$.

Let B be a linear self-adjoint endomorphism on V and consider V_ℓ a B-invariant subspace such that $B_\ell = B|_{V_\ell}$ is a Jordan block of type II in a basis (4). Let $V_\ell^{\mathbb{C}}$ be the complexification of V_ℓ and define the following complex vectors

$$F_j = \begin{cases} \dfrac{1}{\sqrt{2}}(E_j + iE_{j+1}) & \text{for } j \text{ odd},\\ \dfrac{1}{\sqrt{2}}(E_{j-1} - iE_j) & \text{for } j \text{ even}.\end{cases} \tag{5}$$

It is not difficult to see that $\{F_1, \dots F_q\}$ is a basis for $V_\ell^{\mathbb{C}}$ and

$$\begin{aligned} B_\ell^{\mathbb{C}} F_j &= \kappa F_j + F_{j+2}, \quad 1 \le j \le q-3, \quad j \text{ odd}, \\ B_\ell^{\mathbb{C}} F_{q-1} &= \kappa F_{q-1}, \\ B_\ell^{\mathbb{C}} F_j &= \overline{\kappa} F_j + F_{j+2}, \quad 2 \le j \le q-2, \quad j \text{ even}, \\ B_\ell^{\mathbb{C}} F_q &= \overline{\kappa} F_q, \end{aligned}$$

where $\kappa = \alpha + i\beta$. Then we can reorder the basis in such a way that $B_\ell^{\mathbb{C}}$ has matrix of form

$$\left(\begin{array}{cccc|cccc} \kappa & & & & & & & \mathbf{0} \\ 1 & \kappa & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & \kappa & & & & \\ \hline & & & & \overline{\kappa} & & & \\ & & & & 1 & \overline{\kappa} & & \\ & & & & & \ddots & \ddots & \\ \mathbf{0} & & & & & & 1 & \overline{\kappa} \end{array}\right) \tag{6}$$

Therefore every Jordan block of type II can be reduced to two Jordan blocks of type I by the complexification process.

The (possibly complex) eigenvalues of shape operator S are called the principal curvatures of M_s^n. When M_s^n is endowed with an indefinite metric the algebraic and geometric multiplicity of a principal curvature need not coincide. If they coincide, it is called simply the multiplicity of the principal curvature. For every point $x \in M_s^n$, consider the decomposition $T_xM = V_1 \oplus \cdots \oplus V_r$ where subspaces V_ℓ, $\ell = 1, \dots, r$, are mutually orthogonal and S-invariant, and each $S_\ell = S|_{V_\ell}$ is a Jordan block. We can write $S_x = \operatorname{diag}(S_1, \dots, S_r)$ or $S_x = S_1 \oplus \cdots \oplus S_r$. These decompositions of T_xM and S_x also work in a neighborhood of point x. Characteristic polynomial $Q_S(t)$ of S is given by

$$Q_S(t) = \det(tI - S) = \prod_{\ell=1}^{r} \det(tI - S_\ell) = \prod_{\ell=1}^{r} Q_{S_\ell}(t),$$

where characteristic polynomial $Q_{S_\ell}(t)$ of S_ℓ is given by

$$Q_{S_\ell}(t) = \begin{cases} (t-\kappa)^p & \text{if } S_\ell \text{ is of type I}, \\ ((t-\alpha)^2 + \beta^2)^p = (t-\kappa)^p (t-\overline{\kappa})^p & \text{if } S_\ell \text{ is of type II } (q = 2p). \end{cases}$$

If we write

$$Q_S(t) = \prod_{\ell=1}^{n} (t - \kappa_\ell) = \sum_{k=0}^{n} a_k t^{n-k}, \quad \text{with } a_0 = 1,$$

where $\{\kappa_1, \dots, \kappa_n\}$ are the n roots (real or complex) of $Q_S(t)$, then it is not difficult to see that

$$\begin{cases} a_1 = & -\sum_{i=1}^{n} \kappa_i, \\ a_k = & (-1)^k \sum_{i_1<\cdots<i_k}^{n} \kappa_{i_1}\cdots\kappa_{i_k}, \quad k = 2,\dots,n. \end{cases}$$

These equations can be easily obtained by making use of the Leverrier–Faddeev method (see [14, 9]), since coefficients of $Q_S(t)$ can be computed, in terms of the traces of S^j, as follows:

(7) $$a_k = -\frac{1}{k}\sum_{j=1}^{k} a_{k-j}\operatorname{tr}(S^j), \quad k = 1,\dots,n, \quad \text{with } a_0 = 1.$$

From now on, we will write

$$\mu_k = \sum_{i_1<\cdots<i_k}^{n} \kappa_{i_1}\cdots\kappa_{i_k} \qquad \text{and} \qquad \mu_k^J = \sum_{\substack{i_1<\cdots<i_k \\ i_j\notin J}}^{n} \kappa_{i_1}\cdots\kappa_{i_k},$$

where $1 \leq k \leq n$ and $J \subset \{1,\dots,n\}$.

The *k-th mean curvature* H_k or *mean curvature of order* k of M_s^n is defined by

(8) $$\binom{n}{k} H_k = (-\varepsilon)^k a_k = \varepsilon^k \mu_k,$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. In particular, when $k = 1$,

$$nH_1 = -\varepsilon a_1 = \varepsilon \operatorname{tr}(S),$$

and so H_1 is nothing but the usual mean curvature H of M_s^n, which is one of the most important extrinsic curvatures of the hypersurface. The hypersurface M_s^n is said to be k-maximal if $H_{k+1} \equiv 0$.

3. The Newton Transformations

The k-th Newton transformation of M is the operator $P_k : \mathfrak{X}(M_s^n) \longrightarrow \mathfrak{X}(M_s^n)$ defined by

(9) $$P_k = \sum_{j=0}^{k} a_{k-j} S^j.$$

Equivalently, P_k can be defined inductively by

(10) $$P_0 = I \quad \text{and} \quad P_k = a_k I + S \circ P_{k-1}.$$

Note that by Cayley-Hamilton theorem we have $P_n = 0$. The Newton transformations were introduced by Reilly [21] in the Riemannian context; its definition was $\overline{P}_k = (-1)^k P_k$. We have the following properties of P_k (the proof is algebraic and straightforward).

Lemma 3. *Let $\psi : M_s^n \to \mathbb{M}_t^{n+1}(c)$ be an isometric immersion of a hypersurface M_s^n in the pseudo-Riemannian space form $\mathbb{M}_t^{n+1}(c)$. The Newton transformations P_k, $k = 1, \ldots, n-1$, satisfy:*

(a) *P_k is self-adjoint and commutes with S,*

(b) $\operatorname{tr}(P_k) = (n-k)a_k = c_k H_k$,

(c) $\operatorname{tr}(S \circ P_k) = -(k+1)a_{k+1} = \varepsilon c_k H_{k+1}$,

(d) $\operatorname{tr}(S^2 \circ P_k) = a_1 a_{k+1} - (k+2)a_{k+2} = C_k \Big[nH_1 H_{k+1} - (n-k-1)H_{k+2} \Big]$, $1 \le k \le n-2$,

where constants c_k and C_k are given by

$$(k+1)C_k = c_k = (-\varepsilon)^k (n-k)\binom{n}{k} = (-\varepsilon)^k (k+1)\binom{n}{k+1}.$$

In a neighborhood of any point, let $W \subset T_pM$ be an m-dimensional, non-degenerate and S-invariant subspace such that $S|_W$ is a Jordan block. Then its d-power is given by either

$$(S|_W)^d = \begin{pmatrix} \kappa^d & 0 & 0 & \cdots & 0 \\ \binom{d}{1}\kappa^{d-1} & \kappa^d & 0 & \cdots & 0 \\ \binom{d}{2}\kappa^{d-2} & \binom{d}{1}\kappa^{d-1} & \kappa^d & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{d}{m-1}\kappa^{d-m+1} & \binom{d}{m-2}\kappa^{d-m+2} & \binom{d}{m-3}\kappa^{d-m+3} & \cdots & \kappa^d \end{pmatrix}$$

if $S|_W$ is of type I, where $\binom{d}{r} = 0$ when $d < r$, or

$$(S|_W)^d = \begin{pmatrix} [\Lambda_d] & \mathbf{0}_2 & \mathbf{0}_2 & \cdots & \mathbf{0}_2 \\ \binom{d}{1}[\Lambda_{d-1}] & [\Lambda_d] & \mathbf{0}_2 & \cdots & \mathbf{0}_2 \\ \binom{d}{2}[\Lambda_{d-2}] & \binom{d}{1}[\Lambda_{d-1}] & [\Lambda_d] & \cdots & \mathbf{0}_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{d}{m-1}[\Lambda_{d-m+1}] & \binom{d}{m-2}[\Lambda_{d-m+2}] & \binom{d}{m-3}[\Lambda_{d-m+3}] & \cdots & [\Lambda_d] \end{pmatrix}$$

if $S|_W$ is of type II, where $\mathbf{0}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, Λ_0 is the identity map and

$$\Lambda_r = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}^r = \begin{bmatrix} C_r & D_r \\ -D_r & C_r \end{bmatrix} \text{ with } \begin{cases} C_r = \displaystyle\sum_{t=0}^{[\frac{r}{2}]} (-1)^t \binom{r}{2t} \beta^{2t} \alpha^{r-2t} \\ D_r = \displaystyle\sum_{t=0}^{[\frac{r-1}{2}]} (-1)^t \binom{r}{2t+1} \beta^{2t+1} \alpha^{r-(2t+1)} \end{cases}$$

Here $[z]$ stands for the integer part of z.

