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One Existence Theorem for non-CSC Extremal Kähler Metrics with Conical Singularities on S^2

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Abstract. We often call an extremal Kähler metric with finite singularities on a compact Riemann surface an HCMU (the Hessian of the Curvature of the Metric is Umbilical) metric. In this paper we consider the following question: if we give N points $p_1, \ldots, p_N$ on S^2 and N positive real numbers $2\pi\alpha_1, \ldots, 2\pi\alpha_N$ with $\alpha_n \neq 1$, $n = 1, \ldots, N$, what condition can guarantee the existence of a non-CSC HCMU metric which has conical singularities $p_1, \ldots, p_N$ with singular angles $2\pi\alpha_1, \ldots, 2\pi\alpha_N$ respectively. We prove that if there are at least $N-2$ integers in $\alpha_1, \ldots, \alpha_N$ then there exists one non-CSC HCMU metric on S^2 satisfying the condition stated above no matter where the given points are.

1. Introduction

The extremal Kähler metric was defined in [1] by Calabi. The aim is to find the "best" metric in a fixed Kähler class on a compact Kähler manifold $\mathcal{M}$. In a fixed Kähler class, an extremal Kähler metric is the critical point of the following Calabi energy functional

$$\mathcal{C}(g) = \int_{\mathcal{M}} R^2 \, dg,$$

where R is the scalar curvature of the metric g in the Kähler class. The Euler-Lagrange equations of $\mathcal{C}(g)$ are $R_{,\alpha\beta} = 0$ for all α, β the indexes, where $R_{,\alpha\beta}$ is the second-order $(0,2)$ covariant derivative of R. When $\mathcal{M}$ is a compact Riemann surface, Calabi in [1] proves that an extremal Kähler metric is a CSC (constant scalar curvature) metric.

Therefore a natural question is if on a compact Riemann surface an extremal Kähler metric has singularities whether or not it is still a CSC metric. X. Chen in [2] first gives an example of a non-CSC extremal Kähler metric with singularities. We often call an extremal Kähler metric with finite singularities on a compact Riemann surface an HCMU (the Hessian of the Curvature of the Metric is Umbilical) metric. Besides X. Chen's work in [2], some work has been done to study non-CSC HCMU metrics such as [3–5] and so on.

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In [6], Q. Chen and the second author reduce the existence of a non-CSC HCMU metric to the existence of a meromorphic 1-form on the underlying Riemann surface. However the existence of the meromorphic 1-form is still complicated. On S^2 the existence of the meromorphic 1-form can be reduced to an algebraic problem. In this paper we study when the algebraic problem has a solution. Our main theorem is

Theorem 1.1. *Let $p_1, \dots, p_N$ be N ($N \geq 3$) points on S^2 and $2\pi\alpha_1, \dots, 2\pi\alpha_N$ be N positive real numbers with $\alpha_n \neq 1$, $n = 1, 2, \dots, N$. If there are at least $N-2$ integers in $\alpha_1, \dots, \alpha_N$, there exists a non-CSC HCMU metric which has conical singularities $p_1, \dots, p_N$ with singular angles $2\pi\alpha_1, \dots, 2\pi\alpha_N$ respectively.*

Remark 1.2. In general, the existence of a non-CSC HCMU metric is related to both the position of the given points $p_1, \dots, p_N$ and the given conical singular angles $2\pi\alpha_1, \dots, 2\pi\alpha_N$. This theorem shows if the number of integers in $\alpha_1, \dots, \alpha_N$ is large enough, there always exists a non-CSC HCMU metric no matter where the given points are.

On the other hand for $N = 4$ an example in [5] shows if there is only one integer in $\alpha_1, \dots, \alpha_4$ it is possible for some choice of 4 points given on S^2 that there is no non-CSC HCMU metric on S^2 such that the metric has conical singularities at the given points with the singular angles $2\pi\alpha_1, \dots, 2\pi\alpha_4$. Therefore we conjecture that for each $N \geq 5$, there exists no HCMU metric for some examples of $p_1, \dots, p_N$ with angles $2\pi\alpha_1, \dots, 2\pi\alpha_N$, if one only assumes $N-3$ integers in $\alpha_1, \dots, \alpha_N$.

2. Preliminaries

Definition 2.1. [9] Let $\mathcal{M}$ be a Riemann surface, $p \in \mathcal{M}$. A conformal metric g on $\mathcal{M}$ is said to have a conical singularity at p with the singular angle $2\pi\alpha$ ($\alpha > 0$) if in a neighborhood of p

$$g = e^{2\varphi}|dz|^2,$$

where z is a local complex coordinate defined in the neighborhood of p with $z(p) = 0$ and

$$\varphi - (\alpha - 1)\ln|z|$$

is continuous at 0.

Definition 2.2. [3] Let $\mathcal{M}$ be a compact Riemann surface and $p_1, \dots, p_N$ be N points on $\mathcal{M}$. Denote $\mathcal{M} \setminus \{p_1, \dots, p_N\}$ by $\mathcal{M}^*$. Let g be a conformal metric on $\mathcal{M}^*$. If g satisfies

$$K_{,zz} = 0, \tag{2.1}$$

where K is the Gauss curvature of g, we call g an HCMU metric on $\mathcal{M}$.

