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On the Finiteness Results of Generalized Local Cohomology Modules with Respect to a Pair of Ideals

Tran Tuan Nam* and Nguyen Minh Tri

Abstract. We study the finiteness of associated primes and support of the generalized local cohomology modules $H^i_{I,J}(M,N)$ concerning Grothendieck's conjecture and Huneke's question. The paper also discusses the relationship between the vanishing and the finiteness of the generalized local cohomology modules.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring. The local cohomology with respect to a pair of ideals was first introduced and studied by Takahashi, Yoshino and Yoshizawa [13]. Let I, J be two ideals of R, for an R-module M, the (I,J)-torsion submodule $\Gamma_{I,J}(M)$ of M consists of all elements x of M such that $I^n x \subseteq Jx$ for some $n \in \mathbb{N}$. For an integer i, they defined the i-th local cohomology functor $H^i_{I,J}$ to be the i-th right derived functor of $\Gamma_{I,J}$. It is clear that if $J = 0$, then the functor $H^i_{I,J}$ coincides with the ordinary local cohomology functor H^i_I of Grothendieck.

Recently, a natural generalization of local cohomology modules with respect to (I,J) was introduced in [8] as follows: Let M, N be two R-modules, the module $\Gamma_{I,J}(M,N)$ is the (I,J)-torsion submodule of $\operatorname{Hom}_R(M,N)$. For each finitely generated R-module M, the i-th generalized local cohomology functor $H^i_{I,J}(M,-)$ with respect to a pair of ideals (I,J) is the i-th right derived functor of the functor $\Gamma_{I,J}(M,-)$. Clearly, whenever $J = 0$, the functor $H^i_{I,J}(M,-)$ is the generalized local cohomology functor $H^i_I(M,-)$ of J. Herzog [4]. On the other hand, when $M = R$, the generalized local cohomology module $H^i_{I,J}(R,N)$ is the local cohomology module $H^i_{I,J}(N)$.

In [3] A. Grothendieck gave a conjecture: For any ideal I of R and any finitely generated R-module M, the module $\operatorname{Hom}_R(R/I, H^i_I(M))$ is finitely generated, for all i. A lighter question is due to C. Huneke [5]: If M is finitely generated, is the number of associated primes of local cohomology modules $H^i_I(M)$ always finite?

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*Corresponding author.

The purpose of this paper is to show some properties of generalized local cohomology modules $H^i_{I,J}(M,N)$ concerning A. Grothendieck's conjecture and C. Huneke's question. It is well known that if M, N are finitely generated R-modules, then $\mathrm{Ass}_R(H^t_I(M,N))$ is finite in either of the following cases:

(i) $H^i_I(M,N)$ is finitely generated for all $i<t$ (see [1] and [14]);

(ii) $H^i_I(M,N)$ is artinian for all $i<t$ (see [15]);

(iii) $H^i_I(M,N)$ is weakly Laskerian for all $i<t$ (see [7]);

(iv) $\mathrm{Supp}_R(H^i_I(N))$ is finite for all $i<t$ (see [11]).

In this paper, we prove in Theorem 2.2 that the set $\mathrm{Ass}_R(H^t_I(M,N))$ is finite if $\mathrm{Supp}_R(H^i_{I,J}(M,N))$ is finite for all $i<t$, where t is a non-negative integer. We also show in Theorem 2.4 that if $\mathrm{Ass}_R(H^t_{I,J}(N))$ is a finite set and $\mathrm{Ext}^{t-i}_R(M,H^i_{I,J}(N))$ is weakly Laskerian for all $i<t$, then $\mathrm{Ass}_R(H^t_{I,J}(M,N))$ is finite.

There is a similar question: When is the set $\mathrm{Supp}_R(H^i_{I,J}(M,N))$ finite? Theorem 2.6 says that if $\mathrm{Supp}_R(\mathrm{Ext}^i_R(M,N))$ is finite for all $i<t$, then $\mathrm{Supp}_R(H^i_{I,J}(M,N))$ is finite for all $i<t$. We will see in Theorem 2.7 that if $d=\dim R$, then $H^d_{I,J}(M,N)/JH^d_{I,J}(M,N)$ is an artinian R-module and $\mathrm{Supp}_R(H^{d-1}_{I,J}(M,N)/JH^{d-1}_{I,J}(M,N))$ is finite. Next, Theorem 2.9 shows a connection between the finiteness and the vanishing of the module $H^i_{I,J}(M,N)$. This theorem says that in a local ring $(R,\mathfrak{m})$, if $t>\mathrm{pd}\,M$ and $\dim N<\infty$, then $H^i_{I,J}(M,N)$ is finitely generated for all $i\ge t$ if and only if $H^i_{I,J}(M,N)=0$ for all $i\ge t$. Moreover, if $H^i_{I,J}(M,N)$ is finitely generated for all $i>t$, then $H^t_{I,J}(M,N)/\mathfrak{a}H^t_{I,J}(M,N)$ is finitely generated for all $\mathfrak{a}\in\widetilde{W}(I,J)$ (see Theorem 2.11). The paper is closed by Theorem 2.13 which shows that

$$\inf\{i \mid H^i_{I,J}(M,N)\neq 0\}=\inf\{\mathrm{grade}(\mathfrak{a}+\mathrm{Ann}(M),N)\mid \mathfrak{a}\in\widetilde{W}(I,J)\}.$$

Moreover, $\mathrm{Ass}_R(H^t_{I,J}(M,N))\cap V(\mathfrak{a})=\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M,N))$ for some $\mathfrak{a}\in\widetilde{W}(I,J)$, where $t=\inf\{i\mid H^i_{I,J}(M,N)\neq 0\}$.

