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Finding All Salem Numbers of Trace −1 and Degree up to 20

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Abstract. In 1999, C. J. Smyth proved that, for all $d \geq 4$, there are Salem numbers of degree $2d$ and trace -1, and that the number of them is greater than $0.1387d/(\log\log d)^2$. He gave also all Salem numbers of trace -1 and degree $2d = 8, 10, 12, 14$. In this paper, we complete the table of the minimal polynomials of all Salem numbers of trace -1 and degree $2d = 16, 18, 20$, and we conjecture a new lower bound of the numbers of such Salem numbers.

1. Introduction

A Salem number is a real algebraic integer greater than 1 whose other conjugates all lie in the closed disc $|z| \leq 1$, with at least one on the unit circle. Its minimal polynomial is a reciprocal polynomial of degree $2d$ with $d \geq 2$. The smallest known Salem number $1.17628\ldots$, discovered by Lehmer [4] in 1933, has minimal polynomial $x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$.

In 1999, C. J. Smyth [8] studied the set $\mathcal{S}_d$ of Salem numbers of degree $2d$ and trace -1. He proved that for all $d \geq 4$, there are Salem numbers of degree $2d$ and trace -1, all Salem numbers of degree up to 18 have trace at least -1, and the number of elements of the set $\mathcal{S}_d$ satisfies

$$|\mathcal{S}_d| \geq \frac{0.1387d}{(\log\log d)^2}.$$

He also said that it is likely that $|\mathcal{S}_d|$ grows at least exponentially with d. At the same time, he gave all Salem numbers of trace -1 and degree $2d = 8, 10, 12, 14$. In 2004, J. McKee and C. J. Smyth [6] established that the minimal degree for a Salem number of trace -2 is 20 and exhibit all Salem number of degree 20 and trace -2. In 2009, V. Flammang [3] proved that if a Salem number has trace -3 then its degree is at least 30. In 2011, Y. Liang and the third author [5] proved that if a Salem number has trace -4 then its degree is at least 40 and if a Salem number has trace -5 then its degree is at least 50.

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In this paper, with 1.79193 [5] for the lower bound of the absolute trace of totally positive algebraic integers, we find all totally positive algebraic integers α of trace $2d-1$ and degree $d = 8, 9, 10$. There are 151, 686 and 3699 of them respectively. With the transformation $z + 1/z + 2$, we then have all Salem numbers of trace -1 and degree $2d = 16, 18, 20$, the numbers of them are 138, 458 and 1814 respectively. It is only those α with $\alpha > 4$ and other conjugates in $(0, 4)$ that give rise to Salem numbers in this way. For the lower bound of the numbers of such Salem numbers, as it is likely that $|\mathcal{S}_d|$ grows at least exponentially with d [8], using the numerical simulation method, we then make the following conjecture.

Conjecture 1.1. *For every $d \geq 4$,*

$$|\mathcal{S}_d| \geq \frac{(0.1071d)^5 e^d}{(\log d)^4 \log\log\log d^3}.$$

This paper is organized as follows. In Section 2, we briefly describe the research method. Section 3 is devoted to the final computation to get all Salem numbers of trace -1 and degree $2d = 16, 18, 20$.

2. The research method

2.1. The intervals of research

Let α be a totally positive algebraic integer of degree d, with trace $2d-1$, whose conjugates are $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$, and let

$$P(x) = x^d - (2d-1)x^{d-1} + b_2 x^{d-2} + \cdots + b_d$$

be its minimal polynomial.

In 2011, Y. Liang and the third author [5] proved that for $x > 0$,

$$f(x) = x - \sum_{1 \leq j \leq J} c_j \log|Q_j(x)| > 1.79193,$$

where $c_j > 0$ for $1 \leq j \leq J$, and $Q_j(x) \in \mathbb{Z}[x]$ can be found in [5].

Then, for the totally positive algebraic integer $\alpha = \alpha_1$ of degree d and trace $2d-1$, and its conjugates $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d$, we have

$$\alpha = f(\alpha) + \sum_{1 \leq j \leq J} c_j \log|Q_j(\alpha)| \quad \text{and} \quad \alpha_i > 1.79193 + \sum_{1 \leq j \leq J} c_j \log|Q_j(\alpha_i)|$$

for $2 \leq i \leq d$. Adding, we obtain

$$\operatorname{tr}(\alpha) = 2d - 1 > 1.79193(d-1) + f(\alpha) + \sum_{j=1}^{J} c_j \log|\operatorname{Res}(Q_j, P)|.$$

When $P(x)$ doesn't divide any of the $Q_j(x)$ for $1 \le j \le J$, $\mathrm{Res}(P, Q_j)$ is a nonzero integer, thus

$$\sum_{j=1}^{J} c_j \log|\mathrm{Res}(Q_j, P)| \ge 0.$$

Then

$$f(\alpha) < 2d - 1 - 1.79193(d-1) = 0.20807d + 0.79193,$$

i.e., for $d = 8$, $f(\alpha) < 2.45649$; for $d = 9$, $f(\alpha) < 2.66456$; and for $d = 10$, $f(\alpha) < 2.87263$.

So for the totally positive algebraic integer $\alpha = \alpha_1$ of degree d and trace $2d-1$, if $d = 8$, then $0 < \alpha < 6.931$; if $d = 9$, then $0 < \alpha < 7.327$; and if $d = 10$, then $0 < \alpha < 7.699$.

2.2. Newton's relation

Let $s_k = \sum_{i=1}^{d} \alpha_i^k$ for $k \ge 1$. It is clear that $s_1 = \mathrm{tr}(\alpha)$. The s_k are related to the coefficients of $P(x)$ by Newton's relation:

$$s_k + b_1 s_{k-1} + \cdots + b_{k-1}s_1 + kb_k = 0$$

for $k \ge 1$ with $b_k = 0$ for $k > d$. In our cases, $s_1 = 2d - 1$.

2.3. The bound of s_k

2.3.1. The auxiliary function

We use another auxiliary function to get better bounds for s_k for $k > 1$. Let $x \in B_d$ where $B_8 = (0, 6.931)$, $B_9 = (0, 7.327)$, $B_{10} = (0, 7.699)$. Let $\overline{e_k} = (e_{k0}, e_{k1}, e_{k2}, \dots, e_{kJ_k}) \in \mathbb{R}^n$ where $e_{kj} \ge 0$, and $Q_{kj} \in \mathbb{Z}[x]$. For the lower bound of s_k $(k > 1)$, we consider the auxiliary function

$$f_1(x, k, \overline{e_k}) = x^k - e_{k0}x - \sum_{1\le j\le J_k} e_{kj} \log|Q_{kj}(x)|. \tag{2.1}$$

Suppose $m_k(\overline{e_k}) = \min_{x\in B_d} f_1(x, k, \overline{e_k})$. Then we have

$$\alpha_i^k - e_{k0}\alpha_i - \sum_{1\le j\le J_k} e_{kj}\log|Q_{kj}(\alpha_i)| \ge m_k(\overline{e_k}) \quad \text{for } 1 \le i \le d,$$

and therefore,

$$\sum_{i=1}^{d}\alpha_i^k - e_{k0}\sum_{i=1}^{d}\alpha_i - \sum_{1\le j\le J_k} e_{kj}\log\left|\prod_{1\le i\le d} Q_{kj}(\alpha_i)\right| \ge m_k(\overline{e_k})d,$$
$$s_k - e_{k0}s_1 - \sum_{1\le j\le J_k} e_{kj}\log|\mathrm{Res}(P, Q_{kj})| \ge m_k(\overline{e_k})d.$$

When $P(x)$ doesn't divide any of the $Q_{kj}(x)$, $\mathrm{Res}(P, Q_{kj})$ is a nonzero integer, thus for $k \geq 2$,

$$s_k \geq m_k(\overline{e_k})d + e_{k0}(2d-1).$$

Clearly, we then have to solve the following optimization problems:

$$m_k = \max_{\overline{e_k}} m_k(\overline{e_k}) = \max_{\overline{e_k}} \min_{x \in B_d} f_1(x, k, \overline{e_k}).$$

Similarly, if we replace x^k by $-x^k$ and e_{kj} by e'_{kj} in (2.1), we have

$$f'_1(x, k, \overline{e}') = -x^k - e'_{k0}x - \sum_{1 \leq j \leq J'} e'_{kj} \log |Q'_{kj}(x)|. \tag{2.2}$$

We will get the upper bounds for s_k for $k \geq 2$.

