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Pack Graphs with Subgraphs of Size Three

Zhen-Chun Chen, Hung-Lin Fu and Kuo-Ching Huang*

Abstract. An H-packing $\mathcal{F}$ of a graph G is a set of edge-disjoint subgraphs of G in which each subgraph is isomorphic to H. The leave L or the remainder graph L of a packing $\mathcal{F}$ is the subgraph induced by the set of edges of G that does not occur in any subgraph of the packing $\mathcal{F}$. If a leave L contains no edges, or simply $L = \phi$, then G is said to be H-decomposable, denoted by $H \mid G$. In this paper, we prove a conjecture made by Chartrand, Saba and Mynhardt [13]: *If G is a graph of size $q(G) \equiv 0 \pmod 3$ and $\delta(G) \geq 2$, then G is H-decomposable for some graph H of size* 3.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, simple and undirected graph. The *order*, *size*, *maximum* and *minimum degree* of G are denoted by $p(G)$, $q(G)$, $\Delta(G)$ and $\delta(G)$, respectively. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v. The graphs P_n and C_k are a path of order n and a cycle of order $k \geq 3$, respectively. The graph $G_1 \cup G_2$ is the edge disjoint union of G_1 and G_2. The graph tH is the union of t copies of H. For more graph theoretic terminologies we refer to [11].

A graph G is said to be *H-decomposable*, denoted by $H \mid G$, if the edge set $E(G)$ of G can be partitioned into subsets such that the edge-induced subgraph of each subset is isomorphic to H. Graph decomposition is one of the most important topics in the study of both graph theory and combinatorial designs, not to mention their applications on many other fields. Quite a few research results are obtained in considering the decomposition of complete graphs or complete multipartite graphs into complete subgraphs or cycles. See [1–6, 8–10, 18–22, 25–31, 33–35] for references. Decomposition problems of a general graphs could be more complicated, as a result of the failure of the tools and methods used on decomposition of well-structured graphs. On the other hand, if we consider the decomposition, packing or covering of a general graph, it is getting more complicate.

In [13], Chartrand, Saba and Mynhardt study prime graphs and proposed the following:

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*Corresponding author.

Conjecture 1.1. [13] *Suppose G is a graph of size $q(G) \equiv 0 \pmod 3$ and $\delta(G) \geq 2$. Then G is H-decomposable for some graph H of size* 3.

Conjecture 1.2. [13] *Suppose G is a 2-connected graph of order $p(G) \geq 4$ and of size $q(G) \equiv 0 \pmod 3$. Then G is P_4-decomposable.*

These conjectures motivate our study of decomposing a graph of size $3k$ into k copies of isomorphic graphs of size 3. If $q(H) = 3$, then $H = K_3$, P_4, $K_{1,3}$, $P_3 \cup P_2$ or M_3 (a matching of size 3). There are many research results of decomposing graphs into subgraphs of size three. See [7, 12, 14–17, 23, 24, 32]. For convenience, we use $x_1x_2 \cdots x_t$ and $x_1x_2 \cdots x_tx_1$, respectively, to denote a path and a cycle of order t. Since the graph $D = \{x_1x_2x_3x_4x_5x_6x_1\} \cup \{x_1y_1x_2, x_3y_2x_4, x_5y_3x_6\}$ disproves the Conjecture 1.2, we will focus on the Conjecture 1.1. In order to prove the Conjecture 1.1, for each given graph G such that $q(G) \equiv 0 \pmod 3$, we have to find a graph H of size 3 and prove that $H \mid G$. It is not difficult to see that $G \mid G$ if $q(G) = 3$ and the complete graph K_4 is P_4-decomposable. Moreover, the complete bipartite graph $K_{2,3}$ is P_4-decomposable and $(P_3 \cup P_2)$-decomposable and the complete 3-partite graph $K_{1,1,4}$ is P_4-decomposable. Since the graph $K_{1,1,3c+1} = K_{1,1,4} \cup (c-1)K_{2,3}$, we have $P_4 \mid K_{1,1,3c+1}$, $c \geq 1$. In this paper, we prove the following to confirm the Conjecture 1.1.

Theorem 1.3. *If G is a graph of size $6 \leq q(G) \equiv 0 \pmod 3$ and $\delta(G) \geq 2$, then G is $(P_3 \cup P_2)$-decomposable if and only if G is different from K_4 and $K_{1,1,3c+1}$, $c \geq 0$.*

2. Main results

We start this section with the study of $(P_3 \cup P_2)$-packings of graphs. An *H-packing* of a graph G is a set of edge-disjoint subgraphs of G in which each subgraph is isomorphic to H. An H-packing $\mathcal{F}$ is *maximum* if $|\mathcal{F}| \geq |\mathcal{F}'|$ for all other H-packings $\mathcal{F}'$ of G. The *leave* L of an H-packing $\mathcal{F}$ is the subgraph induced by the set of edges of G that does not occur in any subgraph of the H-packing $\mathcal{F}$. Therefore, a maximum packing has a minimum leave. In what follows, all the leaves we consider are minimum. It is easy to see that $H \mid G$ if and only if G has an H-packing with empty leave L, that is, L contains no edge, or simply $L = \phi$.

The following lemmas are essential for proving the main theorem. Since they are easy to be proved, we omit the proofs.

Lemma 2.1. *If $G \cong G_i$, $1 \leq i \leq 18$, given in Figure 2.1, then $P_3 \cup P_2 \mid G$.*

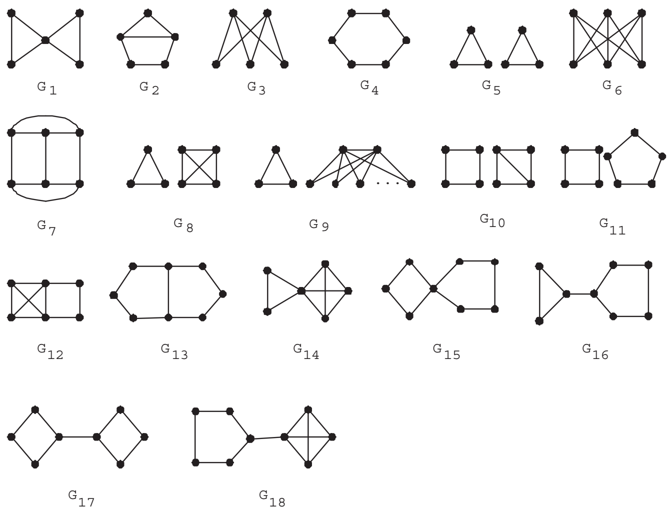

Figure 2.1

Lemma 2.2. *If* $G \cong G_i$, $19 \leq i \leq 26$, *given in Figure* 2.2, *then* G *has a* $(P_3 \cup P_2)$*-packing with a* P_2 *as the leave.*

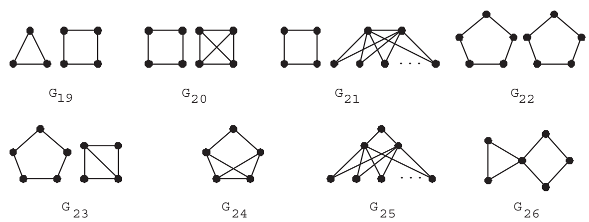

Figure 2.2

Lemma 2.3. *If* $G \cong G_i$, $27 \leq i \leq 40$, *given in Figure* 2.3, *then* G *has a* $(P_3 \cup P_2)$*-packing with a* P_3 *as the leave.*

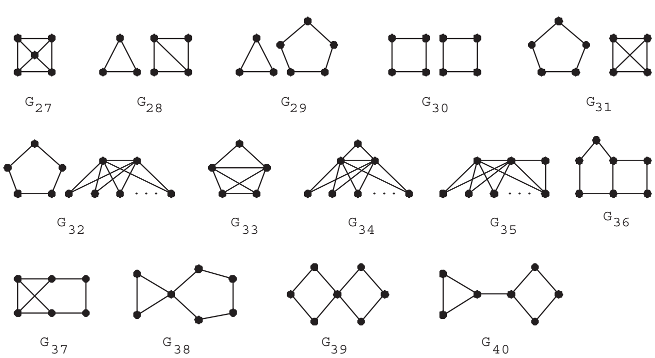

Figure 2.3

The followings are our main results.

