

Inequalities for the Casorati Curvatures of Real Hypersurfaces in Some Grassmannians

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Abstract. In this paper we obtain two types of optimal inequalities consisting of the normalized scalar curvature and the generalized normalized δ -Casorati curvatures for real hypersurfaces of complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians. We also find the conditions on which the equalities hold.

1. Introduction

As we know, S. S. Chern [11] gave an open question in 1968, which deals with the existence of minimal immersions into any Euclidean spaces. To solve such problems, B.-Y. Chen [8] introduced the notion of Chen invariants (or δ -invariants) in 1993 and he obtained some optimal inequalities consisting of intrinsic invariants and extrinsic invariants for Riemannian submanifolds. It is the starting point of the theory of Chen invariants, which are one of the most interesting topics in differential geometry (see [1, 9, 12, 18, 23]).

The Casorati curvature of a submanifold in a Riemannian manifold is the extrinsic invariant, which is the normalized square of the second fundamental form. Some optimal inequalities containing Casorati curvatures were obtained for submanifolds of real space forms, complex space forms, and quaternionic space forms (see [10, 13, 17, 21]). The notion of Casorati curvature is the extended version of the notion of the principal curvatures of a hypersurface of a Riemannian manifold. Hence, it is both important and very interesting to obtain some optimal inequalities for the Casorati curvatures of submanifolds in ambient Riemannian manifolds.

For the real hypersurfaces of both complex space forms and quaternionic space forms, we see that by using the Codazzi equation, there does not exist any real hypersurface with parallel shape operator.

The following are also well-known. A real hypersurface of a complex projective space with a parallel second fundamental form is locally congruent to a tube over some totally geodesic complex submanifold with some radius [16]. There does not exist any real Hopf hypersurface with parallel Ricci tensor of a complex projective space [15].

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A real hypersurface of a quaternionic projective space with the shape operator to be parallel with respect to some almost contact structure vector fields is locally congruent to a tube over some quaternionic projective space with some radius [19]. After these results had been introduced, many geometers studied real hypersurfaces of a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. We know that some natural two distributions of a real hypersurface of $G_2(\mathbb{C}^{m+2})$ with $m \geq 3$ are invariant under the shape operator if and only if either it is an open part of a tube around a totally geodesic submanifold $G_2(\mathbb{C}^{m+1})$ of $G_2(\mathbb{C}^{m+2})$ or it is an open part of a tube around a totally geodesic submanifold $\mathbb{H}P^n$ of $G_2(\mathbb{C}^{m+2})$ [4]. There does not exist any real hypersurface of $G_2(\mathbb{C}^{m+1})$ with parallel second fundamental form [22].

As we know, both a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ and a complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$ are examples of Hermitian symmetric spaces with rank 2. Studying a real hypersurface of Hermitian symmetric spaces with rank 2 is very important and one of the main topics in submanifold theory. Furthermore, the classification of real hypersurfaces of Hermitian symmetric spaces with rank 2 is one of the important subjects in differential geometry.

Many geometers obtained some results on $SU_{2,m}/S(U_2 \cdot U_m)$. The maximal complex subbundle and the maximal quaternionic subbundle of a real hypersurface of $SU_{2,m}/S(U_2 \cdot U_m)$ are invariant under the shape operator if and only if it is locally congruent to an open part of some particular type of hypersurfaces [5]. There does not exist any real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with commuting shape operator [20]. There does not exist any Hopf hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with commuting shape operator on the complex maximal subbundle [20].

As the author knows, there are only examples of optimal inequalities for the submanifolds of constant space forms (i.e., real space forms, complex space forms, and quaternionic space forms). Therefore, the optimal inequalities, which are given here, are both meaningful and very important.

2. Preliminaries

In this section we remind some notions, which will be used in the following sections.