Using auxiliary functions (2.1) and (2.2), we get bounds for s_k ($k \geq 2$), for totally positive algebraic integers of degree $d = 8, 9, 10$ and trace $2d - 1$. We give here, as an example, the bounds for s_k ($1 \leq k \leq 10$) for degree $d = 9$ which we obtain by this method.

Table 2.1: Bounds of s_k for degree $d = 9$.

k	1	2	3	4	5
$s_k \geq$	17	49	158	541	1923
$s_k \leq$	17	68	379	2450	17033
k	6	7	8	9	10
$s_k \geq$	7004	25878	96597	363321	1373766
$s_k \leq$	121432	892882	6441395	47158743	345213844

The Q_j and e_j which are used to compute the lower bound of s_2 for totally positive algebraic integers of degree $d = 9$ are listed in Table 3.3 and Table 3.4.

We also construct an auxiliary function to get better bounds of s_3 for fixed s_2:

$$f_2(x, \overline{e}'') = x^3 - e''_0 x - e''_1 x^2 - \sum_{2 \leq j \leq J''} e''_j \log |Q''_j(x)|, \tag{2.3}$$

where the numbers e''_j and the polynomials Q''_j are defined as above. If m is the minimum of $f_2(x, \overline{e}'')$ for $x \in B_d$, by the same argument, we get

$$s_3 - e''_0 s_1 - e''_1 s_2 = s_3 - e''_0(2d-1) - e''_1 s_2 \geq md.$$

If we assume that s_2 has the value σ, then $s_3 \geq dm + e''_0(2d-1) + e''_1\sigma$. We optimize the numbers $e''_0, \ldots, e''_J$ to get a maximal value of $dm + e''_0(2d-1) + e''_1\sigma$. Therefore we get a lower bound for s_3 depending on the value of σ. This gives a better bound than the one which was given by auxiliary function (2.1), if we take σ close to its upper bound. If we replace x^3 in (2.3) by $-x^3$, we get upper bounds for s_3.

Using auxiliary function (2.3), when s_2 is fixed as in the first row in the following table, we get the new bounds of s_3 (for example $d = 9$):

Table 2.2: Bounds of s_3 for degree $d = 9$ when s_2 is fixed.

s_2	49	51	53	55	57	59	61	63	65	67
$s_3 \geq$	159	170	182	195	210	227	246	264	285	310
$s_3 \leq$	171	194	215	239	262	284	306	328	350	369

2.3.2. The optimization method

For example, in order to get the largest possible lower bound for s_k, we only need to find the greatest $m_k(\overline{e_k})$. If, in the auxiliary function of (2.1), we replace the real numbers e_{kj} by rational numbers we may write

$$f(k, z) = z^k - e_{k0}z - \frac{t}{h_k} \log |H_k(z)|,$$

where H_k is in $\mathbb{Z}[X]$ of degree h_k and t is a positive real number. We want to obtain a function $f(k, z)$ whose minimum m_k in B_d is as large as possible. That is to say, we seek a polynomial $H_k \in \mathbb{Z}[X]$ such that

$$\sup_{z \in B_d} |H_k(z)|^{t/h_k} \exp(z^k - e_{k0}z) \leq e^{-m_k}.$$

Now, if we suppose that t is fixed, it is clear that we need to get an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}(B_d) = \liminf_{\substack{h_k \geq 1 \\ h_k \to \infty}} \inf_{\substack{H_k \in \mathbb{Z}[X] \\ \deg H_k = h_k}} \sup_{z \in B_d} |H_k(z)|^{t/h_k} \varphi(z)$$

in which we use the weight $\varphi(z) = \exp(z^k - e_{k0}z)$. To get an upper bound for $t_{\mathbb{Z},\varphi}(B_d)$, it is sufficient to get an explicit polynomial $H_k \in \mathbb{Z}[X]$ and then to use the sequence of the successive powers of H_k.

The function $t_{\mathbb{Z},\varphi}(B_d)$ is a generalization of the integer transfinite diameter. For any $h \geq 1$, we say that a polynomial H (not always unique) is an *integer Chebyshev polynomial* if the quantity $\sup_{z \in B_d} |H(z)|^{t/h} \varphi(z)$ is minimum. With the third author's algorithm [9], we compute the polynomials H of degree less than 30 and take their irreducible factors as the polynomials Q_j. We start with the polynomial $x - 1$, get the best e_1 and take $t = e_1$.

When we have computed J polynomials, we optimize the numbers e_j with a refinement of the semi-infinite linear programming method that has been introduced into number theory by Smyth [7]. This gives us a new number t. We continue by induction to get $J+1$ polynomials. More details can be found in [9] or [1]. The LLL algorithm is our main method for finding the Q_i. The basic idea is similar to that of V. Flammang in [3], but her method has difficulty for finding the Q_i of higher degree. Moreover, technical improvements allows us to find the polynomials Q_j of higher degree than before.

2.3.3. The Chebyshev polynomial

Let T_k be the Chebyshev polynomial of degree k of the interval (a,b). This is the monic polynomial whose sup norm is the least on (a,b). The polynomials T_k can be defined by the relations $T_1 = x - A$, $T_2 = x^2 - 2Ax + A^2 - 2B^2$ and $T_k = (x-A)T_{k-1} - B^2T_{k-2}$ for $k > 2$, where $A = (a+b)/2$ and $B = (b-a)/4$. Then $\max_{a \le x \le b} |T_k(x)| = 2B^k$ for $k \ge 1$.

Then we have the inequality $\left|\sum_{1 \le i \le d} T_k(\alpha_i)\right| \le 2dB^k$. This gives a lower and upper bound for s_k depending on the known values of s_j for $0 \le j \le k-1$.

Using the Chebyshev Polynomials, we refined the bounds for s_k. For example, when $s_2 = 59$ and $d = 9$, then for each s_3 fixed, we get new bounds for s_4:

Table 2.3: Bounds of s_4 for degree $d = 9$, $s_2 = 59$ when s_3 is fixed.

s_3	227	230	233	236	239	242	245	248	251	254
$s_4 \ge$	634	678	722	766	810	854	898	942	986	1030
$s_4 \le$	1038	1082	1126	1170	1214	1258	1302	1346	1390	1434
s_3	257	260	263	266	269	272	275	278	281	284
$s_4 \ge$	1074	1118	1162	1206	1250	1294	1338	1381	1425	1469
$s_4 \le$	1478	1522	1566	1610	1654	1698	1742	1786	1830	1874

Remark 2.1. All the bounds for s_k in this Section were obtained by the auxiliary functions or by the Chebyshev polynomials. In our computation of totally positive algebraic integers, we refined the bounds of s_k since $s_1, \ldots, s_{k-1}$ determine $b_1, \ldots, b_{k-1}$ by the Newton's relation, it follows that s_k is determined modulo k by $s_1, \ldots, s_{k-1}$.

3. Numerical results

Using the method described in Section 2, we have got all 151, 686 and 3669 totally positive algebraic integers of trace $2d-1$ and degree $d = 8, 9, 10$ respectively. With the transformation $z + 1/z + 2$ we then obtain all Salem numbers of trace -1 and degree $2d = 16, 18, 20$, the numbers of them are 138, 458 and 1814 respectively. We give, in the following table, the numbers of Salem numbers $|\mathcal{S}_d|$ of trace -1 for $8 \le 2d \le 20$, the numerical lower bounds of the numbers of them $B(|\mathcal{S}_d|)$ which is given by Conjecture 1.1. We give also the smallest Salem number SS and the largest Salem number LS in this table.