The following two propositions describe operator P_k in W, according to $S|_W$ is of type I or type II, respectively.

Proposition 4. ($S|_W$ is of type I).
Let $\{E_1, E_2 \ldots, E_m\}$ be a local frame of tangent vector fields on W satisfying (3) such that $S|_W$ is a Jordan block of type I: $SE_i = \kappa E_i + E_{i+1}$, for $1 \leq i \leq m-1$, and $SE_m = \kappa E_m$. Then the k-th Newton transformation P_k in W is given by

$$P_k|_W = (-1)^k \begin{pmatrix} \mu_k^{1} & 0 & \cdots & 0 \\ -\mu_{k-1}^{1,2} & \mu_k^{2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ (-1)^{m-1}\mu_{k-(m-1)}^{1,\ldots,m} & \cdots & -\mu_{k-1}^{m-1,m} & \mu_k^{m} \end{pmatrix},$$

where $\kappa_i = \kappa$ for all i.

Proposition 5. ($S|_W$ is of type II).
Let $\{E_1, E_2 \ldots, E_m\}$ be a local frame of tangent vector fields on W satisfying (4) such that $S|_W$ is a Jordan block of type II (hence necessarily m is even):

$$\begin{aligned} SE_i &= \alpha E_i - \beta E_{i+1} + E_{i+2}, \quad 1 \leq i \ (odd) \leq m-3, \\ SE_{m-1} &= \alpha E_{m-1} - \beta E_m, \\ SE_j &= \beta E_{j-1} + \alpha E_j + E_{j+2}, \quad 2 \leq j \ (even) \leq m-2, \\ SE_m &= \beta E_{m-1} + \alpha E_m. \end{aligned}$$

The k-th Newton transformation P_k in W is given by

$$P_k|_W = \begin{pmatrix} U_0 & Z_0 & & & & & & \\ -Z_0 & U_0 & & & & & & \\ U_1 & Z_1 & U_0 & Z_0 & & & & \\ -Z_1 & U_1 & -Z_0 & U_0 & & & & \\ \vdots & \vdots & & & \ddots & \ddots & & \\ & & \cdots & U_1 & Z_1 & U_0 & Z_0 & \\ & & \cdots & -Z_1 & U_1 & -Z_0 & U_0 & \end{pmatrix},$$

where $U_r = \sum_{j=0}^{k} a_{k-j}\binom{j}{r}C_{j-r}$ and $Z_r = \sum_{j=0}^{k} a_{k-j}\binom{j}{r}D_{j-r}$.

Expression for $P_k|_W$ obtained in Proposition 5 can be reformulated as follows when the tangent frame is complexificated according to (5). The proof is straightforward.

Proposition 6. *Let $\mathcal{B} = \{E_1, E_2 \ldots, E_m\}$ be a local frame of tangent vector fields on W satisfying (4) such that $S|_W$ is a Jordan block of type II (hence $m = 2d$ even). Let $\mathcal{B}^{\mathbb{C}} = \{F_1, F_2 \ldots, F_m\}$ be the complexification of $\mathcal{B}$ such that $(S|_W)^{\mathbb{C}}$ has in this frame a matrix of form (6), with $\kappa = \alpha + i\beta$. Then the k-th Newton transformation P_k in W is given by $P_k|_W = (-1)^k \mathrm{diag}(Z(\kappa), \overline{Z(\kappa)})$ where*

$$Z(\kappa) = \begin{pmatrix} \overset{1}{\mu}_k & 0 & \cdots & 0 \\ -\overset{1,2}{\mu}_{k-1} & \overset{2}{\mu}_k & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ (-1)^{d-1}\overset{1,\ldots,d}{\mu}_{k-(d-1)} & \cdots & -\overset{d-1,d}{\mu}_{k-1} & \overset{d}{\mu}_k \end{pmatrix}.$$

Here $\kappa_1 = \cdots = \kappa_d = \kappa$ and $\kappa_{d+1} = \cdots = \kappa_{2d} = \overline{\kappa}$.

Now, we recall the notion of divergence of a vector field X or an operator T. For any differentiable function $f \in \mathcal{C}^\infty(M^n_s)$, the gradient of f is the vector field ∇f metrically equivalent to df, which is characterized by $\langle \nabla f, X\rangle = X(f)$, for every differentiable vector field $X \in \mathfrak{X}(M^n_s)$. The divergence of a vector field X is the differentiable function defined as the trace of operator ∇X, where $\nabla X(Y) := \nabla_Y X$, that is,

$$\mathrm{div}(X) = \mathrm{tr}(\nabla X) = \sum_{i,j} g^{ij} \langle \nabla_{E_i} X, E_j \rangle,$$

$\{E_i\}$ being any local frame of tangent vectors fields, where (g^{ij}) represents the inverse of the metric $(g_{ij}) = (\langle E_i, E_j\rangle)$. Analogously, the divergence of an operator $T : \mathfrak{X}(M^n_s) \longrightarrow \mathfrak{X}(M^n_s)$ is the vector field $\mathrm{div}(T) \in \mathfrak{X}(M^n_s)$ defined as the trace of ∇T, that is,

$$\mathrm{div}(T) = \mathrm{tr}(\nabla T) = \sum_{i,j} g^{ij} (\nabla_{E_i} T) E_j,$$

where $\nabla T(E_i, E_j) = (\nabla_{E_i} T)E_j$.

In the following lemma we present two interesting properties of the Newton transformations.

Lemma 7. *The Newton transformation P_k, for $k = 0, \ldots, n-1$, satisfies:*
(*a*) $\mathrm{tr}(\nabla_X S \circ P_k) = -X(a_{k+1})$.
(*b*) $\mathrm{div}(P_k) = 0$.

Proof. (a) From definition of P_k (9) we deduce

$$\nabla_X S \circ P_k = \sum_{j=0}^{k} a_{k-j} (\nabla_X S \circ S^j) = \sum_{i=1}^{k+1} \frac{a_{k+1-i}}{i} \nabla_X S^i.$$

By taking traces and using that ∇_X commutes with trace operator we have

$$\text{tr}(\nabla_X S \circ P_k) = \sum_{i=1}^{k+1} \frac{a_{k+1-i}}{i} \text{tr}(\nabla_X S^i) = \sum_{i=1}^{k+1} \frac{a_{k+1-i}}{i} X(\text{tr } S^i). \tag{11}$$

From (7) it is not difficult to see that

$$\frac{1}{i} X(\text{tr } S^i) = \sum_{t=1}^{i} \lambda_{i+1-t} X(a_t),$$

where

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_{b+1} = \sum_{\substack{i_1+\cdots+i_r=b \\ i_j \geq 1}} (-1)^{r+1} a_{i_1} \cdots a_{i_r} \quad \text{for } b \geq 1.$$

That equation, jointly with (11), yields

$$\text{tr}(\nabla_X S \circ P_k) = \sum_{i=1}^{k+1} \sum_{t=1}^{i} \lambda_{i+1-t}\, a_{k+1-i}\, X(a_t) = \sum_{t=1}^{k+1} \beta_t X(a_t), \tag{12}$$

where

$$\beta_t = \sum_{i=t}^{k+1} \lambda_{i+1-t} a_{k+1-i}.$$

It is not difficult to see that

$$\sum_{t=1}^{b} \lambda_t\, a_{b+1-t} = - \sum_{\substack{i_1+\cdots+i_r=b \\ i_j \geq 1}} (-1)^{r+1} a_{i_1} \cdots a_{i_r} = -\lambda_{b+1},$$

and then $\beta_t = 0$ for $t = 1, \ldots, k$. Using this equation in (12) we obtain

$$\text{tr}(\nabla_X S \circ P_k) = \sum_{t=1}^{k+1} \beta_t X(a_t) = \lambda_1 a_0 X(a_{k+1}) = -X(a_{k+1}),$$

and the proof finishes.

(b) From the inductive definition (10) of P_k we have

$$\big(\nabla_X P_k\big) Y = X(a_k) Y + \big(\nabla_X S \circ P_{k-1}\big) Y + \big(S \circ \nabla_X P_{k-1}\big) Y,$$

and then

$$\begin{aligned}
\text{div}(P_k) &= \sum_{i,j=1}^{n} g^{ij} \Big[E_i(a_k) E_j + \big(\nabla_{E_i} S \circ P_{k-1}\big) E_j + \big(S \circ \nabla_{E_i} P_{k-1}\big) E_j \Big] \\
&= \nabla a_k + \sum_{i,j=1}^{n} g^{ij} \big(\nabla_{E_i} S \circ P_{k-1}\big) E_j + S\Big(\sum_{i,j=1}^{n} g^{ij} \big(\nabla_{E_i} P_{k-1}\big) E_j \Big) \\
&= \nabla a_k + \sum_{i,j=1}^{n} g^{ij} \big(\nabla_{E_i} S \circ P_{k-1}\big) E_j + S\big(\text{div}(P_{k-1})\big),
\end{aligned}$$

where $\{E_1,\dots,E_n\}$ is a frame of the tangent space. Then for every tangent vector field $X\in\mathfrak{X}(M^n_s)$ we have

$$\langle \mathrm{div}(P_k),X\rangle=\langle\nabla a_k,X\rangle+\mathrm{tr}\big(\nabla_X S\circ P_{k-1}\big)+\big\langle S\big(\mathrm{div}(P_{k-1})\big),X\big\rangle,$$

which implies from (a) that

$$\langle \mathrm{div}(P_k),X\rangle=\big\langle S\big(\mathrm{div}(P_{k-1})\big),X\big\rangle.$$