Given an almost Hermitian manifold (N, g, J) , i.e., N is a C^∞ -manifold, g is a Riemannian metric on N , and J is a compatible almost complex structure on (N, g) (i.e., $J \in \text{End}(TN)$, $J^2 = -\text{id}$, $g(JX, JY) = g(X, Y)$ for any vector fields $X, Y \in \Gamma(TN)$), we call the manifold (N, g, J) *Kähler* if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g .

Let N be a $4m$ -dimensional C^∞ -manifold and let E be a rank 3 subbundle of $\text{End}(TN)$ such that for any point $p \in N$ with a neighborhood U , there exists a local basis $\{J_1, J_2, J_3\}$

of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -\text{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3. Then we call E an *almost quaternionic structure* on N and (N, E) an *almost quaternionic manifold* [2].

Moreover, let g be a Riemannian metric on N such that for any point $p \in N$ with a neighborhood U , there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$(2.1) \quad J_\alpha^2 = -\text{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

$$(2.2) \quad g(J_\alpha X, J_\alpha Y) = g(X, Y)$$

for all vector fields $X, Y \in \Gamma(TN)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3. Then we call (N, E, g) an *almost quaternionic Hermitian manifold* [14].

For convenience, the above basis $\{J_1, J_2, J_3\}$ satisfying (2.1) and (2.2) is said to be a *quaternionic Hermitian basis*.

Let (N, E, g) be an almost quaternionic Hermitian manifold. We call (N, E, g) a *quaternionic Kähler manifold* if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$$

for any vector field $X \in \Gamma(TN)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3 [14].

If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on N (i.e., $\nabla J_\alpha = 0$ for $\alpha \in \{1, 2, 3\}$, where ∇ is the Levi-Civita connection of the metric g), then (N, E, g) is said to be a *hyperkähler manifold*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on N and g a *hyperkähler metric* [6].

Let $G_2(\mathbb{C}^{m+2})$ be the set of all complex 2-dimensional linear subspaces of \mathbb{C}^{m+2} . Then we know that the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ has some Riemannian symmetric structure (see [3, 22]). Denote by g the corresponding metric. As we know, it is the unique compact irreducible Riemannian manifold such that it has both a Kähler structure J and a quaternionic Kähler structure E with $J \notin E$. And $G_2(\mathbb{C}^{m+2})$ is the unique compact irreducible Kähler quaternionic Kähler manifold such that it is not a hyperkähler manifold.

Given a local quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of E , we have

$$(2.3) \quad J_i \circ J = J \circ J_i$$

for $J_i \in \{J_1, J_2, J_3\}$ and the Riemannian curvature tensor \bar{R} of $(G_2(\mathbb{C}^{m+2}), g)$ is locally

given by

$$\begin{aligned}
\bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\
(2.4) \quad &+ \sum_{\alpha=1}^3 \{g(J_\alpha Y, Z)J_\alpha X - g(J_\alpha X, Z)J_\alpha Y - 2g(J_\alpha X, Y)J_\alpha Z\} \\
&+ \sum_{\alpha=1}^3 \{g(J_\alpha JY, Z)J_\alpha JX - g(J_\alpha JX, Z)J_\alpha JY\}
\end{aligned}$$

for any vector fields $X, Y, Z \in \Gamma(TG_2(\mathbb{C}^{m+2}))$ (see [3, 22]).

Similarly, let $SU_{2,m}/S(U_2 \cdot U_m)$ be the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space \mathbb{C}_2^{m+2} . Then the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$ becomes a connected simply connected irreducible Riemannian symmetric space with noncompact type and rank two [5]. Denote by g the corresponding metric. It is the unique noncompact irreducible manifold with negative scalar curvature such that it has a Kähler structure J and a quaternionic Kähler structure E with $J \notin E$ [5].