Table 3.1: The numbers of Salem number of degree $2d$ and trace -1.

$2d$	8	10	12	14	16	18	20
$\|\mathcal{S}_d\|$	1	3	9	39	138	458	1814
$B(\|\mathcal{S}_d\|)$	0.6	2.1	8.2	31.9	121.7	456.0	1675.7
SS	1.84959	1.17628	1.48856	1.51662	1.30840	1.41126	1.31987
LS	1.84959	2.16305	2.43439	3.34978	3.97837	4.32474	4.46363

The minimal polynomials of Salem numbers of trace -1 and $2d = 16, 18$ are listed in Table 3.2 with coefficient of x^{2d} is equal to 1 and coefficient of x^{2d-1} is equal to -1. As the number of all Salem number is too large, and -1 is not the smallest trace, we don't list all Salem numbers of degree 20 and trace -1 in this paper. One can find the completed table by contacting the corresponding author.

The computing times of all totally positive algebraic integers of trace $2d-1$ and degree $d = 9, 10$ were 14.9 hours and 2276.76 hours respectively on a 3.0Ghz PC. All the computations were realized by the Pascal programming language and Pari/GP [2].

Table 3.2: Minimal polynomials of all Salem numbers of degree $2d = 16$ and trace -1.

α	Coefficients of $x^{2d-2}, \ldots, x^d$						α	Coefficients of $x^{2d-2}, \ldots, x^d$					
3.978378	-7 -29	-59	-83	-90	-84	-79	3.817029	-6 -25	-53	-83	-107	-121	-125
3.628389	-5 -21	-46	-77	-107	-128	-135	3.520614	-5 -20	-40	-59	-71	-75	-75
3.424949	-5 -19	-35	-46	-48	-44	-41	3.403021	-5 -19	-34	-42	-40	-33	-29
3.289959	-5 -18	-29	-29	-17	-2	5	3.553164	-4 -19	-47	-86	-127	-159	-171
3.535392	-4 -19	-46	-82	-118	-145	-155	3.468150	-4 -18	-42	-73	-103	-125	-133
3.346133	-4 -17	-36	-55	-67	-72	-73	3.304705	-4 -16	-33	-53	-71	-82	-85
3.208700	-4 -15	-28	-42	-57	-70	-75	3.190575	-4 -15	-28	-40	-48	-51	-51
3.162532	-4 -15	-27	-36	-40	-40	-39	3.121299	-4 -15	-26	-31	-27	-18	-13
3.094554	-4 -15	-26	-28	-16	1	9	3.078606	-4 -14	-23	-29	-34	-39	-41
3.019761	-4 -14	-22	-23	-17	-9	-5	3.005016	-4 -14	-22	-22	-12	2	9
2.965436	-4 -14	-21	-18	-4	13	21	2.947741	-4 -13	-18	-18	-20	-27	-31
2.871392	-4 -13	-17	-12	-3	3	5	2.741830	-4 -12	-13	-5	3	4	3
3.292706	-3 -15	-37	-67	-98	-122	-131	3.211585	-3 -14	-33	-58	-83	-102	-109
3.185311	-3 -14	-32	-54	-74	-88	-93	3.122359	-3 -13	-29	-49	-68	-82	-87
3.092349	-3 -13	-28	-45	-59	-68	-71	3.053090	-3 -13	-27	-40	-47	-49	-49
3.018784	-3 -13	-26	-36	-38	-35	-33	2.987356	-3 -12	-24	-36	-44	-48	-49
2.952374	-3 -12	-23	-32	-36	-37	-37	2.940363	-3 -12	-23	-31	-32	-29	-27
2.898607	-3 -12	-22	-27	-23	-15	-11	2.888189	-3 -12	-22	-26	-20	-10	-5

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α	Coefficients of $x^{2d-2},...,x^d$							α	Coefficients of $x^{2d-2},...,x^d$						
2.937507	-3	-11	-21	-34	-48	-58	-61	2.900501	-3	-11	-20	-30	-40	-47	-49
2.842944	-3	-11	-19	-25	-27	-25	-23	2.821760	-3	-11	-19	-23	-21	-17	-15
2.773951	-3	-11	-18	-19	-13	-6	-3	2.751584	-3	-11	-18	-18	-8	5	11
2.694436	-3	-11	-17	-14	0	16	23	2.779814	-3	-10	-16	-22	-29	-35	-37
2.730456	-3	-10	-15	-18	-21	-24	-25	2.711708	-3	-10	-15	-17	-17	-16	-15
2.686817	-3	-10	-15	-16	-12	-5	-1	2.651169	-3	-10	-15	-14	-5	6	11
2.623951	-3	-10	-14	-12	-4	6	11	2.627776	-3	-9	-12	-14	-17	-19	-19
2.541199	-3	-9	-11	-9	-6	-4	-3	2.528988	-3	-9	-10	-7	-7	-12	-15
2.498290	-3	-9	-10	-6	-3	-4	-5	2.454453	-3	-9	-10	-5	2	7	9
2.908222	-2	-10	-24	-42	-60	-74	-79	2.879399	-2	-10	-23	-39	-54	-65	-69
2.839128	-2	-10	-22	-35	-45	-51	-53	2.799183	-2	-9	-20	-34	-48	-59	-63
2.795842	-2	-9	-20	-34	-47	-56	-59	2.768586	-2	-9	-19	-31	-43	-53	-57
2.753535	-2	-9	-19	-30	-39	-45	-47	2.741741	-2	-9	-19	-29	-36	-40	-41
2.703105	-2	-9	-18	-26	-30	-31	-31	2.689801	-2	-9	-18	-25	-27	-26	-25
2.662098	-2	-9	-17	-23	-24	-23	-22	2.631076	-2	-9	-17	-21	-18	-12	-9
2.672678	-2	-8	-16	-26	-36	-44	-47	2.654515	-2	-8	-16	-25	-32	-36	-37
2.620305	-2	-8	-15	-22	-28	-33	-35	2.604894	-2	-8	-15	-21	-25	-28	-29
2.599582	-2	-8	-15	-21	-24	-25	-25	2.578986	-2	-8	-15	-20	-20	-18	-17
2.536989	-2	-8	-14	-17	-16	-14	-13	2.530514	-2	-8	-14	-17	-15	-11	-9
2.510791	-2	-8	-14	-16	-12	-6	-3	2.455248	-2	-8	-13	-13	-7	0	3
2.421517	-2	-8	-13	-12	-3	8	13	2.533912	-2	-7	-12	-19	-27	-32	-33
2.500245	-2	-7	-12	-17	-21	-24	-25	2.455630	-2	-7	-11	-14	-17	-21	-23
2.425157	-2	-7	-11	-13	-13	-13	-13	2.381810	-2	-7	-11	-12	-8	-2	1
2.356068	-2	-7	-11	-11	-5	2	5	2.371789	-2	-7	-10	-10	-9	-10	-11
2.332256	-2	-7	-10	-9	-5	-2	-1	2.286228	-2	-7	-10	-8	-1	6	9
2.272337	-2	-7	-10	-8	0	9	13	2.309519	-2	-6	-8	-10	-12	-12	-11
2.243980	-2	-6	-7	-7	-8	-9	-9	2.188897	-2	-6	-7	-6	-4	-1	1
2.170001	-2	-6	-7	-5	-2	-1	-1	2.093324	-2	-6	-6	-3	0	2	3
1.975801	-2	-5	-4	-2	-1	0	1	2.541782	-1	-6	-15	-27	-39	-48	-51
2.500730	-1	-6	-14	-24	-34	-42	-45	2.430264	-1	-6	-13	-20	-25	-28	-29
2.292778	-1	-6	-11	-14	-12	-9	-7	2.378980	-1	-5	-11	-19	-27	-33	-35
2.357020	-1	-5	-11	-18	-24	-28	-29	2.323968	-1	-5	-10	-16	-22	-27	-29
2.298668	-1	-5	-10	-15	-19	-22	-23	2.288889	-1	-5	-10	-15	-18	-19	-19
2.271298	-1	-5	-10	-14	-16	-17	-17	2.231631	-1	-5	-9	-12	-14	-16	-17
2.184960	-1	-5	-9	-11	-10	-8	-7	2.145682	-1	-5	-9	-10	-7	-3	-1
2.075318	-1	-5	-8	-8	-4	0	2	2.010330	-1	-5	-8	-7	-1	6	9
2.233354	-1	-4	-8	-14	-20	-24	-25	2.160459	-1	-4	-7	-11	-15	-18	-19
2.137844	-1	-4	-7	-10	-13	-16	-17	2.127122	-1	-4	-7	-10	-12	-14	-15
2.122872	-1	-4	-7	-10	-12	-13	-13	2.084929	-1	-4	-7	-9	-9	-9	-9
2.066557	-1	-4	-7	-9	-8	-6	-5	2.043202	-1	-4	-6	-7	-8	-10	-11
2.022510	-1	-4	-6	-7	-7	-7	-7	1.961617	-1	-4	-6	-6	-4	-2	-1
1.931830	-1	-4	-6	-6	-3	1	3	1.963146	-1	-4	-5	-5	-5	-7	-8
1.915060	-1	-4	-5	-4	-3	-4	-5	1.763870	-1	-4	-5	-3	1	4	5
1.934484	-1	-3	-4	-6	-8	-9	-9	1.871443	-1	-3	-4	-5	-5	-5	-5
1.812844	-1	-3	-3	-3	-4	-6	-7	1.691801	-1	-3	-3	-2	-1	-1	-1
1.600679	-1	-3	-3	-2	0	2	3	2.073847	0	-3	-7	-12	-17	-21	-23
2.005180	0	-2	-6	-11	-16	-20	-21	1.936997	0	-2	-5	-9	-13	-16	-17
1.881530	0	-2	-5	-8	-10	-11	-11	1.816923	0	-2	-4	-6	-8	-10	-11
1.732376	0	-2	-4	-5	-5	-5	-5	1.675661	0	-2	-3	-4	-4	-5	-5
1.647558	0	-1	-2	-4	-6	-7	-7	1.577562	0	-1	-2	-3	-4	-5	-5
1.527072	0	-1	-2	-3	-3	-3	-3	1.308409	0	-1	-1	-1	-1	-1	-1