2. Main results

We begin by recalling the concept of *weakly Laskerian modules* which was introduced in [2]. An R-module M is said to be *weakly Laskerian* if the set of associated primes of any quotient module of M is finite. In [13], the authors introduced the following sets:

$$W(I,J)=\{\mathfrak{p}\in\mathrm{Spec}(R)\mid I^n\subseteq\mathfrak{p}+J \text{ for some } n\gg 1\}$$

and

$$\widetilde{W}(I,J)=\{\mathfrak{a}\lhd R\mid I^n\subseteq\mathfrak{a}+J \text{ for some } n\gg 1\}.$$

The following result concerning to weakly Laskerian modules will be used to prove Theorem 2.2.

Proposition 2.1. *Let M be finitely generated and t a non-negative integer. If N is a weakly Laskerian R-module and $H^i_{I,J}(M,N)$ is weakly Laskerian for all $i < t$, then* $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M,N))$ *is weakly Laskerian for all* $\mathfrak{a} \in \widetilde{W}(I,J)$.

Proof. We use induction on t. When $t = 0$, we have

$$\operatorname{Hom}_R(R/\mathfrak{a}, H^0_{I,J}(M,N)) \cong \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Hom}_R(M, \Gamma_{I,J}(N))).$$

Since $\Gamma_{I,J}(N) \subseteq N$, $\operatorname{Hom}_R(R/\mathfrak{a}, H^0_{I,J}(M,N))$ is weakly Laskerian by the hypothesis.

Let $t > 0$, set $\overline{N} = N/\Gamma_{I,J}(N)$. The short exact sequence

$$0 \to \Gamma_{I,J}(N) \to N \to \overline{N} \to 0$$

induces a long exact sequence

$$\cdots \to H^t_{I,J}(M, \Gamma_{I,J}(N)) \xrightarrow{f} H^t_{I,J}(M,N) \xrightarrow{g} H^t_{I,J}(M,\overline{N}) \to \cdots .$$

The long exact sequence gives us the following exact sequences

$$0 \to \operatorname{Im} f \to H^t_{I,J}(M,N) \to \operatorname{Im} g \to 0$$

and

$$0 \to \operatorname{Im} g \to H^t_{I,J}(M,\overline{N}) \xrightarrow{h} H^{t+1}_{I,J}(M, \Gamma_{I,J}(N)) \to \cdots .$$

By applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ to the above exact sequences, we obtain exact sequences

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} f) \to \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M,N)) \to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} g) \to \cdots$$

and

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} g) \to \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M,\overline{N})) \to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im} h) \to \cdots .$$

It follows from [8, 2.6] that $H^i_{I,J}(M, \Gamma_{I,J}(N)) \cong \operatorname{Ext}^i_R(M, \Gamma_{I,J}(N))$ for all $i \geq 0$. As N is a weakly Laskerian R-module, $H^i_{I,J}(M, \Gamma_{I,J}(N))$ is weakly Laskerian for all $i \geq 0$. It follows that $\operatorname{Im} f$, $\operatorname{Im} h$ are weakly Laskerian. Thus, the proof is completed by showing that $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M,\overline{N}))$ is weakly Laskerian. Since $\overline{N}$ is (I,J)-torsion-free, $\overline{N}$ is also $\mathfrak{a}$-torsion-free. Then there is an $\overline{N}$-regular element $x \in \mathfrak{a}$. Now, the short exact sequence

$$0 \to \overline{N} \xrightarrow{.x} \overline{N} \to \overline{N}/x\overline{N} \to 0$$

induces the long exact sequence

$$\cdots \to H^{t-1}_{I,J}(M,\overline{N}) \xrightarrow{\alpha} H^{t-1}_{I,J}(M,\overline{N}/x\overline{N}) \xrightarrow{\beta} H^{t}_{I,J}(M,\overline{N}) \xrightarrow{.x} H^{t}_{I,J}(M,\overline{N}) \to \cdots .$$

By the assumption $H^i_{I,J}(M,\overline{N})$ is weakly Laskerian for all $i < t$, so is $H^i_{I,J}(M,\overline{N}/x\overline{N})$ for all $i < t-1$. It follows from the inductive hypothesis that $\operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{I,J}(M,\overline{N}/x\overline{N}))$ is weakly Laskerian. The long exact sequence induces the exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im}\alpha) \to \operatorname{Hom}_R(R/\mathfrak{a}, H^{t-1}_{I,J}(M,\overline{N}/x\overline{N})) \\ &\to \operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im}\beta) \to \operatorname{Ext}^1_R(R/\mathfrak{a}, \operatorname{Im}\alpha) \to \cdots . \end{aligned}$$

As $\operatorname{Im}\alpha$ is weakly Laskerian, $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im}\beta)$ is also weakly Laskerian. By applying the functor $\operatorname{Hom}_R(R/\mathfrak{a}, -)$ to the following exact sequence

$$0 \to \operatorname{Im}\beta \to H^t_{I,J}(M,\overline{N}) \xrightarrow{.x} H^t_{I,J}(M,\overline{N})$$

we get $\operatorname{Hom}_R(R/\mathfrak{a}, \operatorname{Im}\beta) \cong \operatorname{Hom}_R(R/\mathfrak{a}, H^t_{I,J}(M,\overline{N}))$ and the proof is complete. □

In [7, 2.4], if M, N are two finitely generated R-modules such that $\operatorname{Supp}_R(H^i_I(M,N))$ is finite for all $i < t$, then $\operatorname{Ass}_R(H^t_I(M,N))$ is finite. Now, we will extend this property in the case N is weakly Laskerian and $H^i_{I,J}(M,N)$ has finite support for all $i < t$.