We also know that given a local quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of E , we have

$$(2.5) \quad J_i \circ J = J \circ J_i$$

for $J_i \in \{J_1, J_2, J_3\}$ and the Riemannian curvature tensor \bar{R} of $(SU_{2,m}/S(U_2 \cdot U_m), g)$ is locally given by

$$\begin{aligned}
&\bar{R}(X, Y)Z \\
&= -\frac{1}{2} \left[g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \right. \\
(2.6) \quad &+ \sum_{\alpha=1}^3 \{g(J_\alpha Y, Z)J_\alpha X - g(J_\alpha X, Z)J_\alpha Y - 2g(J_\alpha X, Y)J_\alpha Z\} \\
&\left. + \sum_{\alpha=1}^3 \{g(J_\alpha JY, Z)J_\alpha JX - g(J_\alpha JX, Z)J_\alpha JY\} \right]
\end{aligned}$$

for any vector fields $X, Y, Z \in \Gamma(TSU_{2,m}/S(U_2 \cdot U_m))$ [5].

Furthermore, we remind some notions, which will be used later. Let (N, g_N) be a Riemannian manifold and M a submanifold of (N, g_N) with the induced metric g_M . Then the *Gauss* and *Weingarten formula* are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

for $X, Y \in \Gamma(TM)$,

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N$$

for $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, where $\bar{\nabla}$ and ∇ are the Levi-Civita connections of the metrics g_N and g_M , respectively, h is the second fundamental form of M in N , A is the shape operator of M in N , and ∇^\perp is the normal connection of M in N .

We denote by \bar{R} and R the Riemannian curvature tensors of g_N and g_M , respectively. Then the *Gauss equation* is given by

$$(2.7) \quad R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + g_N(h(X, W), h(Y, Z)) - g_N(h(X, Z), h(Y, W))$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$, where $\bar{R}(X, Y, Z, W) := g_N(\bar{R}(X, Y)Z, W)$ and $R(X, Y, Z, W) := g_M(R(X, Y)Z, W)$.

Consider a local orthonormal tangent frame $\{e_1, \dots, e_m\}$ of the tangent bundle TM of M and a local orthonormal normal frame $\{e_{m+1}, \dots, e_n\}$ of the normal bundle TM^\perp of M in N . The *scalar curvature* τ of M is defined by

$$\tau = \sum_{1 \leq i < j \leq m} K(e_i \wedge e_j),$$

where $K(e_i \wedge e_j) := R(e_i, e_j, e_j, e_i)$ for $1 \leq i < j \leq m$. The *normalized scalar curvature* ρ of M is given by

$$\rho = \frac{2\tau}{m(m-1)}.$$

We denote by H the *mean curvature vector field* of M in N , i.e., $H = \frac{1}{m} \sum_{i=1}^m h(e_i, e_i)$. Conveniently, let $h_{ij}^\alpha := g_N(h(e_i, e_j), e_\alpha)$ for $i, j \in \{1, \dots, m\}$ and $\alpha \in \{m+1, \dots, n\}$. Then we have the *squared mean curvature* $\|H\|^2$ of M in N and the *squared norm* $\|h\|^2$ of h as follows:

$$\begin{aligned} \|H\|^2 &= \frac{1}{m^2} \sum_{\alpha=m+1}^n \left(\sum_{i=1}^m h_{ii}^\alpha \right)^2, \\ \|h\|^2 &= \sum_{\alpha=m+1}^n \sum_{i,j=1}^m (h_{ij}^\alpha)^2. \end{aligned}$$

The *Casorati curvature* C of M in N is defined by

$$C := \frac{1}{m} \|h\|^2.$$

The submanifold M is said to be *invariantly quasi-umbilical* if there exists a local orthonormal normal frame $\{e_{m+1}, \dots, e_n\}$ of M in N such that the shape operators A_{e_α} have an eigenvalue of multiplicity $m-1$ for all $\alpha \in \{m+1, \dots, n\}$ and the distinguished eigendirection of A_{e_α} is the same for each $\alpha \in \{m+1, \dots, n\}$ [7].