Lemma 2.4. *Suppose G is a connected 3-regular graph of order $p(G) \geq 8$. Then there is an edge $xy \in E(G)$ with $N(x) = \{y, a, b\}$, $N(y) = \{x, c, d\}$, $ac \notin E(G)$ and $bd \notin E(G)$ such that the graph $G' = (G - \{x, y\}) \cup \{ac, bd\}$ is a connected 3-regular graph of order $p(G') = p(G) - 2$.*

Proof. If G has a cut vertex, since G is 3-regular, G has a cut edge xy such that $G - \{xy\} = H_1 \cup H_2$, where H_1 is a block containing x and H_2 is connected containing y. Let $N(x) = \{y, a, b\}$ and $N(y) = \{x, c, d\}$. Since H_1 is a block, $H_1 - x$ is connected. Hence, a and b are connected in $H_1 - x$ and then the graph $G' = (G - \{x, y\}) \cup \{ac, bd\}$ is a connected 3-regular graph of order $p(G') = p(G) - 2$.

Let G be 2-connected. Suppose there is an edge $xy \in E(G)$ such that $\{x, y\}$ is a cut set. Then $G - \{x, y\}$ contains exact two components H_1 and H_2. Otherwise, there a component H_3 of $G - \{x, y\}$ such that $N(x) \cap V(H_3) = \phi$. Then y is a cut vertex, a contradiction. Moreover, $|N(x) \cap V(H_i)| = |N(y) \cap V(H_i)| = 1$ for $i = 1, 2$. Let $N(x) = \{y, a, b\}$ and $N(y) = \{x, c, d\}$ such that a and c are in H_1 and b and d are in H_2. If a and c are coincide, then a is a cut vertex, a contradiction. Hence, $a \neq c$. Similarly, $b \neq d$. Since H_1 and H_2 are components, the graph $G' = (G - \{x, y\}) \cup \{ad, bc\}$ is a connected 3-regular graph of order $p(G') = p(G) - 2$.

Suppose $G - \{u, v\}$ is connected for every edge $uv \in E(G)$. Choose an edge $xy \in E(G)$ with $N(x) = \{y, a, b\}$ and $N(y) = \{x, c, d\}$. If $\{a, b\} = \{c, d\}$, then $ab \notin E(G)$. Otherwise, $G = K_4$. Let $N(a) = \{x, y, z\}$ and $N(z) = \{a, u, v\}$. If $b \in N(z)$, then z is a cut vertex, a contradiction. Hence, $b \notin N(z)$ and then $N(x) \cap \{u, v\} = N(y) \cap \{u, v\} = \phi$. Thus, the graph $G' = (G - \{a, z\}) \cup \{xu, yv\}$ is a connected 3-regular graph of order $p(G') = p(G) - 2$. Suppose $|\{a, b\} \cap \{c, d\}| = 1$, say $a = c$. If $ab \in E(G)$ (similarly if $ad \in E(G)$), then $N(a) = \{x, y, b\}$. Let $N(b) = \{x, a, z\}$. If $z = d$, then d is a cut vertex, a contradiction. Hence, $z \neq d$. Let $N(z) = \{b, u, v\}$. Then the graph $G' = (G - \{b, z\}) \cup \{xu, av\}$ is a connected 3-regular graph of order $p(G') = p(G) - 2$. Suppose $N(a) \cap \{b, d\} = \phi$. Let $N(a) = \{x, y, z\}$ and $N(z) = \{a, u, v\}$. If $\{u, v\} = \{b, d\}$, then the graph $G' = (G - \{a, z\}) \cup \{xd, yb\}$ is a connected 3-regular graph of order $p(G') = p(G) - 2$. If $|\{u, v\} \cap \{b, d\}| = 1$, say $b = u$, then the graph $G' = (G - \{a, z\}) \cup \{xv, yb\}$ is a connected 3-regular graph of order $p(G') = p(G) - 2$. If $\{u, v\} \cap \{b, d\} = \phi$, then the graph $G' = (G - \{a, z\}) \cup \{xu, yv\}$ is a connected 3-regular graph of order $p(G') = p(G) - 2$. Suppose $\{a, b\} \cap \{c, d\} = \phi$. If $|N(a) \cap \{c, d\}| = 2$ (similarly if $N(b) = \{x, c, d\}$, $N(c) = \{y, a, b\}$ or $N(d) = \{y, a, b\}$), then $|N(b) \cap \{c, d\}| \leq 1$. Otherwise, $G = K_{3,3}$ and $p(G) = 6$, a contradiction. We may assume that $bd \notin E(G)$. Let $N(d) = \{a, y, z\}$ and $N(z) = \{d, u, v\}$. If $z = c$, then x is a cut vertex, a contradiction. Hence, $z \neq c$. Since $N(a) = \{x, c, d\}$ and $N(y) = \{x, c, d\}$, $\{a, y\} \cap \{u, v\} = \phi$ and then the graph $G' = (G - \{d, z\}) \cup \{au, yv\}$ is a connected 3-

regular graph of order $p(G') = p(G) - 2$. Suppose $|N(a) \cap \{c, d\}| \le 1$, $|N(b) \cap \{c, d\}| \le 1$, $|N(c) \cap \{a, b\}| \le 1$ and $|N(d) \cap \{a, b\}| \le 1$. If $ac \in E(G)$ or $bd \in E(G)$, then $ad \notin E(G)$ and $bc \notin E(G)$. If $ad \in E(G)$ or $bc \in E(G)$, then $ac \notin E(G)$ and $bd \notin E(G)$. We may assume $ac \notin E(G)$ and $bd \notin E(G)$. Then the graph $G' = (G - \{x, y\}) \cup \{ac, bd\}$ is a connected 3-regular graph of order $p(G') = p(G) - 2$. □

Theorem 2.5. *Suppose G is a graph different from $K_{1,1,3c+1}$ with $p(G) \ge 5$, $q(G) \ge 6$ and $\delta(G) \ge 2$. Then G has a $(P_3 \cup P_2)$-packing with leave L, where*

$$L = \begin{cases} \phi & \text{if } q(G) \equiv 0 \pmod 3, \\ P_2 & \text{if } q(G) \equiv 1 \pmod 3, \\ P_3 & \text{if } q(G) \equiv 2 \pmod 3). \end{cases}$$

Proof. If $q(G) = 6$, then $G = G_i$, $1 \le i \le 5$, given in Figure 2.1 By Lemma 2.1, we have $P_3 \cup P_2 | G$.