Theorem 2.2. *Let M be a finitely generated R-module and N an R-module. Let t be a non-negative integer such that $\operatorname{Supp}_R(H^i_{I,J}(M,N))$ is finite for all $i < t$. Then the following statements hold:*

(i) $\operatorname{Supp}_R(H^i_{\mathfrak{a}}(M,N))$ *is finite for all $i < t$ and $\mathfrak{a} \in \widetilde{W}(I,J)$.*

(ii) *If N is weakly Laskerian, then $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M,N))$ is finite for all $\mathfrak{a} \in \widetilde{W}(I,J)$. In particular, $\operatorname{Ass}_R(H^t_I(M,N))$ is a finite set.*

Proof. (i) Let $F = \Gamma_{\mathfrak{a}}(-)$ and $G = \Gamma_{I,J}(M,-)$ be functors from the category of R-modules to itself. It is clear that

$$\begin{aligned} FG(N) &= \Gamma_{\mathfrak{a}}(\Gamma_{I,J}(M,N)) = \Gamma_{\mathfrak{a}}(\Gamma_{I,J}(\operatorname{Hom}_R(M,N))) \\ &= \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M,N)) = \Gamma_{\mathfrak{a}}(M,N). \end{aligned}$$

If E is an injective R-module, then $\Gamma_{I,J}(E)$ is also injective. Hence

$$R^iF(G(E)) = R^i\Gamma_{\mathfrak{a}}(\Gamma_{I,J}(M,E)) \cong R^i\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M,\Gamma_{I,J}(E))) = 0$$

for all $i > 0$. By [10, 10.47] there is a Grothendieck spectral sequence

$$E_2^{p,q} = H^p_{\mathfrak{a}}(H^q_{I,J}(M,N)) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{a}}(M,N).$$

It follows from the hypothesis that $\mathrm{Supp}_R(E_2^{p,q})$ is finite for all $q < t$, $p \geq 0$. This implies that $\mathrm{Supp}_R(E_\infty^{p,q})$ is finite for all $q < t$, $p \geq 0$, as $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$. Let $n < t$, there is a filtration Φ of submodules of $H^n = H_{\mathfrak{a}}^n(M,N)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H_{\mathfrak{a}}^n(M,N)$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $i \leq n$. By descending induction on i, we conclude that $\mathrm{Supp}_R(\Phi^i H^n)$ is finite for all $i \leq n$. In particular, $\mathrm{Supp}_R(H_{\mathfrak{a}}^n(M,N))$ is finite.

(ii) We consider the homomorphism of the spectral sequence

$$0 = E_{t+2}^{-t-2,2t+1} \stackrel{d_{t+2}^{-t-2,2t+1}}{\longrightarrow} E_{t+2}^{0,t} \stackrel{d_{t+2}^{0,t}}{\longrightarrow} E_{t+2}^{t+2,-1} = 0.$$

It follows that $E_{t+2}^{0,t} = E_{t+3}^{0,t} = \cdots = E_\infty^{0,t}$. Note that there is a filtration Φ of submodules of $H^t = H_{\mathfrak{a}}^t(M,N)$

$$0 = \Phi^{t+1}H^t \subseteq \Phi^t H^t \subseteq \cdots \subseteq \Phi^1 H^t \subseteq \Phi^0 H^t = H_{\mathfrak{a}}^t(M,N)$$

such that

$$E_\infty^{i,t-i} \cong \Phi^i H^t / \Phi^{i+1} H^t$$

for all $i \leq t$. By descending induction on i, we conclude that $\mathrm{Supp}_R(\Phi^i H^t)$ is finite for all $0 < i \leq t$. The short exact sequence

$$0 \to \Phi^1 H^t \to \Phi^0 H^t \to E_\infty^{0,t} \to 0$$

induces that

$$\mathrm{Ass}_R(H_{\mathfrak{a}}^t(M,N)) \subseteq \mathrm{Ass}_R(\Phi^1 H^t) \cup \mathrm{Ass}_R(E_\infty^{0,t}) = \mathrm{Ass}_R(\Phi^1 H^t) \cup \mathrm{Ass}_R(E_{t+2}^{0,t}).$$

Thus, the proof is completed by showing that $\mathrm{Ass}_R(E_{t+2}^{0,t})$ is finite. Now, we prove that $\mathrm{Ass}_R(E_r^{0,t})$ is finite by induction on r, where $2 \leq r \leq t+2$. When $r = 2$, we see that

$$\begin{aligned} \mathrm{Ass}_R(E_2^{0,t}) &= \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(H_{I,J}^t(M,N))) \\ &= \mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(H_{I,J}^t(M,N)))) \\ &= \mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, H_{I,J}^t(M,N))). \end{aligned}$$

From the hypothesis we conclude that $H_{I,J}^i(M,N)$ is weakly Laskerian for all $i < t$. Hence, $\mathrm{Hom}_R(R/\mathfrak{a}, H_{I,J}^t(M,N))$ is weakly Laskerian for all $\mathfrak{a} \in \widetilde{W}(I,J)$ by Proposition 2.1. In particular, $\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, H_{I,J}^t(M,N)))$ is finite. Assume that $\mathrm{Ass}_R(E_r^{0,t})$ is finite, we will show that $\mathrm{Ass}_R(E_{r+1}^{0,t})$ is finite. Let us consider the homomorphism of the spectral sequence

$$0 = E_r^{-r,t+r-1} \xrightarrow{d_r^{-r,t+r-1}} E_r^{0,t} \xrightarrow{d_r^{0,t}} E_r^{r,t-r+1}.$$

Since $E_{r+1}^{0,t} = \mathrm{Ker}\, d_r^{0,t} \subseteq E_r^{0,t}$, it follows from the inductive hypothesis that $\mathrm{Ass}_R(E_{r+1}^{0,t})$ is finite. Therefore, $\mathrm{Ass}_R(E_{t+2}^{0,t})$ is finite and this finishes the proof. □

Note that the support of an artinian module is finite. By Theorem 2.2, we have the following immediate result, which generalizes [15, 3.3].