Let L be a k -dimensional subspace of T_pM , $k \geq 2$, for $p \in M$ such that $\{e_1, \dots, e_k\}$ is an orthonormal basis of L . Then the *scalar curvature* $\tau(L)$ of the k -plane L is given by

$$\tau(L) := \sum_{1 \leq i < j \leq k} K(e_i \wedge e_j)$$

and the *Casorati curvature* $C(L)$ of the subspace L is defined by

$$C(L) := \frac{1}{k} \sum_{\alpha=m+1}^n \sum_{i,j=1}^k (h_{ij}^\alpha)^2.$$

The *normalized δ -Casorati curvatures* $\delta_c(m-1)$ and $\widehat{\delta}_c(m-1)$ of M in N are given by

$$\begin{aligned} [\delta_c(m-1)](p) &:= \frac{1}{2}C(p) + \frac{m+1}{2m} \inf\{C(L) \mid L \text{ is a hyperplane of } T_pM\}, \\ [\widehat{\delta}_c(m-1)](p) &:= 2C(p) - \frac{2m-1}{2m} \sup\{C(L) \mid L \text{ is a hyperplane of } T_pM\}. \end{aligned}$$

We define the *generalized normalized δ -Casorati curvatures* $\delta_c(r, m-1)$ and $\widehat{\delta}_c(r, m-1)$ of M in N as follows:

$$\begin{aligned} &[\delta_c(r, m-1)](p) \\ &:= rC(p) + \frac{(m-1)(m+r)(m^2-m-r)}{rm} \inf\{C(L) \mid L \text{ is a hyperplane of } T_pM\} \end{aligned}$$

for $0 < r < m^2 - m$,

$$\begin{aligned} &[\widehat{\delta}_c(r, m-1)](p) \\ &:= rC(p) - \frac{(m-1)(m+r)(r-m^2+m)}{rm} \sup\{C(L) \mid L \text{ is a hyperplane of } T_pM\} \end{aligned}$$

for $r > m^2 - m$.

Notice that $[\delta_c(\frac{m(m-1)}{2}, m-1)](p) = m(m-1)[\delta_c(m-1)](p)$ and $[\widehat{\delta}_c(2m(m-1), m-1)](p) = m(m-1)[\widehat{\delta}_c(m-1)](p)$ for $p \in M$ so that the generalized normalized δ -Casorati curvatures $\delta_c(r, m-1)$ and $\widehat{\delta}_c(r, m-1)$ are the generalized versions of the normalized δ -Casorati curvatures $\delta_c(m-1)$ and $\widehat{\delta}_c(m-1)$, respectively.

Throughout this paper, we will use the above notations.

3. Some optimal inequalities

In this section we will obtain some optimal inequalities consisting of the normalized scalar curvature and the generalized normalized δ -Casorati curvatures for real hypersurfaces of complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians.

Theorem 3.1. *Let M be a real hypersurface of a complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with $n = 4m - 1$. Then we have*

(a) *The generalized normalized δ -Casorati curvature $\delta_c(r, n-1)$ satisfies*

$$(3.1) \quad \rho \leq \frac{\delta_c(r, n-1)}{n(n-1)} + \frac{n+9}{n}$$

for any $r \in \mathbb{R}$ with $0 < r < n(n-1)$.

(b) The generalized normalized δ -Casorati curvature $\widehat{\delta}_c(r, n - 1)$ satisfies

$$(3.2) \quad \rho \leq \frac{\widehat{\delta}_c(r, n - 1)}{n(n - 1)} + \frac{n + 9}{n}$$

for any $r \in \mathbb{R}$ with $r > n(n - 1)$.