Let G be a counterexample with fewest edges. We shall prove that the assertion holds for G and obtain a contradiction. There are three cases to be considered.

Case 1: $\Delta(G) \ge 4$ and $\delta(G) \ge 3$.

By degree-sum formula, $q(G) = \frac{1}{2}\sum_{x \in V(G)} d(x) \ge \frac{1}{2}(4 + 3 \times 4) = 8$. If $q(G) = 8$, then $G = G_{27}$. By Lemma 2.3, G has a $(P_3 \cup P_2)$-packing with a P_3 as the leave.

Now, suppose $q(G) > 8$. Let v be a vertex with $d(v) = \Delta(G)$ and $N(v) = \{v_1, v_2, \ldots, v_{\Delta(G)}\}$. If v_1 is adjacent to some v_i for $i \ge 2$, say $v_1v_2 \in E(G)$, let $F_1 = \{v_3vv_4, v_1v_2\}$ and $G' = G - F_1$; otherwise, let u be a neighbor of v_1 which is different from v and $G' = G - F_2$, where $F_2 = \{v_2vv_3, v_1u\}$. Then the assertion holds for G' by the choice of G. Since $G = G' \cup (P_3 \cup P_2)$, the assertion holds for the graph G.

Case 2: G is 3-regular.

Suppose G is connected. If $p(G) = 6$, then $G = G_6$ or G_7. By Lemma 2.1, $P_3 \cup P_2 \mid G$. For $p(G) \ge 8$, by Lemma 2.4, G has an edge xy with $N(x) = \{x_1, x_2, y\}$, $N(y) = \{y_1, y_2, x\}$, $N(x) \cap N(y) = \phi$, $x_1y_1 \notin E(G)$ and $x_2y_2 \notin E(G)$ such that $G' = (G - \{x, y\}) \cup \{x_1y_1, x_2y_2\}$ is a connected 3-regular graph of order $p(G) - 2$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Without loss of generality, we may consider the following cases.

(1) If there is an $F = \{x_1y_1v_1, x_2y_2\}$ in $\mathcal{F}$, then G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F\}) \cup \{x_1xx_2, yy_1\} \cup \{xyy_2, y_1v_1\}$ with empty leave.

(2) If there are $F_1 = \{v_1v_2v_3, x_1y_1\}$ and $F_2 = \{u_1u_2u_3, x_2y_2\}$ in $\mathcal{F}$, then G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{x_1xx_2, yy_1\} \cup \{v_1v_2v_3, xy\} \cup \{u_1u_2u_3, yy_2\}$ with empty leave.

(3) If there are $F_1 = \{v_1v_2v_3, x_1y_1\}$ and $F_2 = \{x_2y_2u_1, u_2u_3\}$ in $\mathcal{F}$, then G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{x_1xx_2, yy_1\} \cup \{v_1v_2v_3, xy\} \cup \{yy_2u_1, u_2u_3\}$ with empty

leave.

(4) Suppose there are $F_1 = \{x_1y_1v_1, v_2v_3\}$ and $F_2 = \{x_2y_2u_1, u_2u_3\}$ (or $F_2 = \{y_2x_2u_1, u_2u_3\}$) in $\mathcal{F}$. If $x_1 \notin \{u_2, u_3\}$, then G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{x_1xy, u_2u_3\} \cup \{yy_1v_1, v_2v_3\} \cup \{yy_2u_1, xx_2\}$ (or $\{xx_2u_1, yy_2\}$) with empty leave. If $x_1 = u_2$ or u_3 (say $x_1 = u_2$) and $u_3 \neq v_1$, then G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{xx_1u_3, y_1v_1\} \cup \{xyy_1, v_2v_3\} \cup \{yy_2u_1, xx_2\}$ (or $\{xx_2u_1, yy_2\}$) with empty leave. If $x_1 = u_2$ and $u_3 = v_1$, then G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F_1, F_2\}) \cup \{x_1xy, y_2u_1$ (or $x_2u_1)\} \cup \{x_1v_1y_1, v_2v_3\} \cup \{y_1yy_2, xx_2\}$ with empty leave. Hence, we have $P_3 \cup P_2 \mid G$ for any connected 3-regular graph G.

If G is disconnected, let $G = (mK_4) \cup H_1 \cup \cdots \cup H_n$ such that each H_i is different from K_4 and a connected 3-regular component, where $m \geq 0$ and $1 \leq i \leq n$. Since $P_3 \cup P_2 \mid H_i$ by the choice of G, $G - mK_4$ has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. If $m = 1$, choose an F in $\mathcal{F}$. Since $K_4 = 3P_3$ and $F = 3P_2$, $K_4 \cup F = 3(P_3 \cup P_2)$. Hence, $P_3 \cup P_2 \mid G$. If $m \neq 1$, then $G = \frac{m}{2}(2K_4) \cup H_1 \cup \cdots \cup H_n$ when m is even and $G = \frac{m-3}{2}(2K_4) \cup (3K_4) \cup H_1 \cup \cdots \cup H_n$ when m is odd. Since $K_4 = 2P_3 \cup 2P_2$, it is not difficult to see that $P_3 \cup P_2 \mid (tK_4)$ for $t = 2$ or 3. Hence, $P_3 \cup P_2 \mid (mK_4)$ for $m \geq 2$ and then $P_3 \cup P_2 \mid G$.

Case 3: $\delta(G) = 2$.

Suppose G has a cycle-component. Let $C_n = x_1x_2 \cdots x_nx_1$ be the minimum cycle-component. If $3 \leq n \leq 5$, let $G' = G - C_n$. Suppose $n = 3$ and $C_n = x_1x_2x_3x_1$. If $G = G_8$, G_9, G_{19}, G_{28} or G_{29}, by Lemmas 2.1, 2.2 and 2.3, the assertion holds for these graphs G. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave L. Choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$. Hence, G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F\}) \cup \{x_1x_2x_3, v_4v_5\} \cup \{v_1v_2v_3, x_1x_3\}$ with leave L.

Suppose $n = 4$ and $C_n = x_1x_2x_3x_4x_1$. If $G = G_{10}$, G_{11}, G_{20}, G_{21} or G_{30}, by Lemmas 2.1, 2.2 and 2.3, the assertion holds for these graphs G. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave L. For $L = \phi$, choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$. Then G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F\}) \cup \{x_1x_2x_3, v_4v_5\} \cup \{v_1v_2v_3, x_3x_4\}$ with leave x_1x_4. For $L = v_1v_2$, G has a $(P_3 \cup P_2)$-packing $\mathcal{F} \cup \{x_1x_2x_3, v_1v_2\}$ with leave $x_3x_4x_1$. For $L = v_1v_2v_3$, G has a $(P_3 \cup P_2)$-packing $\mathcal{F} \cup \{x_1x_2x_3, v_1v_2\} \cup \{x_3x_4x_1, v_2v_3\}$ with empty leave.