Corollary 2.3. *Let M, N be two finitely generated R-modules and t a non-negative integer. Assume that $H_{I,J}^i(M,N)$ is artinian for all $i < t$. Then $\mathrm{Ass}_R(H_{\mathfrak{a}}^t(M,N))$ is finite for all $\mathfrak{a} \in \widetilde{W}(I,J)$. In particular, $\mathrm{Ass}_R(H_I^t(M,N))$ is a finite set.*

The following theorem gives us an answer for the question: When is the set $\mathrm{Ass}_R(H_{I,J}^t(M,N))$ finite?

Theorem 2.4. *Let M be a finitely generated R-module and N an R-module. Let t be a non-negative integer such that*

(i) $\mathrm{Ass}_R(H_{I,J}^t(N))$ *is a finite set;*

(ii) $\mathrm{Ext}_R^{t-i}(M, H_{I,J}^i(N))$ *is weakly Laskerian for all $i < t$.*

Then $\mathrm{Ass}_R(H_{I,J}^t(M,N))$ is finite.

Proof. Let $F = \mathrm{Hom}_R(M,-)$ and $G = \Gamma_{I,J}(-)$ be functors from the category of R-modules to itself. It is clear that $FG(N) = \Gamma_{I,J}(M,N)$ for all R-modules N. Moreover, if E is an injective R-module, then $G(E) = \Gamma_{I,J}(E)$ is also injective. Hence $G(E)$ is right F-acyclic. By [10, 10.47], there is a Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_R^p(M, H_{I,J}^q(N)) \underset{p}{\Rightarrow} H_{I,J}^{p+q}(M,N).$$

Then there is a filtration Φ of submodules of $H^t = H_{I,J}^t(M,N)$

$$0 = \Phi^{t+1}H^t \subseteq \Phi^t H^t \subseteq \cdots \subseteq \Phi^0 H^t = H^t$$

such that

$$E_\infty^{i,t-i} \cong \Phi^i H^t / \Phi^{i+1} H^t$$

for all $i \le t$. By the hypothesis $E_2^{i,t-i}$ is weakly Laskerian for all $0 < i \le t$. Then $E_\infty^{i,t-i}$ is weakly Laskerian for all $0 < i \le t$, as $E_\infty^{i,t-i}$ is a subquotient of $E_2^{i,t-i}$. This implies that $\Phi^t H^t, \Phi^{t-1}H^t, \dots, \Phi^1 H^t$ are weakly Laskerian. Now, the short exact sequence

$$0 \to \Phi^1 H^t \to H^t \to E_\infty^{0,t} \to 0$$

induces

$$\mathrm{Ass}_R(H^t_{I,J}(M,N)) \subseteq \mathrm{Ass}_R(\Phi^1 H^t) \cup \mathrm{Ass}_R(E^{0,t}_\infty).$$

Note that $E^{0,t}_\infty$ is a submodule of $E^{0,t}_2 = \mathrm{Hom}_R(M, H^t_{I,J}(N))$. By the hypothesis, $\mathrm{Ass}_R(\mathrm{Hom}_R(M, H^t_{I,J}(N))) = \mathrm{Supp}_R(M) \cap \mathrm{Ass}_R(H^t_{I,J}(N))$ is finite. Thus $\mathrm{Ass}_R(E^{0,t}_\infty)$ is finite and the proof is complete. □

Corollary 2.5. *Let M be a finitely generated R-module and N an R-module. Let t be a non-negative integer. Assume that $H^i_{I,J}(N)$ is weakly Laskerian for all $i < t$ and $H^t_{I,J}(N)$ has finitely many associated prime ideals. Then $\mathrm{Ass}_R(H^t_{I,J}(M,N))$ is finite.*

Now we have a relationship on the finiteness of the support of $\mathrm{Ext}^i_R(M,N)$ and that of $H^i_{I,J}(M,N)$.

Theorem 2.6. *Let M be a finitely generated R-module and N an R-module. Let t be a non-negative integer such that $\mathrm{Supp}_R(\mathrm{Ext}^i_R(M,N))$ is finite for all $i < t$. Then the following statements hold:*

(i) $\mathrm{Supp}_R(H^i_{I,J}(M,N))$ *is finite for all $i < t$.*

(ii) *In addition, if $\mathrm{Ass}_R(\mathrm{Ext}^t_R(M,N)) \cap W(I,J)$ is finite, then $\mathrm{Ass}_R(H^t_{I,J}(M,N))$ is finite.*

Proof. Let $F = \Gamma_{I,J}(-)$ and $G = \mathrm{Hom}_R(M,-)$ be functors from the category of R-modules to itself. It is clear that $FG = \Gamma_{I,J}(\mathrm{Hom}_R(M,-)) = \Gamma_{I,J}(M,-)$. If E is an injective R-module, then so is $\Gamma_{I,J}(E)$. Let $\mathbf{F}_\bullet$ be a free resolution of M

$$\mathbf{F}_\bullet\colon \ \cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