Minimal polynomials of all Salem numbers of degree $2d = 18$ and trace -1.

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α	Coefficients of $x^{2d-2},\ldots,x^d$						α	Coefficients of $x^{2d-2},\ldots,x^d$					
4.324744	-8 -36 -83	-134	-165	-165	-147	-137	4.174754	-7 -32 -76	-130	-176	-202	-210	-211
4.106176	-7 -31 -70	-112	-139	-143	-133	-127	4.062627	-7 -30 -66	-104	-129	-135	-129	-125
4.029390	-7 -30 -64	-93	-98	-76	-45	-31	3.961838	-6 -27 -64	-113	-163	-203	-227	-235
3.958899	-6 -27 -64	-112	-159	-195	-216	-223	3.946382	-6 -27 -63	-108	-150	-180	-196	-201
3.907495	-6 -26 -59	-102	-150	-197	-233	-247	3.878380	-6 -26 -58	-94	-122	-136	-139	-139
3.874202	-6 -26 -58	-93	-117	-124	-120	-117	3.837824	-6 -26 -56	-83	-90	-73	-47	-35
3.819184	-6 -25 -53	-83	-109	-130	-145	-151	3.804061	-6 -25 -53	-80	-93	-88	-74	-67
3.789766	-6 -25 -52	-76	-85	-77	-62	-55	3.759968	-6 -25 -51	-69	-62	-29	10	27
3.746449	-6 -24 -48	-70	-86	-99	-110	-115	3.745181	-6 -24 -48	-70	-85	-94	-99	-101
3.720413	-6 -24 -47	-64	-68	-63	-57	-55	3.680246	-6 -24 -46	-56	-39	2	45	63
3.692487	-6 -23 -44	-63	-79	-94	-106	-111	3.650478	-6 -23 -42	-53	-55	-57	-63	-67
3.712464	-5 -22 -51	-91	-137	-181	-213	-225	3.683190	-5 -22 -50	-84	-115	-137	-149	-153
3.682089	-5 -22 -50	-84	-114	-133	-141	-143	3.661245	-5 -22 -49	-79	-101	-110	-110	-109
3.631310	-5 -21 -46	-77	-109	-137	-156	-163	3.615240	-5 -21 -45	-73	-101	-127	-147	-155
3.609150	-5 -21 -45	-72	-96	-114	-125	-129	3.596986	-5 -21 -45	-70	-86	-89	-84	-81
3.578345	-5 -21 -44	-66	-77	-74	-64	-59	3.573415	-5 -21 -44	-65	-73	-66	-53	-47
3.554017	-5 -21 -43	-61	-64	-51	-33	-25	3.553523	-5 -20 -41	-65	-90	-114	-132	-139
3.548494	-5 -20 -41	-64	-86	-106	-121	-127	3.541997	-5 -20 -41	-63	-81	-94	-102	-105
3.528820	-5 -20 -40	-60	-77	-91	-101	-105	3.523546	-5 -20 -40	-59	-73	-83	-90	-93
3.489931	-5 -20 -39	-53	-55	-47	-37	-33	3.486399	-5 -20 -39	-53	-53	-38	-18	-9
3.480637	-5 -20 -39	-52	-49	-30	-7	3	3.473144	-5 -20 -39	-51	-44	-18	12	25
3.467207	-5 -20 -39	-50	-40	-10	23	37	3.452263	-5 -20 -38	-47	-36	-7	24	37
3.422299	-5 -20 -37	-42	-23	16	55	71	3.480741	-5 -19 -37	-56	-74	-89	-98	-101
3.462807	-5 -19 -36	-52	-67	-83	-97	-103	3.460815	-5 -19 -36	-52	-66	-78	-86	-89
3.447282	-5 -19 -36	-50	-57	-58	-56	-55	3.436233	-5 -19 -35	-47	-55	-64	-74	-79
3.432360	-5 -19 -35	-47	-53	-55	-55	-55	3.428295	-5 -19 -35	-46	-50	-52	-55	-57
3.426134	-5 -19 -35	-46	-49	-47	-44	-43	3.400833	-5 -19 -35	-43	-34	-10	16	27
3.415166	-5 -19 -34	-43	-47	-54	-65	-71	3.395708	-5 -19 -34	-41	-36	-24	-13	-9
3.388979	-5 -19 -34	-40	-32	-16	-2	3	3.386599	-5 -19 -34	-40	-31	-11	9	17
3.363407	-5 -19 -33	-36	-23	-1	18	25	3.358356	-5 -18 -31	-39	-43	-47	-51	-53
3.336947	-5 -18 -30	-35	-36	-41	-50	-55	3.317177	-5 -18 -30	-33	-26	-16	-9	-7
3.301995	-5 -18 -29	-30	-23	-18	-19	-21	3.294060	-5 -18 -29	-29	-19	-10	-8	-9
3.280551	-5 -18 -29	-28	-13	7	22	27	3.277537	-5 -18 -29	-28	-12	12	33	41
3.255757	-5 -18 -28	-24	-6	13	23	25	3.259365	-5 -17 -26	-28	-29	-36	-46	-51
3.210585	-5 -17 -25	-22	-12	-5	-4	-5	3.500280	-4 -18 -43	-79	-121	-161	-190	-201
3.474295	-4 -18 -42	-74	-107	-134	-151	-157	3.438576	-4 -18 -41	-67	-87	-98	-101	-101
3.388364	-4 -17 -37	-62	-88	-111	-127	-133	3.382278	-4 -17 -37	-61	-84	-103	-116	-121
3.373923	-4 -17 -37	-60	-79	-90	-94	-95	3.359305	-4 -17 -36	-57	-75	-88	-96	-99
3.350549	-4 -17 -36	-56	-70	-75	-74	-73	3.317227	-4 -17 -35	-51	-56	-48	-35	-29
3.289301	-4 -17 -35	-47	-41	-18	9	21	3.283829	-4 -17 -34	-46	-43	-25	-4	5
3.331949	-4 -16 -34	-57	-81	-102	-117	-123	3.315830	-4 -16 -33	-54	-77	-98	-112	-117
3.308987	-4 -16 -33	-53	-73	-90	-101	-105	3.302486	-4 -16 -33	-52	-69	-83	-93	-97
3.283489	-4 -16 -33	-50	-59	-58	-52	-49	3.293290	-4 -16 -32	-50	-69	-88	-103	-109
3.286141	-4 -16 -32	-49	-65	-80	-92	-97	3.283528	-4 -16 -32	-49	-64	-75	-81	-83
3.278398	-4 -16 -32	-48	-61	-71	-78	-81	3.276211	-4 -16 -32	-48	-60	-67	-70	-71
3.268768	-4 -16 -32	-47	-56	-59	-59	-59	3.266032	-4 -16 -32	-47	-55	-54	-48	-45
3.259648	-4 -16 -31	-45	-56	-65	-72	-75	3.251973	-4 -16 -31	-44	-52	-57	-61	-63
3.241259	-4 -16 -31	-43	-47	-44	-39	-37	3.230246	-4 -16 -31	-42	-42	-31	-17	-11
3.221999	-4 -16 -31	-41	-38	-23	-6	1	3.219474	-4 -16 -31	-41	-37	-19	2	11
3.230367	-4 -16 -30	-41	-46	-48	-49	-50	3.191713	-4 -16 -30	-37	-29	-8	14	23
3.188991	-4 -16 -30	-37	-28	-4	22	33	3.192222	-4 -16 -29	-36	-33	-25	-18	-16
3.237398	-4 -15 -29	-46	-66	-85	-97	-101	3.230115	-4 -15 -29	-45	-62	-78	-89	-93
3.204074	-4 -15 -28	-41	-54	-67	-77	-81	3.200986	-4 -15 -28	-41	-53	-62	-66	-67