Suppose $n = 5$ and $C_n = x_1x_2x_3x_4x_5x_1$. If $G = G_{22}$, G_{23}, G_{31} or G_{32}, by Lemmas 2.2 and 2.3, the assertion holds for these graphs G. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave L. Choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$. For $L = \phi$, G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F\}) \cup \{x_1x_2x_3, v_4v_5\} \cup \{v_1v_2v_3, x_3x_4\}$ with leave $x_4x_5x_1$. For $L = u_1u_2$, G has a $(P_3 \cup P_2)$-packing $(\mathcal{F} - \{F\}) \cup \{x_1x_2x_3, v_4v_5\} \cup \{x_3x_4x_5, u_1u_2\} \cup \{v_1v_2v_3, x_1x_5\}$ with empty leave. For $L = u_1u_2u_3$, G has a $(P_3 \cup P_2)$-packing $\mathcal{F} \cup \{x_1x_2x_3, u_1u_2\} \cup$

$\{x_3x_4x_5, u_2u_3\}$ with leave x_1x_5.

For $n \geq 6$, let $C_n = x_1x_2 \cdots x_nx_1$. If $q(G) \equiv 0 \pmod 3$, let $G' = (G - \{x_2, x_3, x_4\}) \cup \{x_1x_5\}$. Then $q(G') = q(G) - 3 \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_1x_5 \in F$. Since $F = \{x_1x_5x_6, v_4v_5\}$, $\{x_nx_1x_5, v_4v_5\}$ or $\{v_1v_2v_3, x_1x_5\}$, $(F - \{x_1x_5\}) \cup \{x_1x_2x_3x_4x_5\}$ $(= P_6 \cup \{v_4v_5\}$ or $P_5 \cup \{v_1v_2v_3\}) = 2(P_3 \cup P_2)$. Hence, G has a $(P_3 \cup P_2)$-packing with empty leave.

If $q(G) \equiv 1 \pmod 3$, let $G' = (G - x_2) \cup \{x_1x_3\}$. Then $q(G') = q(G) - 1 \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ such that $x_1x_3 \in F$. Since $F = \{x_1x_3x_4, v_4v_5\}$, $\{x_nx_1x_3, v_4v_5\}$ or $\{v_1v_2v_3, x_1x_3\}$, $(F - \{x_1x_3\}) \cup \{x_1x_2x_3\}$ $(= P_4 \cup \{v_4v_5\}$ or $P_3 \cup \{v_1v_2v_3\}) = (P_3 \cup P_2) \cup \{L\}$, where $L = x_1x_2$ or x_2x_3. Hence, G has a $(P_3 \cup P_2)$-packing with leave L.

If $q(G) \equiv 2 \pmod 3$, let $G' = (G - \{x_2, x_3\}) \cup \{x_1x_4\}$. Then $q(G') = q(G) - 2 \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ such that $x_1x_4 \in F$. Since $F = \{x_1x_4x_5, v_4v_5\}$, $\{x_nx_1x_4, v_4v_5\}$ or $\{v_1v_2v_3, x_1x_4\}$, $(F - \{x_1x_4\}) \cup \{x_1x_2x_3x_4\}$ $(= P_5 \cup \{v_4v_5\}$ or $P_4 \cup \{v_1v_2v_3\}) = (P_3 \cup P_2) \cup \{L\}$, where $L = x_1x_2x_3$ or $x_2x_3x_4$. Hence, G has a $(P_3 \cup P_2)$-packing with leave L.

Suppose G has no cycle-component. Since $\delta(G) = 2$, there is a path $x_0x_1x_2 \cdots x_t$ (not necessary open), called 2-path, in G with $d(x_0) \geq 3$, $d(x_t) \geq 3$ and $d(x_i) = 2$ for $1 \leq i < t$, where $t \geq 2$. We may choose a 2-path such that t is as small as possible. Note that if $t \geq 3$, then $G_1 = G - \{x_1, x_2, \cdots, x_{t-1}\}$, $G_2 = (G - \{x_1, x_2, \cdots, x_{t-1}\}) \cup \{x_0x_t\}$ and $G_3 = (G - \{x_1, x_2, \cdots, x_{t-2}\}) \cup \{x_0x_{t-1}\}$ are all different from $K_{1,1,3c+1}$, since $K_{1,1,3c+1}$ has a 2-path with $t = 2$. Consider the following cases.

(1) $x_0x_t \in E(G)$.

Suppose $q(G) \equiv 2 \pmod 3$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 0 \pmod 3$. If $G = G_{33}$, G_{34} or G_{35}, by Lemma 2.3, G has a $(P_3 \cup P_2)$-packing with a P_3 as the leave. Otherwise, by the choice of G, $P_3 \cup P_2 \mid G'$. Hence, G has a $(P_3 \cup P_2)$-packing with leave $x_0x_1x_2$.

If $t = 3$, let $G' = G - \{x_1, x_2\}$. Then $q(G') \equiv 2 \pmod 3$. If $G = G_{36}$, by Lemma 2.3, G has a $(P_3 \cup P_2)$-packing with a P_3 as the leave. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with $L = v_1v_2v_3$ as the leave. If $x_0x_3 = v_1v_2$ or v_2v_3, then $\{L\} \cup \{x_0x_1x_2x_3\} = (P_3 \cup P_2) \cup \{L'\}$, where $L' = x_0x_3x_2$ or $x_1x_0x_3$. If $\{x_0, x_3\} \cap \{v_1, v_2, v_3\} = \phi$ or $\{v_2\}$, then $\{L\} \cup \{x_0x_1x_2x_3\} = (P_3 \cup P_2) \cup \{L'\}$, where $L' = x_0x_1x_2$ or $x_1x_2x_3$. If $\{x_0, x_3\} \cap \{v_1, v_2, v_3\} = \{v_1\}$ or $\{v_3\}$, then $\{L\} \cup \{x_0x_1x_2x_3\} = P_6 = (P_3 \cup P_2) \cup \{L'\}$, where $L' = x_0x_1x_2$ or $x_1x_2x_3$. Suppose $\{x_0, x_3\} = \{v_1, v_3\}$. Choose an F in $\mathcal{F}$ with $x_0x_3 \in F$. Then $F = \{x_0x_3u_3, u_4u_5\}$, $\{x_3x_0u_3, u_4u_5\}$ or $\{u_1u_2u_3, x_0x_3\}$. If $F = \{x_0x_3u_3, u_4u_5\}$, then $\{L\} \cup F \cup \{x_0x_1x_2x_3\} = \{x_0v_2x_3, x_1x_2\} \cup \{x_2x_3u_3, u_4u_5\} \cup \{x_1x_0x_3\}$. If $F = \{x_3x_0u_3, u_4u_5\}$, then $\{L\} \cup F \cup \{x_0x_1x_2x_3\} = \{x_0v_2x_3, x_1x_2\} \cup \{x_1x_0u_3, u_4u_5\} \cup \{x_2x_3x_0\}$. If $F = \{u_1u_2u_3, x_0x_3\}$, then $\{L\} \cup F \cup \{x_0x_1x_2x_3\} = \{x_0x_3v_2, x_1x_2\} \cup \{x_1x_0v_2, x_2x_3\} \cup \{u_1u_2u_3\}$. Hence, G has a $(P_3 \cup P_2)$-packing with a P_3 as the leave.