By applying the functor $\mathrm{Hom}_R(-, \Gamma_{I,J}(E))$ to the above exact, we get an exact sequence

$$0 \to \mathrm{Hom}_R(M, \Gamma_{I,J}(E)) \to \mathrm{Hom}_R(F_0, \Gamma_{I,J}(E)) \to \mathrm{Hom}_R(F_1, \Gamma_{I,J}(E)) \to \cdots .$$

Since M is finitely generated, the exact sequence can be rewritten

$$0 \to \Gamma_{I,J}(\mathrm{Hom}_R(M,E)) \to \Gamma_{I,J}(\mathrm{Hom}_R(F_0,E)) \to \Gamma_{I,J}(\mathrm{Hom}_R(F_1,E)) \to \cdots .$$

Note that $\mathrm{Hom}_R(\mathbf{F}_\bullet, E)$ is an injective resolution of $\mathrm{Hom}_R(M,E)$. Then $R^iF(G(E)) = H^i_{I,J}(\mathrm{Hom}_R(M,E)) = 0$ for all $i > 0$. By [10, 10.47] there is a Grothendieck spectral sequence

$$E^{p,q}_2 = H^p_{I,J}(\mathrm{Ext}^q_R(M,N)) \underset{p}{\Rightarrow} H^{p+q}_{I,J}(M,N).$$

(i) Let $n < t$, there is a filtration Φ of submodules of $H^n = H^n_{I,J}(M,N)$

$$0 = \Phi^{n+1}H^n \subseteq \Phi^n H^n \subseteq \cdots \subseteq \Phi^1 H^n \subseteq \Phi^0 H^n = H^n$$

such that

$$E_\infty^{i,n-i} \cong \Phi^i H^n / \Phi^{i+1} H^n$$

for all $i \le n$. By the hypothesis $\mathrm{Supp}_R(E_2^{p,q})$ is finite for all $p \ge 0$, $q < t$. Since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$, $\mathrm{Supp}_R(E_\infty^{p,q})$ is finite for all $p \ge 0$, $q < t$. Consequently, $\mathrm{Supp}_R(\Phi^i H^n)$ is finite for all $0 \le i \le n$. In particular, $\mathrm{Supp}_R(H^n_{I,J}(M,N))$ is finite.

(ii) By a similar argument, we can conclude that $\mathrm{Supp}_R(\Phi^i H^t)$ is finite for all $0 < i \le t$. Now the short exact sequence

$$0 \to \Phi^1 H^t \to H^t_{I,J}(M,N) \to E_\infty^{0,t} \to 0$$

induces that

$$\mathrm{Ass}_R(H^t_{I,J}(M,N)) \subseteq \mathrm{Ass}_R(\Phi^1 H^t) \cup \mathrm{Ass}_R(E_\infty^{0,t}).$$

Note that $E_\infty^{0,t}$ is a submodule of $E_2^{0,t} = \Gamma_{I,J}(\mathrm{Ext}^t_R(M,N))$. By [13, 1.10], we have $\mathrm{Ass}_R(E_2^{0,t}) = \mathrm{Ass}_R(\mathrm{Ext}^t_R(M,N)) \cap W(I,J)$ is finite. This implies that $\mathrm{Ass}_R(E_\infty^{0,t})$ is finite which completes the proof. □

In [9, 5.4], if $\dim R = d$ and M, N are finitely generated R-modules with $\mathrm{pd}\, M < \infty$, then $H^d_I(M,N)$ is artinian and $\mathrm{Supp}_R(H^{d-1}_I(M,N))$ is finite. Now, we will study the modules $H^d_{I,J}(M,N)$ and $H^{d-1}_{I,J}(M,N)$, where $d = \dim R$.

Theorem 2.7. *Let $(R,\mathfrak{m})$ be a local ring and M, N two finitely generated R-modules with $\mathrm{pd}\, M < \infty$, $d = \dim R$. Then the following statements hold:*

(i) $H^d_{I,J}(M,N)/JH^d_{I,J}(M,N)$ *is artinian.*

(ii) $\mathrm{Supp}_R(H^{d-1}_{I,J}(M,N)/JH^{d-1}_{I,J}(M,N))$ *is finite.*

Proof. (i) We use induction on $\dim N$. If $\dim N = 0$, then N is an artinian R-module. It follows from [8, 3.7] that $H^d_{I,J}(M,N)$ is also artinian. Let $\dim N > 0$. From the short exact sequence

$$0 \to \Gamma_J(N) \to N \to N/\Gamma_J(N) \to 0$$

we have the following exact sequence

$$\cdots \to H^i_{I,J}(M,\Gamma_J(N)) \to H^i_{I,J}(M,N) \to H^i_{I,J}(M,N/\Gamma_J(N)) \to \cdots.$$

By [8, 2.7], $H^i_{I,J}(M,\Gamma_J(N)) \cong H^i_I(M,\Gamma_J(N))$ for all $i \ge 0$. Then $H^d_{I,J}(M,\Gamma_J(N))$ is artinian by [9, 5.4(i)]. It follows from [16, 2.7] that $H^i_{I,J}(M,N) = 0$ for all $i > \dim R$. Therefore, the exact sequence

$$H^d_{I,J}(M,\Gamma_J(N)) \xrightarrow{\alpha} H^d_{I,J}(M,N) \to H^d_{I,J}(M,N/\Gamma_J(N)) \to 0$$

induces the following short exact sequence

$$0 \to \operatorname{Im}\alpha \to H^d_{I,J}(M,N) \to H^d_{I,J}(M,N/\Gamma_J(N)) \to 0.$$