Therefore we deduce

$$\mathrm{div}(P_k)=S(\mathrm{div}(P_{k-1}))=S^2(\mathrm{div}(P_{k-2}))=\cdots=S^k(\mathrm{div}(P_0))=0. \qquad \blacksquare$$

Bearing in mind this lemma we obtain

$$\mathrm{div}(P_k(\nabla f))=\mathrm{tr}\big(P_k\circ\nabla^2 f\big),$$

where $\nabla^2 f:\mathfrak{X}(M^n_s)\longrightarrow\mathfrak{X}(M^n_s)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f, given by

$$\big\langle\nabla^2 f(X),Y\big\rangle=\langle\nabla_X(\nabla f),Y\rangle, \qquad X,Y\in\mathfrak{X}(M^n_s).$$

Associated to each Newton transformation P_k, we can define the second-order linear differential operator $L_k:\mathcal{C}^\infty(M^n_s)\longrightarrow\mathcal{C}^\infty(M^n_s)$ by

$$L_k(f)=\mathrm{tr}\big(P_k\circ\nabla^2 f\big). \tag{13}$$

An interesting property of L_k is the following. For every couple of differentiable functions $f,g\in C^\infty(M^n_s)$ we have

$$\begin{aligned}L_k(fg)&=\mathrm{div}\big(P_k\circ\nabla(fg)\big)=\mathrm{div}\big(P_k\circ(g\nabla f+f\nabla g)\big)\\&=gL_k(f)+fL_k(g)+2\langle P_k(\nabla f),\nabla g\rangle.\end{aligned} \tag{14}$$

4. Examples

This section is devoted to show some examples of hypersurfaces in pseudo-Riemannian space forms $\mathbb{M}^{n+1}_t(c)$ satisfying the condition $L_k\psi=A\psi+b$, where $A\in\mathbb{R}^{(n+2)\times(n+2)}$ is a constant matrix and $b\in\mathbb{R}^{n+2}_q$ is a constant vector. Before that, we are going to compute L_k acting on the coordinate components of the immersion ψ, that is, a function given by $\langle\psi,a\rangle$, where $a\in\mathbb{R}^{n+2}_q$ is an arbitrary fixed vector.

A direct computation shows that

$$\nabla\langle\psi,a\rangle=a^\top=a-\varepsilon\langle N,a\rangle N-c\langle\psi,a\rangle\psi, \tag{15}$$

where $a^\top\in\mathfrak{X}(M)$ denotes the tangential component of a. Taking covariant derivative in (15), and using that $\nabla^0_X a=0$, jointly with the Gauss and Weingarten formulae, we obtain

$$\nabla_X \nabla \langle \psi, a \rangle = \nabla_X a^\top = \varepsilon \langle N, a \rangle SX - c \langle \psi, a \rangle X, \tag{16}$$

for every vector field $X \in \mathfrak{X}(M)$. Finally, by using (13) and Lemma 3, we find that

$$\begin{aligned} L_k \langle \psi, a \rangle &= \varepsilon \langle N, a \rangle \operatorname{tr}(P_k \circ S) - c \langle \psi, a \rangle \operatorname{tr}(P_k \circ I) \\ &= c_k H_{k+1} \langle N, a \rangle - c c_k H_k \langle \psi, a \rangle . \end{aligned} \tag{17}$$

This expression allows us to extend operator L_k to vector functions $F = (f_1, \ldots, f_{n+2})$, $f_i \in \mathcal{C}^\infty(M_s^n)$, as follows

$$L_k F := \big(L_k f_1, \ldots, L_k f_{n+2}\big),$$

and then $L_k \psi$ can be computed as

$$\begin{aligned} L_k \psi &= \big(L_k(\varepsilon_1 \langle \psi, e_1 \rangle), \ldots, L_k(\varepsilon_{n+2} \langle \psi, e_{n+2} \rangle)\big) \\ &= c_k H_{k+1} \big(\varepsilon_1 \langle N, e_1 \rangle, \ldots, \varepsilon_{n+2} \langle N, e_{n+2} \rangle\big) \\ &\quad - c c_k H_k \big(\varepsilon_1 \langle \psi, e_1 \rangle, \ldots, \varepsilon_{n+2} \langle \psi, e_{n+2} \rangle\big) \\ &= c_k H_{k+1} N - c c_k H_k \psi, \end{aligned} \tag{18}$$

where $\{e_1, \ldots, e_{n+2}\}$ stands for the standard orthonormal basis in $\mathbb{R}_q^{n+2}$ and $\varepsilon_i = \langle e_i, e_i \rangle$.

Example 1. An easy consequence of (18) is that every hypersurface with $H_{k+1} \equiv 0$ and constant k-th mean curvature H_k trivially satisfies $L_k \psi = A\psi + b$, with $A = -c c_k H_k I_{n+2} \in \mathbb{R}^{(n+2)\times(n+2)}$ and $b = 0$.

Example 2. (Totally umbilical hypersurfaces in $\mathbb{M}_t^{n+1}(c)$) Is is well known that totally umbilical hypersurfaces in $\mathbb{M}_t^{n+1}(c)$ are obtained as the intersection of $\mathbb{M}_t^{n+1}(c)$ with a hyperplane of $\mathbb{R}_q^{n+2}$, and the causal character of the hyperplane determines the type of the hypersurface. More precisely, let $a \in \mathbb{R}_q^{n+2}$ be a non-zero constant vector with $\langle a, a \rangle \in \{1, 0, -1\}$, and take the differentiable function $f_a : \mathbb{M}_t^{n+1}(c) \to \mathbb{R}$ defined by $f_a(x) = \langle x, a \rangle$. It is not difficult to see that for every $\tau \in \mathbb{R}$ with $\langle a, a \rangle - c\tau^2 \neq 0$, the set

$$M_\tau = f_a^{-1}(\tau) = \{x \in \mathbb{M}_t^{n+1}(c) \mid \langle x, a \rangle = \tau\}$$

is a totally umbilical hypersurface in $\mathbb{M}_t^{n+1}(c)$, with Gauss map

$$N(x) = \frac{1}{\sqrt{|\langle a, a \rangle - c\tau^2|}} (a - c\tau x),$$

and shape operator

$$SX = -\nabla^0_X N = \frac{c\tau}{\sqrt{|\langle a, a \rangle - c\tau^2|}} X. \tag{19}$$

Now, by using (8) and (19), we obtain that the k-th mean curvature is given by

$$H_k = \frac{(\varepsilon c\tau)^k}{|\langle a,a\rangle - c\tau^2|^{k/2}}, \quad k = 1,\dots,n, \tag{20}$$

where $\varepsilon = \langle N,N\rangle = \pm 1$. Therefore, by equation (18), we see that M_τ satisfies the condition $L_k\psi = A\psi + b$, for every $k = 0,\dots,n-1$, with

$$A = -\frac{c_k(\varepsilon c\tau)^k\big(\varepsilon\tau^2 + c|\langle a,a\rangle - c\tau^2|\big)}{|\langle a,a\rangle - c\tau^2|^{(k+2)/2}} I_{n+2} \quad \text{and} \quad b = \frac{c_k(\varepsilon c\tau)^{k+1}}{|\langle a,a\rangle - c\tau^2|^{(k+2)/2}} a.$$

In particular, $b = 0$ only when $\tau = 0$, and then M_0 is a totally geodesic hypersurface in $\mathbb{M}^{n+1}_t(c)$.

It is easy to see, from (19), that M_τ has constant curvature

$$K = c + \frac{\tau^2}{\langle a,a\rangle - c\tau^2},$$

and it is a hypersurface of index t or $t-1$ according to $\langle a,a\rangle - c\tau^2$ is negative or positive, respectively.

Next two tables collect the different possibilities.

Table 1. Totally umbilical hypersurfaces in $\mathbb{S}^{n+1}_t \subset \mathbb{R}^{n+2}_t$

$\langle a,a\rangle$	τ	K	ε	Hypersurface
-1	$\forall\tau$	$\frac{1}{\tau^2+1}$	-1	$\mathbb{S}^n_{t-1}(\sqrt{\tau^2+1})$
0	$\tau \neq 0$	0	-1	$\mathbb{R}^n_{t-1}$
1	$\lvert\tau\rvert < 1$	$\frac{1}{1-\tau^2}$	1	$\mathbb{S}^n_t(\sqrt{1-\tau^2})$
1	$\lvert\tau\rvert > 1$	$\frac{-1}{\tau^2-1}$	-1	$\mathbb{H}^n_{t-1}(-\sqrt{\tau^2-1})$

Table 2. Totally umbilical hypersurfaces in $\mathbb{H}^{n+1}_t \subset \mathbb{R}^{n+2}_{t+1}$

$\langle a,a\rangle$	τ	K	ε	Hypersurface
-1	$\lvert\tau\rvert < 1$	$\frac{-1}{1-\tau^2}$	-1	$\mathbb{H}^n_{t-1}(-\sqrt{1-\tau^2})$
-1	$\lvert\tau\rvert > 1$	$\frac{1}{\tau^2-1}$	1	$\mathbb{S}^n_t(\sqrt{\tau^2-1})$
0	$\tau \neq 0$	0	1	$\mathbb{R}^n_t$
1	$\forall\tau$	$\frac{-1}{\tau^2+1}$	1	$\mathbb{H}^n_t(-\sqrt{\tau^2+1})$