If $t \geq 4$, let $G' = (G - \{x_1, x_2\}) \cup \{x_0x_3\}$. Then $q(G') \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_3 \in F$. Then $F = \{x_0x_3x_4, v_4v_5\}$, $\{x_3x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0x_3\}$. Hence, $(F - \{x_0x_3\}) \cup \{x_0x_1x_2x_3\}$ $(= P_5 \cup \{v_4v_5\}$ or $P_4 \cup \{v_1v_2v_3\}) = (P_3 \cup P_2) \cup \{L\}$, where $L = x_0x_1x_2$ or $x_1x_2x_3$. Hence, G has a $(P_3 \cup P_2)$-packing with leave L.

Suppose $q(G) \equiv 1 \pmod 3$. Let $G' = G - \{x_0x_t\}$. Then $q(G') \equiv 0 \pmod 3$. Since x_1 is of degree two in G' and $x_0x_t \notin E(G')$, G' is neither K_4 nor $K_{1,1,3c+1}$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing with empty leave. Hence, G has a $(P_3 \cup P_2)$-packing with leave x_0x_t.

Suppose $q(G) \equiv 0 \pmod 3$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 1 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with an edge e as the leave. If $\{x_0x_1x_2, e\}$ forms a $P_3 \cup P_2$, then $P_3 \cup P_2 \mid G$. If $e = x_0z$, $z \neq x_2$ (similarly if $e = x_2z$, $z \neq x_0$), choose an F in $\mathcal{F}$ with $x_0x_2 \in F$. Then $F = \{x_0x_2v_3, v_4v_5\}$, $\{x_2x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0x_2\}$. If $F = \{x_2x_0v_3, v_4v_5\}$, then $F \cup \{zx_0x_1x_2\} = \{x_1x_0x_2, v_4v_5\} \cup \{zx_0v_3, x_1x_2\}$. Suppose $F = \{v_1v_2v_3, x_0x_2\}$. If $z = v_2$, then $F \cup \{zx_0x_1x_2\} = \{x_1x_0x_2, v_1v_2\} \cup \{x_0zv_3, x_1x_2\}$. If $z = v_1$ (similarly if $z = v_3$), then $F \cup \{zx_0x_1x_2\} = \{x_1x_0x_2, v_2v_3\} \cup \{x_0zv_2, x_1x_2\}$. If $z \neq v_i$, $i = 1, 2, 3$, then $F \cup \{zx_0x_1x_2\} = \{x_0x_1x_2, v_1v_2\} \cup \{zx_0x_2, v_2v_3\}$. Suppose $F = \{x_0x_2v_3, v_4v_5\}$. If $z \neq v_3$, then $F \cup \{zx_0x_1x_2\} = \{x_1x_0x_2, v_4v_5\} \cup \{x_1x_2v_3, x_0z\}$. Let $z = v_3$. Choose an $F_1 = \{u_1u_2u_3, u_4u_5\}$ in $\mathcal{F} - \{F\}$. If $\{x_0, x_2\} \cap V(F_1) = \phi$, then $F \cup F_1 \cup \{zx_0x_1x_2\} = \{x_0x_1x_2, u_4u_5\} \cup \{x_0zx_2, v_4v_5\} \cup \{u_1u_2u_3, x_0x_2\}$. Suppose $\{x_0, x_2\} \cap V(F_1) = \{x_0\}$ (similarly if $\{x_0, x_2\} \cap V(F_1) = \{x_2\}$). If $x_0 = u_4$ (similarly if $x_0 = u_5$), then $F_1 \cup \{zx_0x_1x_2\} = \{zx_0u_5, x_1x_2\} \cup \{u_1u_2u_3, x_0x_1\}$. If $x_0 = u_1$ (similarly if $x_0 = u_3$, then $F \cup F_1 \cup \{zx_0x_1x_2\} = \{x_1x_0x_2, u_4u_5\} \cup \{x_0zx_2, v_4v_5\} \cup \{u_1u_2u_3, x_1x_2\}$. If $x_0 = u_2$, then $F_1 \cup \{zx_0x_1x_2\} = \{zx_0u_1, x_1x_2\} \cup \{x_1x_0u_3, u_4u_5\}$. Suppose $\{x_0, x_2\} \cap V(F_1) = \{x_0, x_2\}$. If $x_0 = u_1$ and $x_2 = u_3$ (similarly if $x_0 = u_3$ and $x_2 = u_1$), then $F \cup F_1 \cup \{zx_0x_1x_2\} = \{x_2x_0u_2, u_4u_5\} \cup \{x_1x_2z, v_4v_5\} \cup \{x_1x_0z, x_2u_2\}$. If $x_0 = u_i$, $i = 1, 2, 3$ and $x_2 = u_4$ or u_5 (similarly if $x_2 = u_i$, $i = 1, 2, 3$ and $x_0 = u_4$ or u_5), then $F_1 \cup \{zx_0x_1x_2\} = \{zx_0x_1, u_4u_5\} \cup \{u_1u_2u_3, x_1x_2\}$. Hence, $P_3 \cup P_2 \mid G$.

Suppose $e = x_0x_2$. Since G is different from $K_{1,1,3c+1}$, there is an edge v_1v_2 such that e and v_1v_2 are vertex disjoint edges. Choose an F in $\mathcal{F}$ with $v_1v_2 \in F$. Then $F = \{u_1u_2u_3, v_1v_2\}$ or $\{v_1v_2v_3, v_4v_5\}$. Suppose $F = \{u_1u_2u_3, v_1v_2\}$. If $u_1u_2u_3 = x_0u_2x_2$, choose an F_1 in $\mathcal{F} - \{F\}$. By the same argument as the last paragraph, $F \cup F_1 \cup \{x_0x_1x_2x_0\} = 3(P_3 \cup P_2)$. Otherwise, $|\{x_0, x_2\} \cap V(F)| \leq 1$. We may assume $x_2 \neq u_i$, $i = 1, 2, 3$. Then $F_1 \cup \{x_0x_1x_2x_0\} = \{x_1x_0x_2, v_1v_2\} \cup \{u_1u_2u_3, x_1x_2\}$. Suppose $F = \{v_1v_2v_3, v_4v_5\}$. If $|\{x_0, x_2\} \cap V(F)| = 2$, then $x_0 = v_3$ and $x_2 = v_4$ or v_5 (similarly if $x_2 = v_3$ and $x_0 = v_4$

or v_5). Hence, $F \cup \{x_0x_1x_2x_0\} = \{x_1x_2x_0, v_1v_2\} \cup \{x_1x_0v_2, v_4v_5\}$. If $\{x_0, x_2\} \cap V(F) = \{x_0\}$ (similarly if$\{x_0, x_2\} \cap V(F) = \{x_2\}$), then $x_0 = v_i$, $i = 3, 4, 5$. If $x_0 = v_3$, then $F \cup \{x_0x_1x_2x_0\} = \{v_1v_2x_0, x_1x_2\} \cup \{x_1x_0x_2, v_4v_5\}$. If $x_0 = v_4$ (similarly if $x_0 = v_5$), $F \cup \{x_0x_1x_2x_0\} = \{x_1x_2x_0, v_1v_2\} \cup \{x_1x_0v_5, v_2v_3\}$. Hence, $P_3 \cup P_2 \mid G$.