Moreover, the equalities hold in the relations (3.1) and (3.2) if and only if M is an invariantly quasi-umbilical submanifold with flat normal connection in $G_2(\mathbb{C}^{m+2})$ such that with some orthonormal tangent frame $\{e_1, \dots, e_n\}$ of TM and orthonormal normal frame $\{e_{n+1} = e\}$ of TM^\perp , the shape operator A_e takes the following form

$$(3.3) \quad A_e = \begin{pmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & \frac{n(n-1)}{r}a \end{pmatrix}.$$

Proof. Since M is a real hypersurface of $G_2(\mathbb{C}^{m+2})$ with a unit normal vector field e , we may choose a local orthonormal tangent frame $\{e_1, \dots, e_n\}$ of TM and an orthonormal normal frame $\{e_{n+1} = e\}$ of TM^\perp such that

$$\begin{aligned} e_{m+i} &= J_1 e_i, & e_{2m+i} &= J_2 e_i, & e_{3m+i} &= J_3 e_i, \\ e_{4m-3} &= \xi_1 = -J_1 e, & e_{4m-2} &= \xi_2 = -J_2 e, & e_{4m-1} &= e_n = \xi_3 = -J_3 e \end{aligned}$$

for $1 \leq i \leq m - 1$, where $\{J_1, J_2, J_3\}$ is a local quaternionic Hermitian basis of E .

Let $\xi := -Je$.

Using (2.4) and (2.7), we get

$$\begin{aligned} 2\tau(p) &= n(n - 1) + 3 \sum_{i,j=1}^n g(e_i, J e_j)^2 \\ &+ \sum_{\alpha=1}^3 \sum_{i,j=1}^n \{3g(e_i, J_\alpha e_j)^2 + g(e_i, J_\alpha J e_i) \cdot g(e_j, J_\alpha J e_j) - g(e_i, J_\alpha J e_j)^2\} \\ &+ n^2 \|H\|^2 - \|h\|^2 \quad \text{for } p \in M. \end{aligned}$$

With some computations, we obtain

$$\begin{aligned}
& 3 \sum_{i,j=1}^n g(e_i, J e_j)^2 + \sum_{\alpha=1}^3 \sum_{i,j=1}^n \{3g(e_i, J_\alpha e_j)^2 + g(e_i, J_\alpha J e_i) \cdot g(e_j, J_\alpha J e_j) - g(e_i, J_\alpha J e_j)^2\} \\
&= 3 \sum_{i,j=1}^n g(e_i, J e_j)^2 + \sum_{\alpha=1}^3 \left\{ 3(n-1) + \left(\sum_{i=1}^n g(e_i, J_\alpha J e_i) \right)^2 \right. \\
&\quad \left. - \sum_{i,j=1}^n g(e_i, J e_j)^2 + \sum_{j=1}^n g(\xi_\alpha, J e_j)^2 - \sum_{j=1}^n g(e, J e_j)^2 \right\} \\
&= 9(n-1) - 3 \sum_{j=1}^n g(e, J e_j)^2 + \sum_{\alpha=1}^3 \left\{ \left(\sum_{i=1}^n g(e_i, J_\alpha J e_i) \right)^2 + \sum_{j=1}^n g(\xi_\alpha, J e_j)^2 \right\}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \sum_{j=1}^n g(e, J e_j)^2 = \sum_{j=1}^n g(\xi, e_j)^2 = \sum_{j=1}^{n+1} g(\xi, e_j)^2 = \|\xi\|^2 = g(e, e) = 1, \\
& \sum_{i=1}^n g(e_i, J_\alpha J e_i) \\
&= - \sum_{i=1}^n g(J_\alpha e_i, J e_i) \\
&= - \sum_{i=1}^{m-1} \{g(J_\alpha e_i, J e_i) + g(J_\alpha J_1 e_i, J J_1 e_i) + g(J_\alpha J_2 e_i, J J_2 e_i) + g(J_\alpha J_3 e_i, J J_3 e_i)\} \\
&\quad - (g(J_\alpha \xi_1, J \xi_1) + g(J_\alpha \xi_2, J \xi_2) + g(J_\alpha \xi_3, J \xi_3)) \\
&= 0 - (g(J_\alpha J_1 e, J J_1 e) + g(J_\alpha J_2 e, J J_2 e) + g(J_\alpha J_3 e, J J_3 e)) \quad (\text{by (2.3)}) \\
&= g(J_\alpha e, J e) = g(\xi_\alpha, \xi) \quad (\text{by (2.3)}), \\
& \sum_{j=1}^n g(\xi_\alpha, J e_j)^2 = \sum_{j=1}^n g(J \xi_\alpha, e_j)^2 = \sum_{j=1}^{n+1} g(J \xi_\alpha, e_j)^2 - g(J \xi_\alpha, e)^2 \\
&= \|J \xi_\alpha\|^2 - g(\xi_\alpha, \xi)^2 = 1 - g(\xi_\alpha, \xi)^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
& 9(n-1) - 3 \sum_{j=1}^n g(e, J e_j)^2 + \sum_{\alpha=1}^3 \left\{ \left(\sum_{i=1}^n g(e_i, J_\alpha J e_i) \right)^2 + \sum_{j=1}^n g(\xi_\alpha, J e_j)^2 \right\} \\
&= 9(n-1).
\end{aligned}$$