Example 3. (Standard pseudo-Riemannian products in $\mathbb{M}_t^{n+1}(c)$). In order to simplify the notation, we will consider in this example that the metric tensor in $\mathbb{R}_q^{n+2}$ is given by

$$\langle,\rangle = \sum_{i=1}^{m+1} \varepsilon_i \,\mathrm{d}x_i \otimes \mathrm{d}x_i + c\,\mathrm{d}x_{m+2} \otimes \mathrm{d}x_{m+2} + \sum_{j=m+3}^{n+2} \varepsilon_j \,\mathrm{d}x_i \otimes \mathrm{d}x_j,$$

where $t = \mathrm{card}\{i \mid \varepsilon_i = -1\}$. Let $f : \mathbb{M}_t^{n+1}(c) \longrightarrow \mathbb{R}$ be the differentiable function defined by

$$f(x) = \delta_1\Big(\sum_{i=1}^{m} \varepsilon_i x_i^2\Big) + \delta_1\delta_2 x_{m+1}^2 + c x_{m+2}^2 + \delta_2\Big(\sum_{j=m+3}^{n+2} \varepsilon_j x_j^2\Big),$$

where $m \in \{1, \ldots, n-1\}$ and $\delta_1, \delta_2 \in \{0, 1\}$ with $\delta_1 + \delta_2 = 1$. In short, $f(x) = \langle Dx, x\rangle$, where D is the diagonal matrix $D = \mathrm{diag}[\delta_1, \ldots, \delta_1, \delta_1\delta_2, 1, \delta_2 \ldots, \delta_2]$. Then, for every $r > 0$ and $\rho = \pm 1$ with $\rho - cr^2 \neq 0$, the level set $M_s^n = f^{-1}(\rho r^2)$ is a hypersurface in $\mathbb{M}_t^{n+1}(c)$, for appropriate values of $(\delta_1, \delta_2, \rho, c)$.

The Gauss map is given by

$$N(x) = \frac{\overline{\nabla} f(x)}{|\overline{\nabla} f(x)|} = \frac{1}{r\sqrt{|\rho - cr^2|}}\,(Dx - \rho c r^2 x), \tag{21}$$

and the shape operator is

$$S = \frac{-1}{r\sqrt{|\rho - cr^2|}} \begin{bmatrix} (\delta_1 - \rho c r^2) I_m & \\ & (\delta_2 - \rho c r^2) I_{n-m} \end{bmatrix}.$$

In other words, M_s^n has two constant principal curvatures

$$\kappa_1 = \frac{\rho c r^2 - \delta_1}{r\sqrt{|\rho - cr^2|}} \quad \text{and} \quad \kappa_2 = \frac{\rho c r^2 - \delta_2}{r\sqrt{|\rho - cr^2|}},$$

with multiplicities m and $n - m$, respectively. In particular, every mean curvature H_k is constant. Therefore, by using (18) and (21), we get that

$$\begin{aligned} L_k\psi &= c_k H_{k+1} N \circ \psi - c c_k H_k \psi \\ &= \Big(\lambda^1\psi_1, \ldots, \lambda^1\psi_m, \theta^0\psi_{m+1}, \theta^1\psi_{m+2}, \lambda^2\psi_{m+3} \ldots, \lambda^2\psi_{n+2}\Big), \end{aligned}$$

where

$$\lambda^i = \frac{c c_k H_{k+1}(\delta_i - \rho c r^2)}{r\sqrt{|\rho - cr^2|}} - c c_k H_k, \quad \text{and} \quad \theta^i = \frac{c c_k H_{k+1}(i - \rho c r^2)}{r\sqrt{|\rho - cr^2|}} - c c_k H_k.$$

That is, M_s^n satisfies the condition $L_k\psi = A\psi + b$, with $b = 0$ and

$$A = \mathrm{diag}[\lambda^1, \ldots, \lambda^1, \theta^0, \theta^1, \lambda^2, \ldots, \lambda^2].$$

Table 3 shows the different hypersurfaces in $\mathbb{M}_t^{n+1}(c)$. Parameters u and v are defined by

$$u = \{i \mid i \leq m, \varepsilon_i = -1\} \quad \text{and} \quad v = \{i \mid i \geq m+3, \varepsilon_i = -1\},$$

where $u + v = t$.

Example 4. (Quadratic hypersurfaces with non-diagonalizable shape operator) The hypersurfaces shown in Examples 2 and 3 have diagonalizable shape operators. However, since we are working in a pseudo-Riemannian space form, it seems natural thinking of hypersurfaces with non-diagonalizable shape operator satisfying $L_k\psi = A\psi + b$. Let R be a self-adjoint endomorphism of $\mathbb{R}_q^{n+2}$, that is, $\langle Rx, y\rangle = \langle x, Ry\rangle$, for all $x, y \in \mathbb{R}_q^{n+2}$. Let $f : \mathbb{M}_t^{n+1}(c) \to \mathbb{R}$ be the quadratic function defined by $f(x) = \langle Rx, x\rangle$, and assume that the minimal polynomial of R is given by $\mu_R(z) = z^2 + a_1 z + a_0$, $a_1, a_0 \in \mathbb{R}$, with $a_1^2 - 4a_0 \leq 0$. Then, by computing the gradient in $\mathbb{M}_t^{n+1}(c)$ at each point $x \in \mathbb{M}_t^{n+1}(c)$, we have $\overline{\nabla} f(x) = 2Rx - 2cf(x)x$.

Let us consider the level set $M_d = f^{-1}(d)$, for a real constant d. Then, at a point x in M_d, we have

$$\left\langle \overline{\nabla} f(x), \overline{\nabla} f(x)\right\rangle = 4\left\langle R^2 x, x\right\rangle - 4cf(x)^2 = -4c\mu_R(cd),$$

Table 3. Standard pseudo-Riemannian products in $\mathbb{M}_t^{n+1}(c)$

δ_1	δ_2	ρ	Hypersurfaces in $\mathbb{S}_t^{n+1}$	Hypersurfaces in $\mathbb{H}_t^{n+1}$
1	0	1	$\mathbb{S}_u^m(r) \times \mathbb{S}_v^{n-m}(\sqrt{1-r^2})$ $\mathbb{S}_u^m(r) \times \mathbb{H}_{v-1}^{n-m}(-\sqrt{r^2-1})$	$\mathbb{S}_{u+1}^m(r) \times \mathbb{H}_{v-1}^{n-m}(-\sqrt{1+r^2})$
0	1	1	$\mathbb{S}_u^m(\sqrt{1-r^2}) \times \mathbb{S}_v^{n-m}(r)$ $\mathbb{H}_{u-1}^m(-\sqrt{r^2-1}) \times \mathbb{S}_v^{n-m}(r)$	$\mathbb{H}_{u-1}^m(-\sqrt{1+r^2}) \times \mathbb{S}_{v+1}^{n-m}(r)$
1	0	-1	$\mathbb{H}_{u-1}^m(-r) \times \mathbb{S}_v^{n-m}(\sqrt{1+r^2})$	$\mathbb{H}_u^m(-r) \times \mathbb{S}_v^{n-m}(\sqrt{r^2-1})$ $\mathbb{H}_u^m(-r) \times \mathbb{H}_{v-1}^{n-m}(-\sqrt{1-r^2})$
0	1	-1	$\mathbb{S}_u^m(\sqrt{1+r^2}) \times \mathbb{H}_{v-1}^{n-m}(-r)$	$\mathbb{S}_u^m(\sqrt{r^2-1}) \times \mathbb{H}_v^{n-m}(-r)$ $\mathbb{H}_{u-1}^m(-\sqrt{1-r^2}) \times \mathbb{H}_v^{n-m}(-r)$

where we have used that $R^2x = -a_1Rx - a_0x$. Then, for every $d \in \mathbb{R}$ with $\mu_R(cd) \neq 0$, $M_d = f^{-1}(d)$ is a pseudo-Riemannian hypersurface in $\mathbb{M}_t^{n+1}(c)$. The Gauss map at a point x is given by

$$N(x) = \frac{1}{|\mu_R(cd)|^{1/2}}\,(Rx - cdx), \tag{22}$$

and thus the shape operator is given by

$$SX = -\frac{1}{|\mu_R(cd)|^{1/2}}(RX - cdX), \tag{23}$$

for every tangent vector field X. From here, and bearing in mind that $R^2 + a_1R + a_0I = 0$, we obtain that

$$S^2X = -\frac{1}{|\mu_R(cd)|}\left((a_1 + 2cd)RX + (a_0 - d^2)X\right),$$

for every tangent vector field X. At this point, it is very easy to deduce that

$$\mu_S(z) = z^2 - \frac{a_1 + 2cd}{|\mu_R(cd)|^{1/2}}z + \frac{a_0 + a_1cd + d^2}{|\mu_R(cd)|}$$

is the minimal polynomial of S, and that every k-th mean curvature is constant. On the other hand, since the discriminant of $\mu_S(t)$ is not positive, the shape operator is non-diagonalizable.