If $t = 3$, let $G' = G - \{x_1, x_2\}$. Then $q(G') \equiv 0 \pmod 3$. If $G = G_{12}$, by Lemma 2.1, $P_3 \cup P_2 \mid G$. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_3 \in F$. Then $F = \{x_0x_3v_3, v_4v_5\}$, $\{x_3x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0x_3\}$. If $F = \{x_0x_3v_3, v_4v_5\}$, then $F \cup \{x_0x_1x_2x_3\} = \{x_0x_1x_2, x_3v_3\} \cup \{x_0x_3x_2, v_4v_5\}$. If $F = \{x_3x_0v_3, v_4v_5\}$, then $F \cup \{x_0x_1x_2x_3\} = \{x_1x_2x_3, x_0v_3\} \cup \{x_1x_0x_3, v_4v_5\}$. If $F = \{v_1v_2v_3, x_0x_3\}$, then $F \cup \{x_0x_1x_2x_3\} = \{x_0x_1x_2, v_1v_2\} \cup \{x_0x_3x_2, v_2v_3\}$. Thus, $P_3 \cup P_2 \mid G$.

If $t = 4$, let $G' = G - \{x_1, x_2, x_3\}$. Then $q(G') \equiv 2 \pmod 3$. If $G = G_{13}$, by Lemma 2.1, $P_3 \cup P_2 \mid G$. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $v_1v_2v_3$. Since $\{v_1v_2v_3\} \cup \{x_0x_1x_2x_3x_4\} = \{x_0x_1x_2, x_3x_4\} \cup \{v_1v_2v_3, x_2x_3\}$, $P_3 \cup P_2 \mid G$.

If $t \geq 5$, let $G' = (G - \{x_1, x_2, x_3\}) \cup \{x_0x_4\}$. Then $q(G') \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_4 \in F$. Then $F = \{x_0x_4x_5, v_4v_5\}$, $\{x_4x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0x_4\}$ and $(F - \{x_0x_4\}) \cup \{x_0x_1x_2x_3x_4\}$ $(= P_6 \cup P_2$ or $P_5 \cup P_3) = 2(P_3 \cup P_2)$. Hence, $P_3 \cup P_2 \mid G$.

(2) $x_0x_t \notin E(G)$ and $x_0 \neq x_t$.

Suppose $q(G) \equiv 2 \pmod 3$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 0 \pmod 3$. If $G' = K_{1,1,3c+1}$, then the three partite sets are $\{u\}$, $\{v\}$ and $\{x_0, x_2, w_3, \ldots, w_{3c+1}\}$. Hence, $G = G' \cup \{x_0x_1x_2\} = \{x_0x_1x_2, uv\} \cup (cK_{2,3}) \cup P_3 = \{x_0x_1x_2, uv\} \cup (2c(P_3 \cup P_2)) \cup P_3$. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Hence, G has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave $x_0x_1x_2$.

If $t \geq 3$, let $G' = (G - \{x_1, x_2\}) \cup \{x_0x_3\}$. Then $q(G') \equiv 0 \pmod 3$. If $G = G_{37}$, by Lemma 2.3, G has a $(P_3 \cup P_2)$-packing with a P_3 as the leave. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_3 \in F$. Then $F = \{x_0x_3v_3, v_4v_5\}, \{x_3x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0x_3\}$. Hence, $(F - \{x_0x_3\}) \cup \{x_0x_1x_2x_3\}$ $(= P_5 \cup P_2$ or $P_4 \cup P_3) = (P_3 \cup P_2) \cup \{L\}$, where $L = x_0x_1x_2$ or $x_1x_2x_3$. Hence, G has a $(P_3 \cup P_2)$-packing with leave L.

Suppose $q(G) \equiv 1 \pmod 3$. Let $G' = (G - x_1) \cup \{x_0x_2\}$. Then $q(G') \equiv 0 \pmod 3$. If $G = G_{24}$ or G_{25}, by Lemma 2.2, G has a $(P_3 \cup P_2)$-packing with a P_2 as the leave. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_2 \in F$. Then $F = \{x_0x_2v_3, v_4v_5\}$, $\{x_2x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0x_2\}$ and $(F - \{x_0x_2\}) \cup \{x_0x_1x_2\}$ $(= P_4 \cup P_2$ or $P_3 \cup P_3) = (P_3 \cup P_2) \cup \{L\}$, where $L = x_0x_1$ or x_1x_2. Hence, G has a $(P_3 \cup P_2)$-packing with leave L.

Suppose $q(G) \equiv 0 \pmod 3$. If $t = 2$, let $G' = G - x_1$. Then $q(G') \equiv 1 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with an edge e as the leave. If $\{x_0x_1x_2, e\}$ forms a $P_3 \cup P_2$, then $P_3 \cup P_2 \mid G$. Let $e = x_0z$ (similarly $e = x_2z$). Choose an F in $\mathcal{F}$ with $x_2 \in V(F)$. Then $F = \{v_1v_2v_3, x_2v_5\}$, $\{v_1x_2v_3, v_4v_5\}$ or $\{x_2v_2v_3, v_4v_5\}$. Suppose $F = \{v_1v_2v_3, x_0v_5\}$. If $z \neq v_5$, then $F \cup \{zx_0x_1x_2\} = \{x_1x_0z, x_2v_5\} \cup \{v_1v_2v_3, x_1x_2\}$. Suppose $z = v_5$. If $x_0 = v_1$ (similarly if $x_0 = v_3$), then $F \cup \{zx_0x_1x_2\} = \{x_1x_0z, v_2v_3\} \cup \{x_1x_2z, x_0v_2\}$. If $x_0 = v_2$, then $F \cup \{zx_0x_1x_2\} = \{x_1x_0v_1, x_2z\} \cup \{zx_0v_3, x_1x_2\}$. If $x_0 \neq v_i$, $i = 1, 2, 3$, then $F \cup \{zx_0x_1x_2\} = \{x_0x_1x_2, v_1v_2\} \cup \{x_0zx_2, v_2v_3\}$. Suppose $F = \{v_1x_2v_3, v_4v_5\}$. Then $z \neq v_1$ (similarly if $z \neq v_3$) and $F \cup \{zx_0x_1x_2\} = \{x_1x_0z, x_2v_1\} \cup \{x_1x_2v_3, v_4v_5\}$. Suppose $F = \{x_2v_2v_3, v_4v_5\}$. If z is neither v_2 nor v_3, then $F \cup \{zx_0x_1x_2\} = \{x_1x_0z, v_2v_3\} \cup \{x_1x_2v_2, v_4v_5\}$. Suppose $z = v_2$. If $x_0 = v_4$ (similarly if $x_0 = v_5$), then $F \cup \{zx_0x_1x_2\} = \{x_2zv_3, x_0x_1\} \cup \{zx_0v_5, x_1x_2\}$. If $x_0 \neq v_i$, $i = 4, 5$, then $F \cup \{zx_0x_1x_2\} = \{x_0x_1x_2, v_2v_3\} \cup \{x_0zx_2, v_4v_5\}$. Suppose $z = v_3$. If $x_0 = v_4$ (similarly if $x_0 = v_5$), then $F \cup \{zx_0x_1x_2\} = \{x_1x_0v_5, v_2z\} \cup \{x_1x_2v_2, x_0z\}$. Suppose $x_0 \neq v_i$, $i = 4, 5$, then $F \cup \{zx_0x_1x_2\}$ is the disjoint union of 5-cycle $x_0x_1x_2v_2zx_0$ and an edge v_4v_5. Since $d(x_2) \geq 3$, there is an F_1 in $\mathcal{F} - \{F\}$ such that $x_2 \in V(F_1)$. By the same argument as above, $F_1 \cup \{zx_0x_1x_2\} = 2(P_3 \cup P_2)$ except $F_1 \cup \{zx_0x_1x_2\}$ is the disjoint union of 5-cycle $x_0x_1x_2u_2zx_0$ and an edge u_4u_5. In this case, if $v_2 = u_4$ (similarly if $v_2 = u_5$, then $F \cup F_1 \cup \{zx_0x_1x_2\} = \{x_0x_1x_2, v_4v_5\} \cup \{x_0zv_2, x_2u_2\} \cup \{x_2v_2u_5, zu_2\}$; otherwise, $F \cup F_1 \cup \{zx_0x_1x_2\} = \{x_0x_1x_2, v_4v_5\} \cup \{u_2x_2v_2, x_0z\} \cup \{u_2zv_2, u_4u_5\}$. Hence, G has a $(P_3 \cup P_2)$-packing with empty leave.