Therefore,

$$(3.4) \quad 2\tau(p) = (n+9)(n-1) + n^2 \|H\|^2 - nC.$$

Conveniently, let $h_{ij} := h_{ij}^{n+1} = g(h(e_i, e_j), e_{n+1})$ for $i, j \in \{1, 2, \dots, n\}$.

Consider the quadratic polynomial in the components of the second fundamental form

$$\mathcal{P} := rC + \frac{(n-1)(n+r)(n^2-n-r)}{rn}C(L) - 2\tau(p) + (n+9)(n-1),$$

where L is a hyperplane of T_pM .

Now, we deal with some linear algebraic properties of the quadratic polynomial \mathcal{P} . Without loss of generality, we may assume that L is spanned by e_1, \dots, e_{n-1} .

With a simple calculation, by (3.4), we have

$$\begin{aligned} \mathcal{P} &= \frac{r}{n} \sum_{i,j=1}^n h_{ij}^2 + \frac{(n+r)(n^2-n-r)}{rn} \sum_{i,j=1}^{n-1} h_{ij}^2 - 2\tau(p) + (n+9)(n-1) \\ &= \frac{n+r}{n} \sum_{i,j=1}^n h_{ij}^2 + \frac{(n+r)(n^2-n-r)}{rn} \sum_{i,j=1}^{n-1} h_{ij}^2 - \left(\sum_{i=1}^n h_{ii} \right)^2 \\ (3.5) \quad &= \sum_{i=1}^{n-1} \left[\frac{n^2+n(r-1)-2r}{r} h_{ii}^2 + \frac{n+r}{n} (h_{in}^2 + h_{ni}^2) \right] \\ &\quad + \frac{(n+r)(n-1)}{r} \sum_{1 \leq i \neq j \leq n-1} h_{ij}^2 - \sum_{1 \leq i \neq j \leq n} h_{ii} h_{jj} + \frac{r}{n} h_{nn}^2. \end{aligned}$$

From (3.5), the critical points $h^c = (h_{11}, h_{12}, \dots, h_{nn})$ of \mathcal{P} are the solutions of the system of linear homogeneous equations:

$$(3.6) \quad \begin{cases} \frac{\partial \mathcal{P}}{\partial h_{ii}} = \frac{2(n+r)(n-1)}{r} h_{ii} - 2 \sum_{k=1}^n h_{kk} = 0, \\ \frac{\partial \mathcal{P}}{\partial h_{nn}} = \frac{2r}{n} h_{nn} - 2 \sum_{k=1}^{n-1} h_{kk} = 0, \\ \frac{\partial \mathcal{P}}{\partial h_{ij}} = \frac{2(n+r)(n-1)}{r} h_{ij} = 0, \\ \frac{\partial \mathcal{P}}{\partial h_{in}} = \frac{2(n+r)}{n} h_{in} = 0, \\ \frac{\partial \mathcal{P}}{\partial h_{ni}} = \frac{2(n+r)}{n} h_{ni} = 0 \end{cases}$$

for $i, j \in \{1, 2, \dots, n-1\}$ with $i \neq j$.