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α	Coefficients of $x^{2d-2},\ldots,x^d$						α	Coefficients of $x^{2d-2},\ldots,x^d$					
3.195701	-4 -15 -28	-40	-50	-59	-66	-69	3.193820	-4 -15 -28	-40	-49	-56	-62	-65
3.192542	-4 -15 -28	-40	-49	-54	-55	-55	3.184515	-4 -15 -28	-39	-45	-47	-47	-47
3.174102	-4 -15 -27	-37	-45	-52	-57	-59	3.167930	-4 -15 -27	-36	-42	-48	-54	-57
3.165170	-4 -15 -27	-36	-41	-44	-46	-47	3.156040	-4 -15 -27	-35	-37	-36	-35	-35
3.152544	-4 -15 -27	-35	-36	-31	-24	-21	3.146700	-4 -15 -27	-34	-33	-28	-24	-23
3.143112	-4 -15 -27	-34	-32	-23	-13	-9	3.139359	-4 -15 -26	-32	-34	-38	-45	-49
3.122924	-4 -15 -26	-31	-28	-21	-15	-13	3.112856	-4 -15 -26	-30	-24	-13	-4	-1
3.108936	-4 -15 -26	-30	-23	-8	7	13	3.098482	-4 -15 -26	-29	-19	0	18	25
3.076510	-4 -15 -25	-26	-15	2	16	21	3.060771	-4 -15 -25	-25	-10	15	38	47
3.101460	-4 -14 -24	-32	-38	-43	-47	-49	3.100588	-4 -14 -24	-32	-38	-42	-43	-43
3.078966	-4 -14 -23	-29	-34	-39	-42	-43	3.072459	-4 -14 -23	-28	-31	-36	-42	-45
3.068293	-4 -14 -23	-28	-30	-31	-31	-31	3.061620	-4 -14 -23	-27	-27	-28	-31	-33
3.054772	-4 -14 -23	-27	-25	-20	-16	-15	3.042361	-4 -14 -23	-26	-21	-11	-1	3
3.027839	-4 -14 -23	-25	-16	0	14	19	3.042902	-4 -14 -22	-24	-24	-30	-41	-47
3.038415	-4 -14 -22	-24	-23	-25	-30	-33	3.026693	-4 -14 -22	-23	-19	-17	-19	-21
3.018085	-4 -14 -22	-23	-17	-8	0	3	3.010623	-4 -14 -22	-22	-14	-5	0	1
3.005663	-4 -14 -22	-22	-13	0	11	15	3.002857	-4 -14 -21	-20	-15	-15	-21	-25
2.990109	-4 -14 -21	-19	-11	-7	-10	-13	2.971472	-4 -14 -21	-18	-6	6	12	13
2.975960	-4 -13 -19	-21	-24	-31	-38	-41	2.970788	-4 -13 -19	-21	-23	-26	-27	-27
2.927630	-4 -13 -18	-16	-13	-17	-26	-31	2.910683	-4 -13 -18	-16	-10	-3	4	7
2.901608	-4 -13 -18	-15	-7	0	4	5	2.881810	-4 -13 -18	-14	-2	11	19	21
2.889817	-4 -13 -17	-12	-6	-11	-25	-33	2.877823	-4 -13 -17	-12	-4	-2	-6	-9
2.861041	-4 -13 -17	-11	0	6	5	3	2.843373	-4 -13 -17	-10	4	14	16	15
2.816751	-4 -13 -16	-7	7	12	6	1	2.721574	-4 -12 -13	-4	7	11	9	7
3.206992	-3 -14 -33	-57	-80	-98	-108	-111	3.099332	-3 -14 -30	-43	-44	-33	-18	-11
3.132411	-3 -13 -29	-50	-72	-91	-104	-109	3.111756	-3 -13 -28	-47	-67	-85	-98	-103
3.102137	-3 -13 -28	-46	-63	-76	-84	-87	3.090021	-3 -13 -28	-45	-58	-64	-65	-65
3.088504	-3 -13 -28	-44	-57	-67	-73	-75	3.072917	-3 -13 -28	-43	-51	-51	-46	-43
3.057039	-3 -13 -27	-41	-49	-49	-45	-43	3.030620	-3 -13 -27	-38	-39	-31	-20	-15
3.043647	-3 -12 -25	-42	-61	-78	-89	-93	3.042091	-3 -12 -25	-42	-60	-76	-88	-93
3.034316	-3 -12 -25	-41	-57	-71	-81	-85	3.020326	-3 -12 -24	-39	-56	-72	-83	-87
3.009755	-3 -12 -24	-38	-52	-64	-72	-75	3.002429	-3 -12 -24	-37	-49	-60	-69	-73
2.998900	-3 -12 -24	-37	-48	-56	-61	-63	2.998080	-3 -12 -24	-37	-48	-55	-58	-59
2.986894	-3 -12 -24	-36	-44	-47	-47	-47	2.978750	-3 -12 -24	-35	-41	-43	-42	-41
2.971565	-3 -12 -24	-35	-39	-35	-28	-25	2.967001	-3 -12 -24	-34	-37	-35	-31	-29
2.977268	-3 -12 -23	-34	-44	-54	-63	-67	2.972642	-3 -12 -23	-34	-43	-49	-52	-53
2.964715	-3 -12 -23	-33	-40	-45	-49	-51	2.960826	-3 -12 -23	-33	-39	-41	-41	-41
2.947690	-3 -12 -23	-32	-35	-32	-27	-25	2.930810	-3 -12 -23	-31	-30	-20	-8	-3
2.906862	-3 -12 -23	-29	-23	-7	11	19	2.937672	-3 -12 -22	-30	-35	-39	-43	-45
2.935279	-3 -12 -22	-30	-34	-37	-40	-42	2.930097	-3 -12 -22	-30	-33	-32	-29	-28
2.919531	-3 -12 -22	-29	-30	-26	-21	-19	2.906065	-3 -12 -22	-28	-26	-18	-10	-7
2.886160	-3 -12 -22	-27	-21	-5	12	19	2.891848	-3 -12 -21	-26	-25	-22	-20	-20
2.930935	-3 -11 -21	-33	-45	-55	-62	-65	2.905185	-3 -11 -21	-31	-37	-39	-39	-39
2.905242	-3 -11 -20	-30	-41	-51	-57	-59	2.891980	-3 -11 -20	-29	-37	-43	-46	-47
2.888779	-3 -11 -20	-29	-36	-40	-42	-43	2.883972	-3 -11 -20	-28	-34	-40	-46	-49
2.879342	-3 -11 -20	-28	-33	-36	-38	-39	2.878236	-3 -11 -20	-28	-33	-35	-35	-35
2.866860	-3 -11 -19	-26	-33	-41	-48	-51	2.860891	-3 -11 -19	-26	-32	-36	-37	-37
2.852354	-3 -11 -19	-25	-29	-33	-37	-39	2.851164	-3 -11 -19	-25	-29	-32	-34	-35
2.848814	-3 -11 -19	-25	-28	-30	-33	-35	2.846111	-3 -11 -19	-25	-28	-28	-26	-25
2.837251	-3 -11 -19	-24	-25	-25	-26	-27	2.836001	-3 -11 -19	-24	-25	-24	-23	-23
2.830704	-3 -11 -19	-24	-24	-20	-15	-13	2.829423	-3 -11 -19	-24	-24	-19	-12	-9
2.814601	-3 -11 -19	-23	-20	-12	-4	-1	2.831383	-3 -11 -18	-23	-27	-32	-36	-38
2.818681	-3 -11 -18	-22	-24	-26	-28	-29	2.807981	-3 -11 -18	-21	-21	-22	-25	-27