Finally, from (18), we obtain that $L_k\psi = A\psi$, where A is the matrix given by

$$A = \frac{c_kH_{k+1}}{|\mu_R(cd)|^{1/2}}R - \left(\frac{c_kH_{k+1}cd}{|\mu_R(cd)|^{1/2}} + cc_kH_k\right)I.$$

5. A Key Lemma

In this section we need to compute L_kN, and to do that we are going to compute the operator L_k acting on the coordinate functions of the Gauss map N, that is, the functions $\langle N, a\rangle$ where $a \in \mathbb{R}_q^{n+2}$ is an arbitrary fixed vector. A straightforward computation yields

$$\nabla \langle N, a\rangle = -Sa^\top.$$

From Weingarten formula and (16), we find that

$$\begin{aligned}\nabla_X\nabla\langle N, a\rangle &= -\nabla_X(Sa^\top) = -(\nabla_XS)a^\top - S(\nabla_Xa^\top)\\ &= -(\nabla_{a^\top}S)X - \varepsilon\langle N, a\rangle S^2X + c\langle \psi, a\rangle SX,\end{aligned}$$

for every tangent vector field X. This equation, jointly with Lemma 3 and (13), yields

$$
\begin{aligned}
&L_k \langle N, a\rangle \\
&= -\mathrm{tr}(P_k \circ \nabla_{a^\top} S) - \varepsilon \langle N, a\rangle \mathrm{tr}(P_k \circ S^2) + c\langle \psi, a\rangle \mathrm{tr}(P_k \circ S) \\
&= -\varepsilon C_k \left\langle \nabla H_{k+1}, a^\top \right\rangle - \varepsilon C_k (nH_1 H_{k+1} - (n-k-1)H_{k+2}) \langle N, a\rangle \\
&\quad + \varepsilon c c_k H_{k+1} \langle \psi, a\rangle .
\end{aligned} \tag{24}
$$

In other words,

$$
L_k N = -\varepsilon C_k \nabla H_{k+1} - \varepsilon C_k \Big(nH_1 H_{k+1} - (n-k-1)H_{k+2} \Big) N + \varepsilon c c_k H_{k+1} \psi. \tag{25}
$$

On the other hand, equations (14) and (17) lead to

$$
\begin{aligned}
L_k(L_k \langle \psi, a\rangle) &= c_k H_{k+1} L_k \langle N, a\rangle + L_k(c_k H_{k+1}) \langle N, a\rangle + 2c_k \big\langle P_k(\nabla H_{k+1}), \nabla \langle N, a\rangle \big\rangle \\
&\quad - c c_k H_k L_k \langle \psi, a\rangle - L_k(c c_k H_k) \langle \psi, a\rangle - 2c c_k \big\langle P_k(\nabla H_k), \nabla \langle \psi, a\rangle \big\rangle,
\end{aligned}
$$

and by using again (17) and (24) we get that

$$
\begin{aligned}
L_k\big(L_k \langle \psi, a\rangle\big) &= -\varepsilon c_k C_k H_{k+1} \langle \nabla H_{k+1}, a\rangle - 2c_k \langle (S \circ P_k)(\nabla H_{k+1}), a\rangle \\
&\quad - 2c c_k \langle P_k(\nabla H_k), a\rangle - \big[\varepsilon C_k H_{k+1}\big(nH_1 H_{k+1} - (n-k-1)H_{k+2}\big) \\
&\quad + c c_k H_k H_{k+1} - L_k(H_{k+1})\big] c_k \langle N, a\rangle \\
&\quad + \big[\varepsilon c c_k H_{k+1}^2 + c_k H_k^2 - c L_k(H_k)\big] c_k \langle \psi, a\rangle .
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
L_k\big(L_k \psi\big) &= -\varepsilon c_k C_k H_{k+1} \nabla H_{k+1} - 2c_k (S \circ P_k)(\nabla H_{k+1}) - 2c c_k P_k(\nabla H_k) \\
&\quad - \big[\varepsilon C_k H_{k+1}\big(nH_1 H_{k+1} - (n-k-1)H_{k+2}\big) \\
&\quad + c c_k H_k H_{k+1} - L_k(H_{k+1})\big] c_k N \\
&\quad + \big[\varepsilon c c_k H_{k+1}^2 + c_k H_k^2 - c L_k(H_k)\big] c_k \psi.
\end{aligned} \tag{26}
$$

Let us assume that, for a fixed $k = 0, 1, \ldots, n-1$, the immersion $\psi : M_s^n \longrightarrow \mathbb{M}_t^{n+1}(c)$ satisfies the condition

$$
L_k \psi = A\psi + b, \tag{27}
$$

for a constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and a constant vector $b \in \mathbb{R}_q^{n+2}$. Then we have $L_k\big(L_k \psi\big) = A L_k \psi$, that, jointly with (26) and (18), yields

$$
\begin{aligned}
&H_{k+1}AN - cH_kA\psi\\
&= -\varepsilon C_k H_{k+1}\nabla H_{k+1} - 2(S\circ P_k)(\nabla H_{k+1}) - 2cP_k(\nabla H_k)\\
&\quad -\big[\varepsilon C_k H_{k+1}\big(nH_1H_{k+1} - (n-k-1)H_{k+2}\big)\\
&\quad +cc_kH_kH_{k+1} - L_k(H_{k+1})\big]N\\
&\quad +\big[\varepsilon cc_kH_{k+1}^2 + c_kH_k^2 - cL_k(H_k)\big]\psi.
\end{aligned} \tag{28}
$$

On the other hand, from (27), and using again (18), we have

$$
\begin{aligned}
A\psi &= c_kH_{k+1}N - cc_kH_k\psi - b^\top - \varepsilon\langle b,N\rangle N - c\langle b,\psi\rangle\psi\\
&= -b^\top + \big[c_kH_{k+1} - \varepsilon\langle b,N\rangle\big]N - \big[cc_kH_k + c\langle b,\psi\rangle\big]\psi,
\end{aligned} \tag{29}
$$

where $b^\top \in \mathfrak{X}(M_s^n)$ denotes the tangential component of b. Finally, from here and (28), we get

$$
\begin{aligned}
&H_{k+1}AN\\
&= -\varepsilon C_k H_{k+1}\nabla H_{k+1} - 2(S\circ P_k)(\nabla H_{k+1}) - 2cP_k(\nabla H_k) - cH_kb^\top\\
&\quad -\big[\varepsilon C_kH_{k+1}\big(nH_1H_{k+1}-(n-k-1)H_{k+2}\big)+\varepsilon cH_k\langle b,N\rangle - L_k(H_{k+1})\big]N\\
&\quad +\big[\varepsilon cc_kH_{k+1}^2 - H_k\langle b,\psi\rangle - cL_k(H_k)\big]\psi.
\end{aligned} \tag{30}
$$

If we take covariant derivative in (27), and use equation (18) as well as Weingarten formula, we obtain

$$
AX = -c_kH_{k+1}SX - cc_kH_kX + c_k\langle\nabla H_{k+1},X\rangle N - cc_k\langle\nabla H_k,X\rangle\psi, \tag{31}
$$

for every tangent vector field X, and therefore

$$
\langle AX,Y\rangle = \langle X,AY\rangle, \tag{32}
$$

for every tangent vector fields $X,Y\in\mathfrak{X}(M_s^n)$. That means A is a self-adjoint endomorphism when it is restricted to the tangent space.

By taking covariant derivative in (32) we obtain

$$
\begin{aligned}
&\varepsilon(\langle AN,Y\rangle - \langle N,AY\rangle)\langle SX,Z\rangle - c(\langle A\psi,Y\rangle - \langle\psi,AY\rangle)\langle X,Z\rangle\\
=&\varepsilon(\langle AN,X\rangle - \langle N,AX\rangle)\langle SY,Z\rangle - c(\langle A\psi,X\rangle - \langle\psi,AX\rangle)\langle Y,Z\rangle,
\end{aligned}
$$

for every tangent vector field $Z\in\mathfrak{X}(M_s^n)$, and then

$$
\begin{aligned}
&\varepsilon(\langle AN,Y\rangle - \langle N,AY\rangle)SX - c(\langle A\psi,Y\rangle - \langle\psi,AY\rangle)X\\
&= \varepsilon(\langle AN,X\rangle - \langle N,AX\rangle)SY - c(\langle A\psi,X\rangle - \langle\psi,AX\rangle)Y.
\end{aligned} \tag{33}
$$

Lemma 8. *Let $\psi : M_s^n \longrightarrow \mathbb{M}_t^{n+1}(c) \subset \mathbb{R}_q^{n+2}$ be an orientable hypersurface satisfying the condition $L_k\psi = A\psi + b$, for a fixed $k = 0, 1, \ldots, n-1$, some constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some constant vector $b \in \mathbb{R}_q^{n+2}$. If H_k is constant and H_{k+1} is non-constant, then $b = 0$.*

Proof. Consider the open set

$$\mathcal{U}_{k+1} = \{p \in M_s^n \mid \nabla H_{k+1}^2(p) \neq 0\},$$

which is non-empty by hypothesis. From (31) we have $\langle AX, \psi\rangle = 0$ on $\mathcal{U}_{k+1}$, and by taking covariant derivative here we obtain

$$\varepsilon \langle SX, Y\rangle \langle AN, \psi\rangle - c \langle X, Y\rangle \langle A\psi, \psi\rangle + \langle AX, Y\rangle = 0 \quad \text{on } \mathcal{U}_{k+1}.$$

This equation, jointly with (29)–(31), leads to

$$(H_k \langle SX, Y\rangle - \varepsilon H_{k+1} \langle X, Y\rangle) \langle b, \psi\rangle = 0 \quad \text{on } \mathcal{U}_{k+1}, \tag{34}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(M_s^n)$. Let us consider the open set

$$\mathcal{V} = \{p \in \mathcal{U}_{k+1} \mid \langle b, \psi\rangle (p) \neq 0\}.$$

Our goal is to show that $\mathcal{V}$ is empty. Otherwise, from (34) we get

$$H_k \langle SX, Y\rangle - \varepsilon H_{k+1} \langle X, Y\rangle = 0 \quad \text{on } \mathcal{V},$$

which implies that $H_k \neq 0$, and therefore

$$SX = \lambda X, \quad \lambda = \varepsilon \frac{H_{k+1}}{H_k}, \quad \text{on } \mathcal{V}.$$

This equation yields $\mathcal{V}$ is totally umbilical in $\mathbb{M}_t^{n+1}(c)$ and then λ (and H_{k+1}) is constant, which is a contradiction.