From (3.6), any solutions h^c satisfy $h_{ij} = 0$ for $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$.

Moreover, we get the Hessian matrix $\mathcal{H}(\mathcal{P})$ of \mathcal{P} as follows:

$$\mathcal{H}(\mathcal{P}) = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} \frac{2(n+r)(n-1)}{r} - 2 & -2 & \cdots & -2 & -2 \\ -2 & \frac{2(n+r)(n-1)}{r} - 2 & \cdots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \cdots & \frac{2(n+r)(n-1)}{r} - 2 & -2 \\ -2 & -2 & \cdots & -2 & \frac{2r}{n} \end{pmatrix},$$

0 denotes the zero matrices with the corresponding sizes, and the diagonal matrices H_2 , H_3 are given by

$$H_2 = \text{diag} \left(\frac{2(n+r)(n-1)}{r}, \frac{2(n+r)(n-1)}{r}, \dots, \frac{2(n+r)(n-1)}{r} \right),$$

$$H_3 = \text{diag} \left(\frac{2(n+r)}{n}, \frac{2(n+r)}{n}, \dots, \frac{2(n+r)}{n} \right).$$

Then we can find that the Hessian matrix $\mathcal{H}(\mathcal{P})$ has the following eigenvalues

$$\lambda_{11} = 0, \quad \lambda_{22} = \frac{2(n^3 - n^2 + r^2)}{rn}, \quad \lambda_{33} = \cdots = \lambda_{nn} = \frac{2(n+r)(n-1)}{r},$$

$$\lambda_{ij} = \frac{2(n+r)(n-1)}{r}, \quad \lambda_{in} = \lambda_{ni} = \frac{2(n+r)}{n}$$

for $i, j \in \{1, 2, \dots, n-1\}$ with $i \neq j$.

Thus, we know that \mathcal{P} is parabolic and has a minimum $\mathcal{P}(h^c)$ at any solution h^c of the system (3.6). Applying (3.6) to (3.5), we obtain $\mathcal{P}(h^c) = 0$. So, $\mathcal{P} \geq 0$ and this implies

$$2\tau(p) \leq rC + \frac{(n-1)(n+r)(n^2 - n - r)}{rn}C(L) + (n+9)(n-1).$$

Therefore, we get

$$(3.7) \quad \rho \leq \frac{r}{n(n-1)}C + \frac{(n+r)(n^2 - n - r)}{rn^2}C(L) + \frac{n+9}{n}$$

for any hyperplane L of T_pM so that both inequalities (3.1) and (3.2) easily follow from (3.7).

Furthermore, we see that the equalities hold at the relations (3.1) and (3.2) if and only if

$$h_{ij} = 0 \quad \text{for } i, j \in \{1, 2, \dots, n\} \text{ with } i \neq j,$$

$$h_{nn} = \frac{n(n-1)}{r}h_{11} = \frac{n(n-1)}{r}h_{22} = \cdots = \frac{n(n-1)}{r}h_{n-1, n-1}.$$

Therefore, we get that the equalities hold at (3.1) and (3.2) if and only if the submanifold M is invariantly quasi-umbilical with flat normal connection in $G_2(\mathbb{C}^{m+2})$ such that the shape operator takes the form (3.3) with respect to some orthonormal tangent and normal frames. \square

In the same way, by using (2.5) and (2.6), we obtain

Theorem 3.2. *Let M be a real hypersurface of a complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$ with $n = 4m - 1$. Then we have*

(a) *The generalized normalized δ -Casorati curvature $\delta_c(r, n - 1)$ satisfies*

$$(3.8) \quad \rho \leq \frac{\delta_c(r, n - 1)}{n(n - 1)} - \frac{n + 9}{2n}$$

for any $r \in \mathbb{R}$ with $0 < r < n(n - 1)$.