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α	Coefficients of $x^{2d-2},\dots,x^d$								α	Coefficients of $x^{2d-2},\dots,x^d$							
2.802289	-3	-11	-18	-21	-20	-18	-17	-17	2.785092	-3	-11	-18	-20	-16	-10	-6	-5
2.783622	-3	-11	-18	-20	-16	-9	-3	-1	2.777447	-3	-11	-18	-20	-15	-5	5	9
2.765421	-3	-11	-18	-19	-12	-1	8	11	2.758860	-3	-11	-18	-19	-11	3	16	21
2.739162	-3	-11	-18	-18	-7	11	27	33	2.761012	-3	-11	-17	-18	-14	-9	-5	-4
2.732749	-3	-11	-17	-16	-8	1	6	7	2.725562	-3	-11	-17	-16	-7	5	14	17
2.797013	-3	-10	-17	-24	-29	-32	-35	-37	2.784678	-3	-10	-16	-22	-30	-38	-42	-43
2.769133	-3	-10	-16	-21	-26	-31	-34	-35	2.763429	-3	-10	-16	-21	-25	-27	-27	-27
2.751361	-3	-10	-16	-20	-22	-23	-23	-23	2.727889	-3	-10	-16	-19	-17	-12	-8	-7
2.737352	-3	-10	-15	-18	-22	-28	-33	-35	2.729119	-3	-10	-15	-18	-21	-23	-22	-21
2.725037	-3	-10	-15	-17	-19	-24	-30	-33	2.718181	-3	-10	-15	-17	-18	-20	-22	-23
2.716477	-3	-10	-15	-17	-18	-19	-19	-19	2.709401	-3	-10	-15	-17	-17	-15	-11	-9
2.700121	-3	-10	-15	-16	-14	-13	-15	-17	2.697858	-3	-10	-15	-16	-14	-12	-11	-11
2.692660	-3	-10	-15	-16	-13	-9	-7	-7	2.688443	-3	-10	-15	-16	-13	-7	0	3
2.670610	-3	-10	-15	-15	-9	-1	4	5	2.668081	-3	-10	-15	-15	-9	0	8	11
2.644142	-3	-10	-15	-14	-5	8	19	23	2.670729	-3	-10	-14	-13	-11	-14	-21	-25
2.660635	-3	-10	-14	-13	-10	-9	-10	-11	2.652255	-3	-10	-14	-13	-9	-5	-2	-1
2.638976	-3	-10	-14	-12	-6	-2	-2	-3	2.636754	-3	-10	-14	-12	-6	-1	1	1
2.627625	-3	-10	-14	-12	-5	3	9	11	2.621150	-3	-10	-14	-12	-4	6	13	15
2.615729	-3	-10	-14	-12	-4	8	20	25	2.613326	-3	-10	-14	-11	-2	6	9	9
2.610874	-3	-10	-14	-11	-2	7	12	13	2.608710	-3	-10	-13	-9	-3	-4	-12	-17
2.596384	-3	-10	-13	-9	-2	1	-1	-3	2.567429	-3	-10	-13	-8	2	9	10	9
2.546118	-3	-10	-12	-6	3	5	0	-4	2.519573	-3	-10	-12	-5	6	11	8	5
2.576695	-3	-9	-11	-10	-11	-15	-18	-19	2.563883	-3	-9	-11	-10	-10	-10	-7	-5
2.561848	-3	-9	-11	-9	-8	-12	-18	-21	2.548374	-3	-9	-11	-9	-7	-7	-7	-7
2.540731	-3	-9	-11	-9	-6	-4	-3	-3	2.504324	-3	-9	-11	-8	-2	5	12	15
2.486089	-3	-9	-10	-6	-2	0	2	3	2.481009	-3	-9	-10	-5	0	-1	-6	-9
2.466859	-3	-9	-10	-5	1	3	2	1	2.461682	-3	-9	-10	-4	3	2	-6	-11
2.427357	-3	-9	-9	-2	4	1	-8	-13	2.400378	-3	-9	-9	-1	7	5	-5	-11
2.381402	-3	-9	-9	-1	8	9	3	-1	3.057163	-2	-11	-29	-56	-89	-122	-146	-155
2.910267	-2	-10	-24	-42	-60	-75	-84	-87	2.870906	-2	-10	-23	-38	-51	-60	-64	-65
2.844132	-2	-10	-22	-35	-46	-55	-61	-63	2.829303	-2	-10	-22	-34	-42	-46	-47	-47
2.812418	-2	-9	-20	-35	-51	-65	-75	-79	2.790739	-2	-9	-20	-33	-45	-55	-61	-63
2.782706	-2	-9	-19	-32	-46	-59	-69	-73	2.766304	-2	-9	-19	-31	-42	-50	-55	-57
2.759581	-2	-9	-19	-30	-40	-49	-55	-57	2.747822	-2	-9	-19	-29	-37	-44	-49	-51
2.742116	-2	-9	-19	-29	-36	-40	-41	-41	2.723742	-2	-9	-19	-28	-32	-31	-27	-25
2.689340	-2	-9	-18	-25	-27	-26	-24	-23	2.673557	-2	-9	-18	-24	-24	-20	-15	-13
2.660946	-2	-9	-18	-24	-22	-13	-2	3	2.690291	-2	-9	-17	-25	-30	-33	-34	-35
2.574623	-2	-9	-17	-19	-10	7	24	31	2.723427	-2	-8	-17	-30	-44	-56	-65	-69
2.698989	-2	-8	-17	-28	-38	-47	-53	-55	2.675998	-2	-8	-16	-26	-36	-45	-52	-55
2.674377	-2	-8	-16	-26	-36	-44	-49	-51	2.666610	-2	-8	-16	-25	-34	-43	-49	-51
2.655440	-2	-8	-16	-25	-32	-36	-38	-39	2.641878	-2	-8	-16	-24	-29	-32	-33	-33
2.596246	-2	-8	-16	-22	-21	-15	-7	-3	2.636942	-2	-8	-15	-23	-31	-38	-43	-45
2.635123	-2	-8	-15	-23	-31	-37	-40	-41	2.621970	-2	-8	-15	-22	-28	-33	-37	-39
2.616192	-2	-8	-15	-22	-27	-30	-32	-33	2.614238	-2	-8	-15	-22	-27	-29	-29	-29
2.607364	-2	-8	-15	-21	-25	-29	-32	-33	2.591985	-2	-8	-15	-21	-23	-21	-18	-17
2.591110	-2	-8	-15	-20	-22	-24	-26	-27	2.582567	-2	-8	-15	-20	-21	-20	-18	-17
2.565801	-2	-8	-15	-20	-19	-12	-4	-1	2.590029	-2	-8	-14	-20	-25	-29	-31	-32
2.580115	-2	-8	-14	-19	-23	-27	-31	-33	2.571401	-2	-8	-14	-19	-22	-23	-23	-23
2.566527	-2	-8	-14	-19	-21	-21	-20	-20	2.546491	-2	-8	-14	-18	-18	-15	-12	-11
2.516737	-2	-8	-14	-17	-14	-6	2	5	2.516262	-2	-8	-14	-16	-13	-9	-6	-5
2.508201	-2	-8	-14	-16	-12	-6	-1	1	2.474312	-2	-8	-14	-15	-8	3	13	17
2.471064	-2	-8	-14	-15	-8	4	16	21	2.507015	-2	-8	-13	-15	-14	-13	-14	-15
2.486885	-2	-8	-13	-15	-12	-6	0	2	2.461417	-2	-8	-13	-14	-9	0	8	11