By applying the functor $R/J \otimes_R -$, there is an exact sequence

$$\begin{aligned}\cdots \to \operatorname{Im}\alpha/J\operatorname{Im}\alpha &\to H^d_{I,J}(M,N)/JH^d_{I,J}(M,N)\\ &\to H^d_{I,J}(M,N/\Gamma_J(N))/JH^d_{I,J}(M,N/\Gamma_J(N)) \to 0.\end{aligned}$$

Since $\operatorname{Im}\alpha$ is an artinian R-module, the proof is completed by showing that $H^d_{I,J}(M,N/\Gamma_J(N))/JH^d_{I,J}(M,N/\Gamma_J(N))$ is artinian. Set $\overline{N} = N/\Gamma_J(N)$, it is clear that N is J-torsion-free. There is a $\overline{N}$-regular element $x \in J$. Now the short exact sequence

$$0 \to \overline{N} \overset{\cdot x}{\to} \overline{N} \to \overline{N}/x\overline{N} \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^d_{I,J}(M,\overline{N}) \overset{\cdot x}{\to} H^d_{I,J}(M,\overline{N}) \to H^d_{I,J}(M,\overline{N}/x\overline{N}) \to 0.$$

Again, by applying the functor $R/J \otimes_R -$ to the above exact sequence, we get

$$H^d_{I,J}(M,\overline{N})/JH^d_{I,J}(M,\overline{N}) \cong H^d_{I,J}(M,\overline{N}/x\overline{N})/JH^d_{I,J}(M,\overline{N}/x\overline{N}).$$

Thus the assertion follows from the inductive hypothesis.

(ii) We consider the long exact sequence

$$\cdots \to H^{d-1}_{I,J}(M,\Gamma_J(N)) \overset{\beta}{\to} H^{d-1}_{I,J}(M,N) \to H^{d-1}_{I,J}(M,\overline{N}) \to \cdots.$$

Combining [8, 2.7] with [9, 5.4], we see that $H^d_{I,J}(M,\Gamma_J(N))$ is artinian and $\operatorname{Supp}_R(H^{d-1}_{I,J}(M,\Gamma_J(N)))$ is finite. If we prove that $\operatorname{Supp}_R(H^{d-1}_{I,J}(M,\overline{N})/JH^{d-1}_{I,J}(M,\overline{N}))$ is a finite set, then the assertion follows. Now we can proceed analogously to the proof of (i). □

Many properties on the finiteness of support of generalized local cohomology modules with respect to an ideal were studied in a local ring. We see in [6, 2.10] that $\operatorname{Supp}_R(H_I^{\dim R+\operatorname{pd} M-1}(M,N))$ is finite. An improvement of this result was shown in [11, 4.4] which says that $\operatorname{Supp}_R(H_I^{\operatorname{pd} M+\dim N-1}(M,N))$ is finite. The following proposition is an extension of [6, 2.10] and [11, 4.4].

Proposition 2.8. *Let M, N be two finitely generated R-modules with $p = \operatorname{pd} M < \infty$. Then*

$$\operatorname{Supp}_R(H^{p+d-1}_{I,J}(M,N)/JH^{p+d-1}_{I,J}(M,N))$$

is a finite set, where $d = \dim N$ or $d = \dim R$.

Proof. The proof is by induction on $\dim N$. When $\dim N = 0$, N is an artinian R-module. Hence $H^{p-1}_{I,J}(M,N)$ is artinian by [8, 3.7]. Therefore $\mathrm{Supp}_R(H^{p-1}_{I,J}(M,N)) \subseteq \mathrm{Max}(R)$ is a finite set.

We now assume that $\dim N > 0$. The rest of the proof is similar to that in the proof of Theorem 2.7(i). □

Next, we show some results concerning to the finiteness of $H^i_{I,J}(M,N)$.

Theorem 2.9. *Let $(R,\mathfrak{m})$ be a local ring and M, N two finitely generated R-modules with $\mathrm{pd}\, M < \infty$. If t is an integer such that $t > \mathrm{pd}\, M$, then the following statements are equivalent:*

(i) *$H^i_{I,J}(M,N)$ is finitely generated for all $i \geq t$.*

(ii) *$H^i_{I,J}(M,N) = 0$ for all $i \geq t$.*

Proof. (ii) ⇒ (i). It is clear.

(i) ⇒ (ii). We prove by induction on $n = \dim N$. Combining [16, 2.2] with [12, 3.1] we get $H^i_{I,J}(M,N) = 0$ for all $i > \mathrm{pd}\, M + \dim N$. If $n = 0$, then $H^i_{I,J}(M,N) = 0$ for all $i \geq t$. Let $n > 0$, the short exact sequence

$$0 \to \Gamma_{I,J}(N) \to N \to N/\Gamma_{I,J}(N) \to 0$$

induces a long exact sequence

$$\cdots \to H^i_{I,J}(M,\Gamma_{I,J}(N)) \to H^i_{I,J}(M,N) \to H^i_{I,J}(M,N/\Gamma_{I,J}(N)) \to \cdots .$$

It follows from [8, 2.6] that

$$H^i_{I,J}(M,\Gamma_{I,J}(N)) \cong \mathrm{Ext}^i_R(M,\Gamma_{I,J}(N))$$

for all $i \geq 0$. Then $H^i_{I,J}(M,\Gamma_{I,J}(N)) = 0$ for all $i > \mathrm{pd}\, M$. By the assumption, $H^i_{I,J}(M,N/\Gamma_{I,J}(N)) \cong H^i_{I,J}(M,N)$ for all $i > \mathrm{pd}\, M$. Thus we may assume, by replacing N with $N/\Gamma_{I,J}(N)$, that N is (I,J)-torsion-free. Since $\mathfrak{m} \in \widetilde{W}(I,J)$, N is $\mathfrak{m}$-torsion-free. Then there is an N-regular element $x \in \mathfrak{m}$. Now the short exact sequence

$$0 \to N \xrightarrow{.x} N \to N/xN \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^i_{I,J}(M,N) \xrightarrow{.x} H^i_{I,J}(M,N) \to H^i_{I,J}(M,N/xN) \to \cdots .$$