Therefore $\mathcal{V} = \varnothing$ and then we have $b = \varepsilon \langle b, N\rangle N$. But N is a non-constant vector field (otherwise $\mathcal{U}_{k+1}$ should be totally umbilical with constant $(k+1)$-th mean curvature), which implies $b = 0$. ■

The following auxiliar result is the key point in the proof of the main theorems.

Lemma 9. *Let $\psi : M_s^n \longrightarrow \mathbb{M}_t^{n+1}(c) \subset \mathbb{R}_q^{n+2}$ be an orientable hypersurface satisfying the condition $L_k\psi = A\psi + b$, for a fixed $k = 0, 1, \ldots, n-1$, some constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some constant vector $b \in \mathbb{R}_q^{n+2}$. If H_k is constant then H_{k+1} is constant.*

Proof. Let us assume that H_k is constant, and consider the open set

$$\mathcal{U}_{k+1} = \{p \in M_s^n \mid \nabla H_{k+1}^2(p) \neq 0\}.$$

Our goal is to show that $\mathcal{U}_{k+1}$ is empty. Otherwise, from Lemma 8 we have that $b = 0$ and then from (29) we get

$$\langle A\psi, X\rangle = 0,$$

for every tangent vector field X. Since H_k is constant, from (31) we get $\langle AX, \psi\rangle = 0$, and thus (33) is equivalent to

$$(\langle AN, Y\rangle - \langle N, AY\rangle)SX = (\langle AN, X\rangle - \langle N, AX\rangle)SY, \tag{35}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(M_s^n)$. From equation (30), we get that the tangential component of AN is given in $\mathcal{U}_{k+1}$ by

$$(AN)^\top = -\varepsilon C_k \nabla H_{k+1} - \frac{2}{H_{k+1}}(S \circ P_k)(\nabla H_{k+1}).$$

Now, bearing in mind (31) and (35), we find that

$$\langle T_k(\nabla H_{k+1}), Y\rangle\, SX = \langle X, T_k(\nabla H_{k+1})\rangle\, SY, \quad X, Y \in \mathfrak{X}(M), \tag{36}$$

where T_k is the linear self-adjoint operator defined by

$$T_k = \varepsilon(k+2)C_k I + \frac{2}{H_{k+1}}(S \circ P_k). \tag{37}$$

We claim that $T_k(\nabla H_{k+1}) = 0$ on $\mathcal{U}_{k+1}$. Indeed, if $T_k(\nabla H_{k+1})(p_0) \neq 0$ at some point $p_0 \in \mathcal{U}_{k+1}$, then there exists a neighborhood of p_0 where $T_k(\nabla H_{k+1}) \neq 0$, and we may choose a local orthonormal (or pseudo-orthonormal, respectively) frame $\{E_1, E_2, \ldots, E_n\}$ with E_1 in the direction of $T_k(\nabla H_{k+1})$. As a consequence, equation (36) implies that $SE_i = 0$ for every $i \neq 1$ (or $i \neq 2$, respectively), and then $\operatorname{rank}(S) \leq 1$ on $\mathcal{U}_{k+1}$. But this implies that $H_{k+1} = 0$ for every $k \geq 1$, which is not possible. Therefore, $T_k(\nabla H_{k+1}) = 0$ on $\mathcal{U}_{k+1}$, which implies by (37) that

$$(S \circ P_k)(\nabla H_{k+1}) = -\frac{\varepsilon(k+2)C_k}{2} H_{k+1} \nabla H_{k+1} \quad \text{on } \mathcal{U}_{k+1}. \tag{38}$$

This equation leads to the proof in the case where $k = n-1$. In fact, from the inductive definition we see that $P_n = a_n I + S \circ P_{n-1}$, and then $S \circ P_{n-1} = -a_n I = -(-\varepsilon)^n H_n I$. From this we have

$$S \circ P_{n-1}(\nabla H_n) = -(-\varepsilon)^n H_n \nabla H_n,$$

that jointly with (38) implies $H_n \nabla H_n = 0$ on $\mathcal{U}_n$, which is not possible.

Now consider the case where $1 \leq k \leq n-2$ (and $n \geq 3$ necessarily). From the inductive definition of P_{k+1} and (38) we obtain

$$P_{k+1}(\nabla H_{k+1}) + \overline{D}_k\, H_{k+1}\nabla H_{k+1} = 0 \quad \text{on } \mathcal{U}_{k+1}, \tag{39}$$

where $\overline{D}_k = \frac{\varepsilon}{2}(k+4)C_k$.

Let us assume that the tangent space is $V = V_1 \oplus \cdots \oplus V_m$ where each V_i is S-invariant and $S_i = S|_{V_i}$ is a Jordan block of type I or II. Then

$$\nabla H_{k+1} = \begin{pmatrix} \nabla H_{k+1}|_{V_1} \\ \vdots \\ \nabla H_{k+1}|_{V_m} \end{pmatrix},$$

and therefore (39) is equivalent to

$$\big(P_{k+1}|_{V_i} + \overline{D}_k H_{k+1} I\big)(\nabla H_{k+1}|_{V_i}) = 0 \quad \text{on } \mathcal{U}_{k+1},$$

for every $i = 1, \ldots, m$.

When S_i is a Jordan block of type II we can complexify and then S_i is reduced to two Jordan blocks of type I. In consequence and without loss of generality, in what follows we shall consider that every S_i is a Jordan block of type I associated to a (real or complex) root κ of S.

Let $\{E_{i_1}, \ldots, E_{i_p}\}$ be a tangent frame of subspace $V_i = V_i(\kappa)$, where $S_i = S|_{V_i}$ is a Jordan block associated to κ. From Propositions 4 and 6 we deduce

$$\begin{pmatrix} \mu^{i_1}_{k+1} + D_k H_{k+1} & & & & \\ -\mu^{i_1,i_2}_{k} & \mu^{i_2}_{k+1} + D_k H_{k+1} & & & \\ \mu^{i_1,i_2,i_3}_{k-1} & -\mu^{i_2,i_3}_{k} & \mu^{i_3}_{k+1} + D_k H_{k+1} & & \\ \vdots & \ddots & \ddots & & \\ (-1)^{p+1}\mu^{i_1,\ldots,i_p}_{k-(p-2)} & \cdots & & -\mu^{i_{p-1},i_p}_{k} & \mu^{i_p}_{k+1} + D_k H_{k+1} \end{pmatrix} \begin{pmatrix} \langle \nabla H_{k+1}, E_{i_p}\rangle \\ \langle \nabla H_{k+1}, E_{i_{p-1}}\rangle \\ \langle \nabla H_{k+1}, E_{i_{p-2}}\rangle \\ \vdots \\ \langle \nabla H_{k+1}, E_{i_1}\rangle \end{pmatrix} = 0,$$

where $D_k = (-1)^{k+1}\overline{D}_k$. Since $\kappa_{i_1} = \cdots = \kappa_{i_p} = \kappa$, then last equation is equivalent to

$$\begin{pmatrix} \langle \nabla H_{k+1}, E_{i_p}\rangle & & & \\ \langle \nabla H_{k+1}, E_{i_{p-1}}\rangle & \langle \nabla H_{k+1}, E_{i_p}\rangle & & \\ \vdots & \vdots & \ddots & \\ \langle \nabla H_{k+1}, E_{i_1}\rangle & \langle \nabla H_{k+1}, E_{i_2}\rangle & \ldots & \langle \nabla H_{k+1}, E_{i_p}\rangle \end{pmatrix} \begin{pmatrix} \mu^{i_1}_{k+1} + D_k H_{k+1} \\ -\mu^{i_1,i_2}_{k} \\ \mu^{i_1,i_2,i_3}_{k-1} \\ \vdots \\ (-1)^{p+1}\mu^{i_1,\ldots,i_p}_{k-(p-2)} \end{pmatrix} = 0.$$

As a consequence, if $\langle \nabla H_{k+1}, E_{i_p} \rangle \neq 0$, then

$$(40)\qquad \begin{cases} \mu_{k+1}^{i_1} + D_k H_{k+1} = 0, & (e_1) \\ \mu_k^{i_1,i_2} = 0, & (e_2) \\ \mu_{k-1}^{i_1,i_2,i_3} = 0, & (e_3) \\ \vdots & \\ \mu_{k-(p-2)}^{i_1,\ldots,i_p} = 0. & (e_p) \end{cases}$$

Equations $(e_2) - (e_p)$ yield

$$(41)\qquad \mu_{(k+2)-l}^{i_1,\ldots,i_q} = 0, \quad \text{for } 2 \leq l \leq q \leq p.$$

We can easily prove (41) by induction on $q-l = 0, \ldots, p-2$. If $q-l = 0$ then equation (41) follows from (40). Let us assume that (41) holds for $q - l = 0, 1 \ldots, s < p - 2$, and consider $q - l = s + 1$. Observe that

$$\mu_{(k+2)-l}^{i_1,\ldots,i_{l+s}} = \kappa_{i_{l+s+1}} \mu_{(k+2)-(l+1)}^{i_1,\ldots,i_{l+s+1}} + \mu_{(k+2)-l}^{i_1,\ldots,i_{l+s+1}},$$

then by using the induction hypothesis on both sides of this equation we find that $\mu_{(k+2)-l}^{i_1,\ldots,i_{l+s+1}} = 0$. That concludes the proof of (41).