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α	Coefficients of $x^{2d-2}, \ldots, x^d$								α	Coefficients of $x^{2d-2}, \ldots, x^d$							
2.467953	-2	-8	-12	-13	-11	-10	-10	-11	2.568329	-2	-7	-13	-21	-29	-36	-42	-45
2.544063	-2	-7	-13	-20	-25	-28	-30	-31	2.525410	-2	-7	-12	-18	-25	-31	-34	-35
2.521328	-2	-7	-12	-18	-24	-29	-33	-35	2.504213	-2	-7	-12	-17	-21	-25	-29	-31
2.501579	-2	-7	-12	-17	-21	-24	-26	-27	2.494482	-2	-7	-12	-17	-20	-21	-22	-23
2.479865	-2	-7	-12	-16	-18	-19	-18	-17	2.472989	-2	-7	-12	-16	-17	-16	-15	-15
2.476550	-2	-7	-11	-15	-20	-25	-28	-29	2.455187	-2	-7	-11	-14	-17	-20	-22	-23
2.446262	-2	-7	-11	-14	-16	-17	-17	-17	2.443060	-2	-7	-11	-14	-16	-16	-14	-13
2.426136	-2	-7	-11	-13	-13	-13	-14	-15	2.422689	-2	-7	-11	-13	-13	-12	-11	-11
2.412641	-2	-7	-11	-13	-12	-9	-6	-5	2.408985	-2	-7	-11	-13	-12	-8	-3	-1
2.403167	-2	-7	-11	-13	-11	-6	-2	-1	2.374627	-2	-7	-11	-12	-8	-1	5	7
2.370349	-2	-7	-11	-12	-8	0	8	11	2.359290	-2	-7	-11	-11	-6	0	5	7
2.341645	-2	-7	-11	-11	-5	4	13	17	2.411451	-2	-7	-10	-12	-14	-16	-16	-16
2.400887	-2	-7	-10	-11	-12	-15	-19	-21	2.382560	-2	-7	-10	-11	-11	-10	-8	-7
2.378415	-2	-7	-10	-10	-9	-11	-16	-19	2.362783	-2	-7	-10	-10	-8	-7	-8	-9
2.358472	-2	-7	-10	-10	-8	-6	-5	-5	2.341383	-2	-7	-10	-10	-7	-2	3	5
2.332419	-2	-7	-10	-9	-5	-2	-2	-3	2.313470	-2	-7	-10	-9	-4	2	6	7
2.292723	-2	-7	-10	-9	-3	6	14	17	2.320502	-2	-7	-9	-8	-6	-6	-7	-8
2.281633	-2	-7	-9	-7	-3	0	1	1	2.270108	-2	-7	-9	-7	-2	2	4	4
2.248355	-2	-7	-9	-6	0	4	4	3	2.355594	-2	-6	-8	-11	-17	-22	-22	-21
2.280430	-2	-6	-8	-9	-9	-8	-7	-7	2.266913	-2	-6	-8	-9	-8	-5	-3	-3
2.248400	-2	-6	-8	-8	-6	-4	-3	-3	2.233065	-2	-6	-8	-8	-5	-1	1	1
2.258484	-2	-6	-7	-7	-9	-12	-13	-13	2.233388	-2	-6	-7	-6	-6	-9	-13	-15
2.227083	-2	-6	-7	-6	-6	-8	-10	-11	2.202172	-2	-6	-7	-6	-5	-4	-2	-1
2.194810	-2	-6	-7	-6	-5	-3	1	3	2.184261	-2	-6	-7	-6	-4	-1	2	3
2.199336	-2	-6	-7	-5	-3	-5	-10	-13	2.151176	-2	-6	-7	-5	-1	2	2	1
2.129161	-2	-6	-7	-5	-1	4	9	11	2.125309	-2	-6	-6	-3	-1	-2	-4	-5
2.074749	-2	-6	-6	-3	0	3	7	9	2.074992	-2	-6	-6	-2	2	2	-1	-3
2.011508	-2	-6	-6	-1	5	6	2	-1	1.959801	-2	-6	-5	1	6	4	-3	-7
2.119581	-2	-5	-4	-4	-9	-13	-10	-7	2.007276	-2	-5	-4	-2	-2	-3	-3	-3
1.819823	-2	-5	-3	1	2	1	2	3	2.603482	-1	-7	-17	-29	-40	-48	-52	-53
2.551527	-1	-6	-15	-27	-40	-52	-60	-63	2.505045	-1	-6	-14	-24	-34	-43	-49	-51
2.492348	-1	-6	-14	-23	-32	-40	-45	-47	2.485451	-1	-6	-14	-23	-31	-37	-40	-41
2.461018	-1	-6	-13	-21	-29	-37	-43	-45	2.431102	-1	-6	-13	-20	-25	-28	-29	-29
2.398027	-1	-6	-13	-19	-21	-19	-15	-13	2.406160	-1	-5	-11	-20	-30	-39	-46	-49
2.370933	-1	-5	-11	-18	-25	-32	-37	-39	2.361668	-1	-5	-11	-18	-24	-29	-32	-33
2.337022	-1	-5	-11	-17	-21	-24	-25	-25	2.317270	-1	-5	-10	-16	-21	-24	-26	-27
2.304042	-1	-5	-10	-15	-19	-23	-26	-27	2.299928	-1	-5	-10	-15	-19	-22	-23	-23
2.284074	-1	-5	-10	-14	-17	-20	-22	-23	2.271845	-1	-5	-10	-14	-16	-17	-17	-17
2.258993	-1	-5	-10	-14	-15	-14	-12	-11	2.220934	-1	-5	-10	-13	-12	-8	-3	-1
2.205151	-1	-5	-10	-13	-11	-5	2	5	2.277349	-1	-5	-9	-14	-18	-22	-25	-27
2.253374	-1	-5	-9	-13	-16	-18	-20	-21	2.222790	-1	-5	-9	-12	-13	-14	-14	-14
2.182283	-1	-5	-9	-11	-10	-8	-6	-5	2.160083	-1	-5	-9	-10	-8	-6	-5	-5
2.132925	-1	-5	-9	-10	-7	-2	3	5	2.110889	-1	-5	-9	-10	-6	1	8	11
2.146625	-1	-5	-8	-10	-9	-7	-5	-5	2.026292	-1	-5	-8	-7	-2	3	6	7
2.241321	-1	-4	-8	-14	-20	-25	-29	-31	2.213171	-1	-4	-8	-13	-17	-21	-24	-25
2.178070	-1	-4	-8	-12	-14	-16	-17	-17	2.164245	-1	-4	-7	-11	-15	-18	-20	-21
2.139991	-1	-4	-7	-10	-13	-16	-17	-17	2.125814	-1	-4	-7	-10	-12	-13	-14	-15
2.106487	-1	-4	-7	-10	-11	-10	-9	-9	2.085297	-1	-4	-7	-9	-9	-9	-9	-9
2.076646	-1	-4	-7	-9	-9	-8	-6	-5	2.030336	-1	-4	-7	-8	-6	-4	-2	-1
2.000933	-1	-4	-7	-8	-5	-1	3	5	1.991657	-1	-4	-7	-8	-5	0	5	7
2.076860	-1	-4	-6	-8	-10	-12	-14	-15	2.068278	-1	-4	-6	-8	-10	-11	-11	-11
2.061442	-1	-4	-6	-8	-9	-10	-11	-12	2.052216	-1	-4	-6	-8	-9	-9	-8	-8
2.025872	-1	-4	-6	-7	-7	-7	-8	-9	1.997667	-1	-4	-6	-7	-6	-4	-3	-3