By the hypothesis, $H^i_{I,J}(M,N/xN)$ is finitely generated for all $i \geq t$. Since N/xN is finitely generated and $\dim N/xN = n-1$, it follows by the inductive hypothesis that $H^i_{I,J}(M,N/xN) = 0$ for all $i \geq t$. Therefore

$$H^i_{I,J}(M,N) = xH^i_{I,J}(M,N)$$

for all $i \geq t$. By Nakayama's Lemma $H^i_{I,J}(M,N) = 0$ for all $i \geq t$, as $x \in \mathfrak{m}$. The proof is complete. □

The generalized local cohomological dimension of M, N with respect to (I,J) is defined as follows

$$\operatorname{cd}(I,J,M,N) = \sup\{n \mid H^n_{I,J}(M,N) \neq 0\}.$$

We have an immediate consequence.

Corollary 2.10. *Let $(R,\mathfrak{m})$ be a local ring and M, N two finitely generated R-modules with $\operatorname{pd} M < \infty$. If $\operatorname{cd}(I,J,M,N) > \operatorname{pd} M$, then $H^{\operatorname{cd}(I,J,M,N)}_{I,J}(M,N)$ is not finitely generated.*

Let $\mathfrak{a} \in \widetilde{W}(I,J)$, the following theorem discusses the finiteness of $H^t_{I,J}(M,N)/\mathfrak{a}H^t_{I,J}(M,N)$.

Theorem 2.11. *Let M, N be two finitely generated R-modules with $\operatorname{pd} M < \infty$, $\dim N < \infty$ and t an integer such that $t > \operatorname{pd} M$. If $H^i_{I,J}(M,N)$ is finitely generated for all $i > t$, then $H^t_{I,J}(M,N)/\mathfrak{a}H^t_{I,J}(M,N)$ is finitely generated for all $\mathfrak{a} \in \widetilde{W}(I,J)$.*

Proof. We use induction on $n = \dim N$. Note that $H^i_{I,J}(M,N) = 0$ for all $i > \operatorname{pd} M + \dim N$. Then the statement is true when $n = 0$. Let $n > 0$, the short exact sequence

$$0 \to \Gamma_{I,J}(N) \to N \to N/\Gamma_{I,J}(N) \to 0$$

induces a long exact sequence

$$\cdots \to H^i_{I,J}(M,\Gamma_{I,J}(N)) \to H^i_{I,J}(M,N) \to H^i_{I,J}(M,N/\Gamma_{I,J}(N)) \to \cdots.$$

From [8, 2.6] we have $H^i_{I,J}(M,\Gamma_{I,J}(N)) \cong \operatorname{Ext}^i_R(M,\Gamma_{I,J}(N))$ for all $i \geq 0$. By the assumption $H^i_{I,J}(M,N/\Gamma_{I,J}(N)) \cong H^i_{I,J}(M,N)$ for all $i > t$. Thus we may assume, by replacing N with $N/\Gamma_{I,J}(N)$, that N is (I,J)-torsion-free. Let $\mathfrak{a} \in \widetilde{W}(I,J)$, N is $\mathfrak{a}$-torsion-free, so there is an N-regular element $x \in \mathfrak{a}$. Now the short exact sequence

$$0 \to N \overset{\cdot x}{\to} N \to N/xN \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^t_{I,J}(M,N) \overset{\cdot x}{\to} H^t_{I,J}(M,N) \to H^t_{I,J}(M,N/xN) \to \cdots.$$

By the hypothesis, $H^i_{I,J}(M,N/xN)$ is finitely generated for all $i > t$. Since N/xN is finitely generated and $\dim N/xN = n-1$, it follows from the inductive hypothesis that $H^t_{I,J}(M,N/xN)/\mathfrak{a}H^t_{I,J}(M,N/xN)$ is finitely generated. Now, the exact sequence

$$\cdots \to H^t_{I,J}(M,N) \overset{\cdot x}{\to} H^t_{I,J}(M,N) \overset{f}{\to} H^t_{I,J}(M,N/xN) \overset{g}{\to} H^{t+1}_{I,J}(M,N) \to \cdots$$

induces two exact sequences

$$0 \to \operatorname{Im} f \to H^t_{I,J}(M, N/xN) \to \operatorname{Im} g \to 0$$

and

$$H^t_{I,J}(M,N) \xrightarrow{.x} H^t_{I,J}(M,N) \to \operatorname{Im} f \to 0.$$

Then we get the following exact sequences

$$\begin{aligned}\cdots \to \operatorname{Tor}^R_1(R/\mathfrak{a}, \operatorname{Im} g) \to \operatorname{Im} f/\mathfrak{a} \operatorname{Im} f \\ \to H^t_{I,J}(M, N/xN)/\mathfrak{a}H^t_{I,J}(M, N/xN) \to \operatorname{Im} g/\mathfrak{a} \operatorname{Im} g \to 0\end{aligned}$$

and

$$H^t_{I,J}(M,N)/\mathfrak{a}H^t_{I,J}(M,N) \xrightarrow{.x} H^t_{I,J}(M,N)/\mathfrak{a}H^t_{I,J}(M,N) \to \operatorname{Im} f/\mathfrak{a} \operatorname{Im} f \to 0.$$

Note that $\operatorname{Tor}^R_1(R/\mathfrak{a}, \operatorname{Im} g)$ is finitely generated, so is $\operatorname{Im} f/\mathfrak{a} \operatorname{Im} f$. As $x \in \mathfrak{a}$, we conclude that

$$H^t_{I,J}(M,N)/\mathfrak{a}H^t_{I,J}(M,N) \cong \operatorname{Im} f/\mathfrak{a} \operatorname{Im} f,$$

which completes the proof. □

Combining Theorems 2.9 with 2.11 we get an immediate consequence.