Claim 1. *Let $\{E_{i_1}, \ldots, E_{i_p}\}$ be a tangent frame of an S-invariant subspace $V_i(\kappa)$, where $S|_{V_i}$ is a Jordan block of type I associated to a root κ. If $\langle \nabla H_{k+1}, E_{i_p} \rangle \neq 0$ then*

$$(42)\qquad \mu_{k+1}^{J} + D_k H_{k+1} = 0,$$

for every $J \subseteq \{i_i, \ldots, i_p\} := J_i(\kappa)$.

We shall prove (42) by induction on the cardinality of J, card(J). If card(J)=1, then (42) is nothing but equation (e_1) in (40). If card(J)=2, $J = \{i_1, i_2\}$, then (42) is a consequence of (e_1) and (e_2) in (40), since we have

$$0 = \mu_{k+1}^{i_1} + D_k H_{k+1} = \left(\kappa_{i_2} \mu_k^{i_1,i_2} + \mu_{k+1}^{i_1,i_2}\right) + D_k H_{k+1} = \mu_{k+1}^{i_1,i_2} + D_k H_{k+1}.$$

Let us assume that (42) is true for every subset J with card$(J) = 1, 2, \ldots, m < p$ and consider a set $J_0 = \{i_1, \ldots, i_{m+1}\}$ with cardinality $m + 1 \leq p$. Let J_1 be the set of cardinality m such that $J_0 = J_1 \cup \{i_{m+1}\}$. By the induction hypothesis applied to J_1 and bearing in mind (41) we get

$$0 = \mu_{k+1}^{J_1} + D_k H_{k+1} = \left(\kappa_{i_{m+1}} \mu_k^{J_0} + \mu_{k+1}^{J_0}\right) + D_k H_{k+1} = \mu_{k+1}^{J_0} + D_k H_{k+1},$$

and that concludes the proof of Claim 1.

An immediate and important consequence of this claim is that $\langle \nabla H_{k+1}, E_i\rangle = 0$ for some i. Otherwise, from Claim 1 we deduce

$$\mathrm{tr}(P_{k+1}) = \sum_{\ell,j=1}^{n} g^{\ell j}\left\langle P_{k+1}E_\ell, E_j\right\rangle = \sum_{\ell=1}^{n}(-1)^{k+1}\mu^{\ell}_{k+1} = (-1)^k n D_k H_{k+1},$$

that jointly with Lemma 3 leads to $H_{k+1} = 0$ on $\mathcal{U}_{k+1}$, which is a contradiction.

Claim 2. *Let $\{E_{i_1}, \ldots, E_{i_p}\}$ and $\{E_{j_1}, \ldots, E_{j_q}\}$ be tangent frames of two S-invariant subspaces $V_i(\kappa_1)$ and $V_j(\kappa_2)$, where $S|_{V_i}$ and $S|_{V_j}$ are Jordan blocks associated to two distinct roots κ_1 and κ_2, respectively. If $\left\langle \nabla H_{k+1}, E_{i_p}\right\rangle \neq 0$ and $\left\langle \nabla H_{k+1}, E_{j_q}\right\rangle \neq 0$ then*

$$\mu^{J}_{k+1} + D_k H_{k+1} = 0, \tag{43}$$

for every set $J \subseteq \{i_1, \ldots, i_p, j_1, \ldots, j_q\} = J_i(\kappa_1)\cup J_j(\kappa_2)$.

We can write $J = J_1 \cup J_2$, where $J_1 \subseteq J_i(\kappa_1)$ and $J_2 \subseteq J_j(\kappa_2)$, and then card$(J) = m_1 + m_2$, with $m_1 = \mathrm{card}(J_1)$ and $m_2 = \mathrm{card}(J_2)$. We shall prove (43) by induction on $m = m_1 + m_2$. If $m = 1$, then (43) is nothing but (42).

Let us assume that (43) holds for every set J with card$(J) = 1, 2, \ldots, r < p + q$ and consider a set $J_0 = \{h_1, \ldots, h_{r+1}\} \subseteq \{i_1, \ldots, i_p, j_1, \ldots, j_q\}$ with cardinality $r + 1 \leq p + q$. In the case where J_0 is a subset either of J_1 or J_2, there is nothing to prove. Thus let us assume that J_0 has elements of J_1 and J_2.

Without loss of generality, we can assume that $h_1 \in J_1$ and $h_{r+1} \in J_2$, and let I_1 and I_2 be the two sets of cardinality r such that $J_0 = I_1 \cup \{h_{r+1}\} = \{h_1\} \cup I_2$. From the induction hypothesis we deduce

$$\begin{aligned}
0 &= \mu^{I_1}_{k+1} + D_k H_{k+1} = \left(\kappa_{h_{r+1}}\mu^{J_0}_{k} + \mu^{J_0}_{k+1}\right) + D_k H_{k+1},\\
0 &= \mu^{I_2}_{k+1} + D_k H_{k+1} = \left(\kappa_{h_1}\mu^{J_0}_{k} + \mu^{J_0}_{k+1}\right) + D_k H_{k+1},
\end{aligned}$$

and then $\left(\kappa_{h_1} - \kappa_{h_{r+1}}\right)\mu^{J_0}_{k} = 0$. Since $\kappa_{h_1} \neq \kappa_{h_{r+1}}$ we obtain $0 = \mu^{J_0}_{k+1} + D_k H_{k+1}$, as desired. That concludes the proof of Claim 2.

Claim 3. *Let $\{E_{i_1}, \ldots, E_{i_p}\}$ and $\{E_{j_1}, \ldots, E_{j_q}\}$ be tangent frames of two S-invariant subspaces $V_i(\kappa)$ and $V_j(\kappa)$, where $S|_{V_i}$ and $S|_{V_j}$ are Jordan blocks associated to the same root κ. Then there exists a tangent vector $\widetilde{E}$ such that*

$$S\widetilde{E} = \kappa\widetilde{E} \quad \textit{and} \quad \langle \nabla H_{k+1}, \widetilde{E}\rangle = 0.$$

To prove this claim, we distinguish two cases:
(a) If $\left\langle \nabla H_{k+1}, E_{i_p} \right\rangle = 0$ (or $\left\langle \nabla H_{k+1}, E_{j_q} \right\rangle = 0$, respectively), there is nothing to prove, we can take $\widetilde{E} = E_{i_p}$ (or $\widetilde{E} = E_{j_q}$, respectively).
(b) If $\left\langle \nabla H_{k+1}, E_{i_p} \right\rangle \neq 0$ and $\left\langle \nabla H_{k+1}, E_{j_q} \right\rangle \neq 0$, then we take

$$\widetilde{E} = -\left\langle \nabla H_{k+1}, E_{j_q} \right\rangle E_{i_p} + \left\langle \nabla H_{k+1}, E_{i_p} \right\rangle E_{j_q}.$$

Two consequences can be obtained from this claim.

(C1) If κ is real, then from (31) we get

$$A\widetilde{E} = -c_k H_{k+1} \kappa \widetilde{E},$$

and then there exists a constant eigenvalue η of matrix A such that

$$\kappa = \frac{\eta}{-c_k H_{k+1}}. \tag{44}$$

(C2) If $\kappa = \alpha + i\beta$ is complex, then there exist two (real) tangent vectors $\widetilde{E}_1, \widetilde{E}_2$ such that $\widetilde{E} = \widetilde{E}_1 + i\widetilde{E}_2$ and $\langle \nabla H_{k+1}, \widetilde{E}_i \rangle = 0$ for $i = 1, 2$. In this case, $W = \text{span}\{\widetilde{E}_1, \widetilde{E}_2\}$ is an S-invariant subspace and $S|_W$ has matrix of form

$$S|_W = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

By using (31) we get that W is also an A-invariant subspace with matrix of form

$$A|_W = \begin{pmatrix} -c_k H_{k+1}\alpha & -c_k H_{k+1}\beta \\ c_k H_{k+1}\beta & -c_k H_{k+1}\alpha \end{pmatrix}.$$

As a consequence, we obtain that

$$\theta = \text{tr}(A|_W) \quad \text{and} \quad \rho = \det(A|_W)$$

are invariants of A (and constant). Explicitly, they are given by $\theta = -2(c_k H_{k+1}\alpha)$ and $\rho = (c_k H_{k+1})^2(\alpha^2 + \beta^2)$, and then it is easy to see that there exist two constants s_1 and s_2 such that

$$\alpha = \frac{s_1}{-c_k H_{k+1}} \quad \text{and} \quad \beta = \frac{s_2}{-c_k H_{k+1}}.$$

Thus we can write

$$\kappa = \frac{\eta}{-c_k H_{k+1}}, \quad \eta = s_1 + i s_2. \tag{45}$$

To finish the proof of Lemma, let K be the following subset of roots of $Q_S(t)$:

$$K = \{\kappa \mid \mathrm{JB}(\kappa) = 1 \text{ and } \langle \nabla H_{k+1}, E_{i_p} \rangle \neq 0\},$$

where $\mathrm{JB}(\kappa)$ stands for the number of Jordan blocks associated to the root κ. From Claim 2 we deduce

$$\mu^J_{k+1} + D_k H_{k+1} = 0,$$

for every subset $J \subseteq \bigcup_{\kappa_i \in K} J(\kappa_i)$. In particular, for $J = \bigcup_{\kappa_i \in K} J(\kappa_i)$ we obtain

$$-D_k H_{k+1} = \mu^J_{k+1} = \sum_{\substack{i_1 < \cdots < i_{k+1} \\ i_j \notin J}}^{n} \kappa_{i_1} \ldots \kappa_{i_{k+1}} = \sum_{\substack{i_1 < \cdots < i_{k+1} \\ \kappa_{i_j} \notin K}}^{n} \kappa_{i_1} \ldots \kappa_{i_{k+1}}$$

that jointly with (44) and (45) lead to

$$-D_k H_{k+1} = \frac{\sum_{i_1 < \cdots < i_{k+1}} \eta_{i_1} \cdots \eta_{i_{k+1}}}{\big(- c_k H_{k+1} \big)^{k+1}} \quad \text{on } \mathcal{U}_{k+1},$$

showing that H_{k+1} is locally constant on $\mathcal{U}_{k+1}$, which is a contradiction. ■

6. Main Results

This section is devoted to prove the main result of this paper.