If $t = 3$, let $G' = G - \{x_1, x_2\}$. Then $q(G') \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0 \in V(F)$. Then $F = \{x_0v_2x_3, v_4v_5\}$, $\{x_0v_2v_3, v_4v_5\}$ ($v_3 \neq x_3$), $\{v_1x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0v_5\}$. If $F = \{x_0v_2x_3, v_4v_5\}$, then $F \cup \{x_0x_1x_2x_3\}$ is a union of 5-cycle $x_0x_1x_2x_3v_2x_0$ and an edge v_4v_5. By the same argument as above, we have $P_3 \cup P_2 \mid G$. If $F = \{x_0v_2v_3, v_4v_5\}$ or $\{v_1x_0v_3, v_4v_5\}$, then $F \cup \{x_0x_1x_2x_3\} = \{x_0x_1x_2, v_4v_5\} \cup \{x_0v_2v_3$ (or $v_1x_0v_3), x_2x_3\}$. If $F = \{v_1v_2v_3, x_0v_5\}$, then $F \cup \{x_0x_1x_2x_3\} = \{x_1x_2x_3, x_0v_5\} \cup \{v_1v_2v_3, x_0x_1\}$. Hence, G has a $(P_3 \cup P_2)$-packing with empty leave.

If $t \geq 4$, let $G' = G - \{x_1, x_2, x_3\} \cup \{x_0x_4\}$. Then $q(G') \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_0x_4 \in F$. Then $F = \{x_0x_4v_3, v_4v_5\}$, $\{x_4x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0x_4\}$ and $(F - \{x_0x_4\}) \cup \{x_0x_1x_2x_3x_4\}$ ($= P_6 \cup P_2$ or $P_5 \cup P_3$) $= 2(P_3 \cup P_2)$. Hence, G has a $(P_3 \cup P_2)$-packing with empty leave.

(3) $x_0 = x_t$ and $t \geq 3$.

Suppose $q(G) \equiv 2 \pmod 3$. For $t = 3$ or 4, if $d(x_0) \geq 4$, let $G' = G - \{x_1, x_2, \ldots, x_{t-1}\}$. If $G = G_{38}$ or G_{39}, by Lemma 2.3, G has a $(P_3 \cup P_2)$-packing with a P_3 as the leave. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave L. If $t = 3$, then $L = P_3$ and $\{L\} \cup \{x_0x_1x_2x_0\} = \{L, x_1x_2\} \cup \{x_1x_0x_2\}$. If $t = 4$, then $L = P_2$ and $\{L\} \cup \{x_0x_1x_2x_3x_0\} = \{L, x_1x_2x_3\} \cup \{x_1x_0x_3\}$. Hence, G has a $(P_3 \cup P_2)$-packing with a P_3 as the leave. Suppose $d(x_0) = 3$. Let $N(x_0) = \{x_1, x_{t-1}, z\}$. In this case, $d(z) \geq 3$. Let $G' = G - \{x_0, x_1, \ldots, x_{t-1}\}$. If $G = G_{40}$, by Lemma 2.3, G has a $(P_3 \cup P_2)$-packing with a P_3 as the leave. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave L. If $t = 3$, then $L = P_2$ and $\{L\} \cup \{x_0x_1x_2x_0\} \cup \{x_0z\} = \{L, x_0x_1x_2\} \cup \{x_2x_0z\}$. If $t = 4$, then $L = \phi$ and $\{x_0x_1x_2x_3x_0\} \cup \{x_0z\} = \{x_1x_2x_3, x_0z\} \cup \{x_1x_0x_3\}$. Hence, G has a $(P_3 \cup P_2)$-packing with a P_3 as the leave.

For $t \geq 5$, let $G' = (G - \{x_2, x_3\}) \cup \{x_1x_4\}$. Then $q(G') \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_1x_4 \in F$. Then $F = \{x_0x_1x_4, v_4v_5\}$, $\{x_1x_4x_5, v_4v_5\}$ or $\{v_1v_2v_3, x_1x_4\}$ and $(F - \{x_1x_4\}) \cup \{x_1x_2x_3x_4\}$ $(= P_5 \cup P_2$ or $P_4 \cup P_3) = (P_3 \cup P_2) \cup \{L\}$, where $L = x_1x_2x_3$ or $x_2x_3x_4$. Hence, G has a $(P_3 \cup P_2)$-packing with leave L.

Suppose $q(G) \equiv 1 \pmod 3$. For $t = 3$, if $d(x_0) \geq 4$, let $G' = G - \{x_1, x_2\}$. If $G = G_{26}$, by Lemma 2.2, G has a $(P_3 \cup P_2)$-packing with a P_2 as the leave. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with a P_2 as the leave. Choose an F in $\mathcal{F}$ with $x_0 \in V(F)$. Then $F = \{x_0v_2v_3, v_4v_5\}$, $\{v_1x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0v_5\}$. If $F = \{x_0v_2v_3, v_4v_5\}$ or $\{v_1x_0v_3, v_4v_5\}$, then $F \cup \{x_0x_1x_2x_0\} = \{x_1x_0x_2, v_4v_5\} \cup \{x_0v_2v_3$ (or $v_1x_0v_3$)$, x_1x_2\}$. If $F = \{v_1v_2v_3, x_0v_5\}$, then $F \cup \{x_0x_1x_2x_0\} = \{x_0x_1x_2, v_1v_2\} \cup \{x_2x_0v_5, v_2v_3\}$. Hence, G has a $(P_3 \cup P_2)$-packing with a P_2 as the leave. Suppose $d(x_0) = 3$. Let $N(x_0) = \{x_1, x_2, z\}$. In this case, $d(z) \geq 3$. Let $G' = G - x_0z$. Then $q(G') \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Hence, G has a $(P_3 \cup P_2)$-packing with leave x_0z.