Corollary 2.12. *Let $(R, \mathfrak{m})$ be a local ring and M, N two finitely generated R-modules with $\operatorname{pd} M < \infty$. If $\operatorname{cd}(I,J,M,N) > \operatorname{pd} M$, then $H^{\operatorname{cd}(I,J,M,N)}_{I,J}(M,N)/\mathfrak{a}H^{\operatorname{cd}(I,J,M,N)}_{I,J}(M,N)$ is finitely generated for all $\mathfrak{a} \in \widetilde{W}(I,J)$.*

Denote by $\operatorname{grade}(\mathfrak{a} + \operatorname{Ann}(M), N) = \inf\{i \mid H^i_{\mathfrak{a}}(M,N) \neq 0\}$. We have the following last result.

Theorem 2.13. *Let M be a finitely generated R-module, N an R-module. Then*

$$\inf\{i \mid H^i_{I,J}(M,N) \neq 0\} = \inf\{\operatorname{grade}(\mathfrak{a} + \operatorname{Ann}(M), N) \mid \mathfrak{a} \in \widetilde{W}(I,J)\}.$$

Moreover, $\operatorname{Ass}_R(H^t_{I,J}(M,N)) \cap V(\mathfrak{a}) = \operatorname{Ass}_R(H^t_{\mathfrak{a}}(M,N))$ for some $\mathfrak{a} \in \widetilde{W}(I,J)$, where $t = \inf\{i \mid H^i_{I,J}(M,N) \neq 0\}$.

Proof. Set $k = \inf\{\operatorname{grade}(\mathfrak{a}+\operatorname{Ann}(M), N) \mid \mathfrak{a} \in \widetilde{W}(I,J)\}$ and $t = \inf\{i \mid H^i_{I,J}(M,N) \neq 0\}$. If $H^i_{\mathfrak{a}}(M,N) = 0$ for all $\mathfrak{a} \in \widetilde{W}(I,J)$, then $H^i_{I,J}(M,N) = 0$ by [16, 2.2]. It is clear that $t \geq k$. We now assume that there are some $\mathfrak{a} \in \widetilde{W}(I,J)$ such that $H^k_{\mathfrak{a}}(M,N) \neq 0$. We will prove that $H^k_{I,J}(M,N) \neq 0$. By [10, 10.47] there is a Grothendieck spectral sequence

$$E^{p,q}_2 = H^p_{\mathfrak{a}}(H^q_{I,J}(M,N)) \underset{p}{\Rightarrow} H^{p+q}_{\mathfrak{a}}(M,N).$$

We consider the filtration Φ of submodules of $H^k = H^k_{\mathfrak{a}}(M,N)$

$$0 = \Phi^{k+1}H^k \subseteq \Phi^k H^k \subseteq \cdots \subseteq \Phi^1 H^k \subseteq \Phi^0 H^k = H^k$$

such that

$$E_\infty^{i,k-i} \cong \Phi^i H^k / \Phi^{i+1} H^k$$

for all $i \le k$. Note that $H^i_{I,J}(M,N) = 0$ for all $i < k$. Since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$, it follows that $E_\infty^{p,q} = 0$ for all $q < k$, $p \ge 0$. Therefore $\Phi^1 H^k = \Phi^2 H^k = \cdots = \Phi^{k+1} H^k = 0$ and $E_\infty^{0,k} \cong \Phi^0 H^k/\Phi^1 H^k = \Phi^0 H^k = H^k_{\mathfrak{a}}(M,N) \neq 0$. The homomorphisms of spectral sequence

$$0 = E_r^{-r,k+r-1} \xrightarrow{d_r^{-r,k+r-1}} E_r^{0,k} \xrightarrow{d_r^{0,k}} E_r^{r,k-r+1} = 0$$

implies $E_2^{0,k} = E_3^{0,k} = \cdots = E_\infty^{0,k}$. It follows that

$$E_2^{0,k} = \Gamma_{\mathfrak{a}}(H^k_{I,J}(M,N)) \cong H^k_{\mathfrak{a}}(M,N) \neq 0.$$

Consequently, we conclude that $H^k_{I,J}(M,N) \neq 0$. Moreover

$$\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M,N)) = \mathrm{Ass}_R(\Gamma_{\mathfrak{a}}(H^t_{I,J}(M,N))) = V(\mathfrak{a}) \cap \mathrm{Ass}_R(H^t_{I,J}(M,N))$$

as required. □

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Tran Tuan Nam
Department of Mathematics-Informatics, Ho Chi Minh University of Pedagogy, Ho Chi Minh city, Vietnam
E-mail address: namtuantran@gmail.com

Nguyen Minh Tri
Department of Natural Science Education, Dong Nai University, Dong Nai, Vietnam
and

Faculty of Mathematics & Computer Science, University of Science, VNU-HCM, Ho Chi Minh City, Vietnam
E-mail address: triminhng@gmail.com