Theorem 1. *Let $\psi : M^n_s \to \mathbb{M}^{n+1}_t(c) \subset \mathbb{R}^{n+2}_q$ be an orientable hypersurface immersed into the pseudo-Riemannian space form $\mathbb{M}^{n+1}_t(c)$, and let L_k be the linearized operator of the $(k+1)$-th mean curvature of M^n_s, for some fixed $k = 0, 1, \ldots, n-1$. Assume that H_k is constant. Then the immersion satisfies the condition $L_k \psi = A\psi + b$, for some constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some constant vector $b \in \mathbb{R}^{n+2}_q$, if and only if it is one of the following hypersurfaces:*

(1) *a hypersurface having zero $(k+1)$-th mean curvature and constant k-th mean curvature.*
(2) *an open piece of one of the following totally umbilical hypersurfaces in $\mathbb{S}^{n+1}_t$: $\mathbb{S}^n_{t-1}(r)$, $r > 1$; $\mathbb{S}^n_t(r)$, $0 < r < 1$; $\mathbb{H}^n_{t-1}(-r)$, $r > 0$; $\mathbb{R}^n_{t-1}$.*
(3) *an open piece of one of the following totally umbilical hypersurfaces in $\mathbb{H}^{n+1}_t$: $\mathbb{H}^n_t(-r)$, $r > 1$; $\mathbb{H}^n_{t-1}(-r)$, $0 < r < 1$; $\mathbb{S}^n_t(r)$, $r > 0$; $\mathbb{R}^n_t$.*
(4) *an open piece of a standard pseudo-Riemannian product in $\mathbb{S}^{n+1}_t$: $\mathbb{S}^m_u(r) \times \mathbb{S}^{n-m}_v(\sqrt{1-r^2})$, $\mathbb{H}^m_{u-1}(-r) \times \mathbb{S}^{n-m}_v(\sqrt{1+r^2})$, $\mathbb{S}^m_u(r) \times \mathbb{H}^{n-m}_{v-1}(-\sqrt{r^2-1})$.*
(5) *an open piece of a standard pseudo-Riemannian product in $\mathbb{H}^{n+1}_t$: $\mathbb{H}^m_u(-r) \times \mathbb{S}^{n-m}_v(\sqrt{r^2-1})$, $\mathbb{S}^m_u(r) \times \mathbb{H}^{n-m}_v(-\sqrt{1+r^2})$, $\mathbb{H}^m_u(-r) \times \mathbb{H}^{n-m}_{v-1}(-\sqrt{1-r^2})$.*

(6) *an open piece of a quadratic hypersurface* $\{x \in \mathbb{M}_t^{n+1}(c) \subset \mathbb{R}_q^{n+2} \mid \langle Rx, x\rangle = d\}$*, where* R *is a self-adjoint constant matrix whose minimal polynomial is* $z^2 + az + b$*,* $a^2 - 4b \leq 0$*.*

Proof. We have already checked in Section 4 that each one of the hypersurfaces mentioned in Theorem 1 does satisfy the condition $L_k\psi = A\psi + b$, for a constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some constant vector $b \in \mathbb{R}_q^{n+2}$.

Conversely, let us assume that $\psi : M_s^n \to \mathbb{M}_t^{n+1}(c) \subset \mathbb{R}_q^{n+2}$ satisfies the condition $L_k\psi = A\psi + b$, for some constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some constant vector $b \in \mathbb{R}_q^{n+2}$. Since H_k is constant on M_s^n, from Lemma 9 we know that H_{k+1} is also constant on M_s^n. Let us assume that H_{k+1} is a non-zero constant (otherwise, there is nothing to prove).

From (31) and (28) we have

$$AX = -c_k H_{k+1} SX - cc_k H_k X, \tag{46}$$

$$AN = (\lambda - cc_k H_k)N + c_k(\varepsilon c H_{k+1} + \frac{H_k^2}{H_{k+1}})\psi + \frac{cH_k}{H_{k+1}} A\psi, \tag{47}$$

with $\lambda = -\varepsilon C_k(nH_1H_{k+1} - (n-k-1)H_{k+2})$. Taking covariant derivative in (47), and using (46), we have

$$\nabla_X^0(AN) = \langle \nabla\lambda, X\rangle N - \lambda SX + \varepsilon cc_k H_{k+1} X,$$

but also from (46) we obtain

$$\nabla_X^0(AN) = A(\nabla_X^0 N) = -A(SX) = c_k H_{k+1} S^2 X + cc_k H_k SX.$$

From the last two equations we deduce that λ is constant on M_s^n, and also that the shape operator S satisfies the equation

$$S^2 + a_1 S - \varepsilon c I = 0, \qquad a_1 = \frac{\lambda + cc_k H_k}{c_k H_{k+1}} = \text{constant}. \tag{48}$$

As a consequence, M_s^n is an isoparametric hypersurface in $\mathbb{M}_t^{n+1}(c)$ and the minimal polynomial of its shape operator S is of degree at most two. If the degree of that polynomial is one, then M_s^n is totally umbilical (but not totally geodesic) in $\mathbb{M}_t^{n+1}(c)$ and so it is one of the hypersurfaces listed in paragraphs (2) or (3) of the theorem, according to $c = 1$ or $c = -1$, respectively (Example 2). Let us assume that the minimal polynomial of S is exactly of degree two. If S is diagonalizable, then M_s^n has exactly two distinct constant principal curvatures, and then from standard arguments (similar to those used in [13, 23, 19, 18, 25, 26]) it is an open piece of a standard pseudo-Riemannian product (Example 3).

Suppose now that S is not diagonalizable, so that the minimal polynomial of S is given by $\mu_S(z) = z^2 + a_1 z - \varepsilon c$, with discriminant $d_S = a_1^2 + 4\varepsilon c \leq 0$. From above equations we easily deduce that the minimal polynomial of A is given by $\mu_A(z) = z^2 + b_1 z + b_0$, where $b_1 = 2cc_k H_k - a_1 c_k H_{k+1}$ and $b_0 = c_k^2 H_k^2 - a_1 cc_k^2 H_k H_{k+1} - \varepsilon cc_k^2 H_{k+1}^2$ are constants. Since the discriminant d_A of $\mu_A(z)$ is given by $d_A = c_k^2 H_{k+1}^2 d_S$, then A also is not diagonalizable. Since $\langle A\psi, \psi\rangle = -c_k H_k$ is constant and $\mu_A(-cc_k H_k) \neq 0$, then M_s^n is an open piece of a quadratic hypersurface as in Example 4. That concludes the proof. ∎

As an easy consequence of this theorem we obtain the following result.

Theorem 2. *Let $\psi : M_s^n \to \mathbb{M}_t^{n+1}(c) \subset \mathbb{R}_q^{n+2}$ be an orientable hypersurface immersed into the pseudo-Riemannian space form $\mathbb{M}_t^{n+1}(c)$, and let L_k be the linearized operator of the $(k+1)$-th mean curvature of M_s^n, for some fixed $k = 0, 1, \ldots, n-1$. Then the immersion satisfies the condition $L_k\psi = A\psi$, for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$, if and only if it is one of the following hypersurfaces:*

(1) *a hypersurface having zero $(k+1)$-th mean curvature and constant k-th mean curvature;*

(2) *an open piece of a standard pseudo-Riemannian product in $\mathbb{S}_t^{n+1}$: $\mathbb{S}_u^m(r) \times \mathbb{S}_v^{n-m}(\sqrt{1-r^2})$, $\mathbb{H}_{u-1}^m(-r) \times \mathbb{S}_v^{n-m}(\sqrt{1+r^2})$, $\mathbb{S}_u^m(r) \times \mathbb{H}_{v-1}^{n-m}(-\sqrt{r^2-1})$.*

(3) *an open piece of a standard pseudo-Riemannian product in $\mathbb{H}_t^{n+1}$: $\mathbb{H}_u^m(-r) \times \mathbb{S}_v^{n-m}(\sqrt{r^2-1})$, $\mathbb{S}_u^m(r) \times \mathbb{H}_v^{n-m}(-\sqrt{1+r^2})$, $\mathbb{H}_u^m(-r) \times \mathbb{H}_{v-1}^{n-m}(-\sqrt{1-r^2})$.*

(4) *an open piece of a quadratic hypersurface $\{x \in \mathbb{M}_t^{n+1}(c) \subset \mathbb{R}_q^{n+2} \mid \langle Rx, x\rangle = d\}$, where R is a self-adjoint constant matrix whose minimal polynomial is $z^2 + az + b$, $a^2 - 4b \leq 0$.*

Proof. Since A is a self-adjoint matrix we have $\langle AX, \psi\rangle = \langle X, A\psi\rangle$, and by using (29) and (31) we deduce

$$\nabla \langle b, \psi\rangle = b^\top = c_k \nabla H_k,$$

which implies that H_k is constant. Now the result follows from Theorem 1. ∎

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