(b) *The generalized normalized δ -Casorati curvature $\widehat{\delta}_c(r, n - 1)$ satisfies*

$$(3.9) \quad \rho \leq \frac{\widehat{\delta}_c(r, n - 1)}{n(n - 1)} - \frac{n + 9}{2n}$$

for any $r \in \mathbb{R}$ with $r > n(n - 1)$.

Moreover, the equalities hold in the relations (3.8) and (3.9) if and only if M is an invariantly quasi-umbilical submanifold with flat normal connection in $SU_{2,m}/S(U_2 \cdot U_m)$ such that with some orthonormal tangent frame $\{e_1, \dots, e_n\}$ of TM and orthonormal normal frame $\{e_{n+1} = e\}$ of TM^\perp , the shape operator A_e takes the following form

$$A_e = \begin{pmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & \frac{n(n-1)}{r}a \end{pmatrix}.$$

Using the relations $[\delta_c(\frac{n(n-1)}{2}, n - 1)](p) = n(n - 1)[\delta_c(n - 1)](p)$ and $[\widehat{\delta}_c(2n(n - 1), n - 1)](p) = n(n - 1)[\widehat{\delta}_c(n - 1)](p)$ for $p \in M$, we easily have

Corollary 3.3. *Let M be a real hypersurface of a complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with $n = 4m - 1$. Then we get*

(a) *The normalized δ -Casorati curvature $\delta_c(n - 1)$ satisfies*

$$\rho \leq \delta_c(n - 1) + \frac{n + 9}{n}.$$

Moreover, the equality holds if and only if M is an invariantly quasi-umbilical submanifold with flat normal connection in $G_2(\mathbb{C}^{m+2})$ such that with some orthonormal

tangent frame $\{e_1, \dots, e_n\}$ of TM and orthonormal normal frame $\{e_{n+1} = e\}$ of TM^\perp , the shape operator A_e takes the following form

$$A_e = \begin{pmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & 2a \end{pmatrix}.$$

(b) The normalized δ -Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(n-1) + \frac{n+9}{n}.$$

Moreover, the equality holds if and only if M is an invariantly quasi-umbilical submanifold with flat normal connection in $G_2(\mathbb{C}^{m+2})$ such that with some orthonormal tangent frame $\{e_1, \dots, e_n\}$ of TM and orthonormal normal frame $\{e_{n+1} = e\}$ of TM^\perp , the shape operator A_e takes the following form

$$A_e = \begin{pmatrix} 2a & 0 & \cdots & 0 & 0 \\ 0 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2a & 0 \\ 0 & 0 & \cdots & 0 & a \end{pmatrix}.$$

Corollary 3.4. Let M be a real hypersurface of a complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2 \cdot U_m)$ with $n = 4m - 1$. Then we obtain

(a) The normalized δ -Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) - \frac{n+9}{2n}.$$

Moreover, the equality holds if and only if M is an invariantly quasi-umbilical submanifold with flat normal connection in $SU_{2,m}/S(U_2 \cdot U_m)$ such that with some orthonormal tangent frame $\{e_1, \dots, e_n\}$ of TM and orthonormal normal frame $\{e_{n+1} = e\}$ of TM^\perp , the shape operator A_e takes the following form

$$A_e = \begin{pmatrix} a & 0 & \cdots & 0 & 0 \\ 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ 0 & 0 & \cdots & 0 & 2a \end{pmatrix}.$$

(b) The normalized δ -Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(n-1) - \frac{n+9}{2n}.$$

Moreover, the equality holds if and only if M is an invariantly quasi-umbilical submanifold with flat normal connection in $SU_{2,m}/S(U_2 \cdot U_m)$ such that with some orthonormal tangent frame $\{e_1, \dots, e_n\}$ of TM and orthonormal normal frame $\{e_{n+1} = e\}$ of TM^\perp , the shape operator A_e takes the following form

$$A_e = \begin{pmatrix} 2a & 0 & \cdots & 0 & 0 \\ 0 & 2a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2a & 0 \\ 0 & 0 & \cdots & 0 & a \end{pmatrix}.$$

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