For $t \geq 4$, let $G' = (G - x_2) \cup \{x_1x_3\}$. Then $q(G') \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_1x_3 \in F$. Then $F = \{x_0x_1x_3, v_4v_5\}$, $\{x_1x_3x_4, v_4v_5\}$ or $\{v_1v_2v_3, x_1x_3\}$ and $(F - \{x_1x_3\}) \cup \{x_1x_2x_3\}$ $(= P_4 \cup P_2$ or $P_3 \cup P_3) = (P_3 \cup P_2) \cup \{L\}$, where $L = x_1x_2$ or x_2x_3. Hence, G has a $(P_3 \cup P_2)$-packing with a P_2 as the leave.

Suppose $q(G) \equiv 0 \pmod 3$. For $3 \leq t \leq 5$, if $d(x_0) \geq 4$, let $G' = G - \{x_1, x_2, \ldots, x_{t-1}\}$. If $G = G_{14}$ or G_{15}, by Lemma 2.1, G has a $(P_3 \cup P_2)$-packing with empty leave. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave L. If $t = 3$, then $L = \phi$. Choose an F in $\mathcal{F}$ with $x_0 \in V(F)$. Then $F = \{x_0v_2v_3, v_4v_5\}$, $\{v_1x_0v_3, v_4v_5\}$ or $\{v_1v_2v_3, x_0v_5\}$. If $F = \{x_0v_2v_3, v_4v_5\}$ or $\{v_1x_0v_3, v_4v_5\}$, then $F \cup \{x_0x_1x_2x_0\} = \{x_1x_0x_2, v_4v_5\} \cup \{x_0v_2v_3$ (or $v_1x_0v_3$)$, x_1x_2\}$. If $F = \{v_1v_2v_3, x_0v_5\}$, then $F \cup \{x_0x_1x_2x_0\} = \{x_0x_1x_2, v_1v_2\} \cup \{x_2x_0v_5, v_2v_3\}$. If $t = 4$, then $L = P_3 = x_0$ $(= v_1)$ v_2v_3, $v_1v_2v_3$ or $v_1x_0v_3$. If $L = x_0v_2v_3$ or $v_1v_2v_3$, then $\{L\} \cup \{x_0x_1x_2x_3x_0\} = \{x_1x_2x_3, v_1v_2\} \cup \{x_1x_0x_3, v_2v_3\}$. If $L = v_1x_0v_3$, then $\{L\} \cup \{x_0x_1x_2x_3x_0\} = \{x_1x_0v_1, x_2x_3\} \cup \{x_3x_0v_3, x_1x_2\}$. If $t = 5$, then $L = uv$. If x_0 is incident with uv, say $x_0 = u$, then $\{x_0x_1x_2x_3x_4x_0\} \cup \{uv\} = \{x_0x_1x_2, x_3x_4\} \cup \{vx_0x_4, x_2x_3\}$. Otherwise, choose an $F = \{v_1v_2v_3, v_4v_5\}$ in $\mathcal{F}$ with $x_0 \in V(F)$. If $x_0 = v_4$ or v_5, then $F \cup \{x_0x_1x_2x_3x_4x_0\} \cup \{uv\} = \{x_0x_1x_2, uv\} \cup \{x_2x_3x_4, v_4v_5\} \cup \{v_1v_2v_3, x_4x_0\}$. If $x_0 = v_1$, v_2 or v_3, then $F \cup \{x_0x_1x_2x_3x_4x_0\} \cup \{uv\} = \{x_0x_1x_2, uv\} \cup \{x_3x_4x_0, v_4v_5\} \cup \{v_1v_2v_3, x_2x_3\}$. Hence, G has a $(P_3 \cup P_2)$-packing with empty leave.

Suppose $d(x_0) = 3$. Let $N(x_0) = \{x_1, x_{t-1}, z\}$. In this case, $d(z) \geq 3$. Let $G' = G - \{x_0, x_1, \ldots, x_{t-1}\}$. If $G = G_{16}$, G_{17} or G_{18}, by Lemma 2.1, G has a $(P_3 \cup P_2)$-packing with empty leave. Otherwise, by the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with leave L. If $t = 3$, then $L = zv_2v_3$, v_1zv_3 or $v_1v_2v_3$. If $L = zv_2v_3$, then $\{L\} \cup \{x_0x_1x_2x_0, x_0z\} = \{x_0x_1x_2, zv_2\} \cup \{x_2x_0z, v_2v_3\}$. If $L = v_1zv_3$, then $\{L\} \cup \{x_0x_1x_2x_0, x_0z\} = \{x_1x_0x_2, zv_1\} \cup \{x_0zv_3, x_1x_2\}$. If $L = v_1v_2v_3$, then $\{L\} \cup \{x_0x_1x_2x_0, x_0z\} = \{x_0x_1x_2, v_1v_2\} \cup \{x_2x_0z, v_2v_3\}$. If $t = 4$, then $L = v_1v_2$ and $\{L\} \cup \{x_0x_1x_2x_3x_0, x_0z\} = \{x_1x_2x_3, x_0z\} \cup \{x_1x_0x_3, v_1v_2\}$. If $t = 5$, then $L = \phi$ and $\{x_0x_1x_2x_3x_4x_0, x_0z\} = \{x_0x_1x_2, x_3x_4\} \cup \{x_4x_0z, x_2x_3\}$. Hence, G has a $(P_3 \cup P_2)$-packing with empty leave.

For $t \geq 6$, let $G' = (G - \{x_2, x_3, x_4\}) \cup \{x_1x_5\}$. Then $q(G') \equiv 0 \pmod 3$. By the choice of G, G' has a $(P_3 \cup P_2)$-packing $\mathcal{F}$ with empty leave. Choose an F in $\mathcal{F}$ with $x_1x_5 \in F$. Then $F = \{x_0x_1x_5, v_4v_5\}$, $\{x_1x_5x_6, v_4v_5\}$ or $\{v_1v_2v_3, x_1x_5\}$ and $(F - \{x_1x_5\}) \cup \{x_0x_1x_2x_3x_4x_5\}$ ($= P_6 \cup P_2$ or $P_5 \cup P_3$) $= 2(P_3 \cup P_2)$. Hence, G has a $(P_3 \cup P_2)$-packing with empty leave.

Therefore, we complete the proof. □

Now, we are ready to prove the Conjecture 1.1.

Theorem 2.6. *If G is a graph with $q(G) \equiv 0 \pmod 3$ and $\delta(G) \geq 2$, then $H \mid G$ for some graph H of size 3.*

Proof. If $q(G) = 3$, then it is trivial that $G \mid G$. It have been argued that $P_4 \mid G$ if $G = K_4$ or $K_{1,1,3c+1}$. By Theorem 2.5, we have $P_3 \cup P_2 \mid G$. Therefore, we complete the proof. □

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Zhen-Chun Chen and Hung-Lin Fu
Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30010, Taiwan
E-mail address: `amco0624@yahoo.com.tw`, `hlfu@math.nctu.edu.tw`

Kuo-Ching Huang
Department of Financial and Computational Mathematics, Providence University, Shalu 43301, Taichung, Taiwan
E-mail address: `kchuang@gm.pu.edu.tw`