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α	Coefficients of $x^{2d-2}, \ldots, x^d$								α	Coefficients of $x^{2d-2}, \ldots, x^d$							
1.985167	-1	-4	-6	-7	-6	-3	0	1	1.954229	-1	-4	-6	-6	-4	-2	0	1
1.861829	-1	-4	-6	-5	-1	3	6	7	1.924288	-1	-4	-5	-5	-4	-3	-2	-2
1.884497	-1	-4	-5	-4	-2	-1	-2	-3	1.800386	-1	-4	-5	-4	-1	3	6	7
1.940283	-1	-3	-4	-6	-8	-9	-10	-11	1.877269	-1	-3	-4	-5	-5	-5	-6	-7
1.813716	-1	-3	-4	-5	-4	-1	1	1	1.754623	-1	-3	-4	-4	-2	0	2	3
1.874779	-1	-3	-3	-4	-7	-9	-8	-7	1.778259	-1	-3	-3	-3	-4	-4	-2	-1
1.737326	-1	-3	-3	-2	-1	-2	-5	-7	1.702524	-1	-3	-3	-2	-1	-1	-2	-3
1.568609	-1	-3	-3	-2	0	2	3	3	1.412219	-1	-3	-2	0	1	1	1	1
2.133431	0	-3	-8	-14	-20	-25	-28	-29	2.070699	0	-3	-7	-12	-16	-20	-22	-23
2.037013	0	-3	-7	-11	-14	-16	-17	-17	1.877198	0	-3	-6	-8	-7	-5	-2	-1
2.017289	0	-2	-6	-11	-16	-21	-24	-25	1.951716	0	-2	-5	-9	-13	-17	-20	-21
1.913927	0	-2	-5	-8	-11	-14	-16	-17	1.893218	0	-2	-5	-8	-10	-12	-13	-13
1.884082	0	-2	-5	-8	-10	-11	-11	-11	1.805477	0	-2	-5	-7	-7	-6	-4	-3
1.822736	0	-2	-4	-6	-8	-10	-11	-11	1.795134	0	-2	-4	-6	-7	-8	-8	-8
1.733929	0	-2	-4	-5	-5	-5	-5	-5	1.666159	0	-2	-4	-5	-4	-2	0	1
1.663018	0	-1	-2	-4	-6	-7	-8	-9	1.585260	0	-1	-2	-3	-4	-5	-5	-5
1.531221	0	-1	-2	-3	-3	-3	-3	-3	1.411261	1	0	-1	-2	-3	-4	-5	-5

Table 3.3: The Q_j $(1 \leq j \leq 34)$ used in the auxiliary function of s_2 for $d = 9$.

Q_j	d	coefficients of Q_j (from a_0 to a_d)														
Q_1	1	0	1													
Q_2	1	-1	1													
Q_3	1	-3	1													
Q_4	1	-2	1													
Q_5	2	1	-3	1												
Q_6	2	2	-4	1												
Q_7	2	5	-5	1												
Q_8	3	-1	6	-5	1											
Q_9	3	-1	9	-6	1											
Q_{10}	3	-7	14	-7	1											
Q_{11}	4	2	-16	20	-8	1										
Q_{12}	4	1	-16	20	-8	1										
Q_{13}	5	-1	12	-31	27	-9	1									
Q_{14}	5	-1	15	-35	28	-9	1									
Q_{15}	6	1	-33	104	-112	54	-12	1								
Q_{16}	6	1	-21	70	-84	45	-11	1								
Q_{17}	6	1	-27	81	-90	46	-11	1								
Q_{18}	6	-1	-18	69	-84	45	-11	1								
Q_{19}	8	1	-30	175	-406	459	-276	90	-15	1						
Q_{20}	8	1	-36	210	-462	495	-286	91	-15	1						
Q_{21}	9	-5	77	-409	1010	-1332	1012	-456	120	-17	1					
Q_{22}	9	-1	30	-247	724	-1034	825	-388	107	-16	1					
Q_{23}	9	-1	21	-184	593	-913	770	-376	106	-16	1					
Q_{24}	9	-1	57	-376	986	-1324	1011	-456	120	-17	1					
Q_{25}	9	-1	45	-330	924	-1287	1001	-455	120	-17	1					
Q_{26}	10	1	-36	294	-1036	1937	-2104	1388	-562	136	-18	1				
Q_{27}	11	-1	66	-715	3003	-6435	8008	-6188	3060	-969	190	-21	1			
Q_{28}	11	-1	48	-508	2120	-4627	5960	-4812	2499	-833	172	-20	1			
Q_{29}	11	-1	96	-890	3379	-6839	8248	-6268	3074	-970	190	-21	1			
Q_{30}	11	-1	42	-479	2285	-5402	7183	-5800	2953	-953	189	-21	1			
Q_{31}	12	1	-39	491	-2686	7601	-12540	12972	-8752	3906	-1142	210	-22	-1		
Q_{32}	12	1	-52	674	-3714	10495	-17006	17043	-11040	4707	-1312	230	-23	1		
Q_{33}	14	1	-63	1053	-7493	27973	-62293	89402	-86662	58277	-27492	9066	-2046	301	-26	1
Q_{34}	14	1	-56	942	-6725	25314	-57060	83057	-81712	55752	-26657	8894	-2026	300	-26	1

Table 3.4: The e_j $(1 \le j \le 34)$ in the auxiliary function of s_2 for $d = 9$.

$e_1 = 0.43231844$	$e_2 = 1.36593017$	$e_3 = 0.12345548$	$e_4 = 1.06191258$	$e_5 = 0.59624651$	$e_6 = 0.08905366$
$e_7 = 0.04082208$	$e_8 = 0.34272417$	$e_9 = 0.07577310$	$e_{10} = 0.02213387$	$e_{11} = 0.01535757$	$e_{12} = 0.01310109$
$e_{13} = 0.04506541$	$e_{14} = 0.12653908$	$e_{15} = 0.01247560$	$e_{16} = 0.06558438$	$e_{17} = 0.02754194$	$e_{18} = 0.00908793$
$e_{19} = 0.00185753$	$e_{20} = 0.02579579$	$e_{21} = 0.00074156$	$e_{22} = 0.00766732$	$e_{23} = 0.01068804$	$e_{24} = 0.03972082$
$e_{25} = 0.01063783$	$e_{26} = 0.01187668$	$e_{27} = 0.01154893$	$e_{28} = 0.00718248$	$e_{29} = 0.00222161$	$e_{30} = 0.00235504$
$e_{31} = 0.01505873$	$e_{32} = 0.00302307$	$e_{33} = 0.00763423$	$e_{34} = 0.00030669$		

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