In this paper we always consider non-CSC HCMU metrics with conical singularities which have finite area and finite Calabi energy, that is,

$$\int_{\mathcal{M}^*} dg < +\infty, \quad \int_{\mathcal{M}^*} K^2\, dg < +\infty.$$

There are many results in studying this kind of metrics. First the equation (2.1) is equivalent to

$$\nabla K \triangleq \sqrt{-1}e^{-2\varphi}K_{\overline{z}}\frac{\partial}{\partial z}$$

is a holomorphic vector field on $\mathcal{M}^*$. In [3] X. Chen proves that the Gauss curvature K can be continuously extended to $\mathcal{M}$ and there are finite smooth extremal points of K on $\mathcal{M}^*$. In [4] Q. Chen, X. Chen and the second author prove the following fact: each smooth extremal point of K is either the maximum point of K or the minimum point of K, and if we denote the maximum of K by K_1 and the minimum of K by K_2 then

$$K_1 > 0, \quad K_1 > K_2 > -(K_1 + K_2).$$

In [7] C. S. Lin and X. Zhu prove that ∇K is actually a meromorphic vector field on $\mathcal{M}$. In [6] Q. Chen and the second author define the dual 1-form of ∇K by $\omega(\nabla K) = \sqrt{-1}/4$. They call ω the character 1-form of the metric. Denote $\mathcal{M}^* \setminus \{$ smooth extremal points of $K\}$ by $\mathcal{M}'$. Then on $\mathcal{M}'$

$$\begin{aligned} &\frac{dK}{-\frac{1}{3}(K-K_1)(K-K_2)(K+K_1+K_2)} = \omega + \overline{\omega}, \\ &g = -\frac{4}{3}(K-K_1)(K-K_2)(K+K_1+K_2)\omega\overline{\omega}. \end{aligned} \tag{2.2}$$

By (2.2) some properties of ω are got in [6]:

- All of the zeros of ω are the conical singularities of g. For each zero of ω the corresponding singular angle is of the form $2\pi\alpha$ where α is an integer and the order of ω at the zero is $\alpha - 1$. K can be smoothly extended to the zeros of ω at which $dK = 0$. At each zero of ω the value of K is between K_1 and K_2 so we call zeros of ω saddle points of K.

- ω only has simple poles which consist of smooth extremal points of K and conical singularities of g except the zeros of ω. Moreover these poles of ω are just all of the maximum points and the minimum points of K. The residue of ω at each pole is a real number. Denote $-\frac{3}{(K_1-K_2)(K_2+2K_1)}$ by σ and $-\frac{2K_1+K_2}{2K_2+K_1}$ by λ. At a maximum point of K the residue of ω is $\sigma\alpha$ if at this point g has the conical singular angle $2\pi\alpha$ or the residue of ω is σ if this maximum point of K is the smooth point of g. At a minimum point of K the residue of ω is $\sigma\lambda\alpha$ if at this point g has the conical singular angle $2\pi\alpha$ or the residue of ω is $\sigma\lambda$ if this minimum point of K is the smooth point of g.

- $\omega+\overline{\omega}$ is exact on $\mathcal{M}\setminus\{\text{poles of }\omega\}$.

3. Proof of Theorem 1.1

3.1. Reduce the existence of non-CSC HCMU metrics to the existence of some kind of meromorphic 1-forms

First by a theorem in [8] one can get the following theorem.

Theorem 3.1. [6] *Let $\mathcal{M}$ be a Riemann surface, $p_1,\dots,p_L$ be L $(L\geq 2)$ points on $\mathcal{M}$ and $d_1,\dots,d_L$ be L nonzero real numbers with $d_1+\cdots+d_L=0$. Then there exists a meromorphic 1-form ω on $\mathcal{M}$ such that*

(1) *ω only has L simple poles at $p_1,\dots,p_L$ with $\mathrm{Res}_{p_l}(\omega)=d_l$, $l=1,2,\dots,L$,*

(2) *$\omega+\overline{\omega}$ is exact on $\mathcal{M}\setminus\{p_1,\dots,p_L\}$.*

Then in [6] Q. Chen and the second author prove the following theorem.

Theorem 3.2. [6] *Let $\mathcal{M}$ be a compact Riemann surface and ω be a meromorphic 1-form on $\mathcal{M}$ satisfying the conditions:*

(1) *ω only has simple poles,*

(2) *At each pole the residue of ω is a real number,*

(3) *$\omega+\overline{\omega}$ is exact on $\mathcal{M}\setminus\{\text{poles of }\omega\}$.*

Then there exists a non-CSC HCMU metric such that ω is the character 1-form of the metric.

Proof. First by Theorem 3.1 a meromorphic 1-form on $\mathcal{M}$ satisfying the conditions (1), (2), (3) in Theorem 3.2 always exists. Suppose $p_1,\dots,p_L$ are the poles of ω in which $p_1,\dots,p_J$ are the poles where the residues of ω are negative and $p_{J+1},\dots,p_L$ are the poles where the residues of ω are positive. Let K_1, K_2 be two real numbers satisfying:

$$K_1>0,\quad K_1>K_2>-(K_1+K_2).$$

Then denote $-\frac{3}{(K_1-K_2)(K_2+2K_1)}$ by σ and $-\frac{2K_1+K_2}{2K_2+K_1}$ by λ. Consider the following equation:

$$\frac{dK}{-\frac{1}{3}(K-K_1)(K-K_2)(K+K_1+K_2)}=\omega+\overline{\omega}\quad\text{and}\quad K(p_0)=K_0,\tag{3.1}$$

where $K_2<K_0<K_1$ and $p_0\in\mathcal{M}\setminus\{p_1,\dots,p_L\}$. One can prove there exists a unique solution K of (3.1) on $\mathcal{M}$ which satisfies K is smooth on $\mathcal{M}\setminus\{p_1,\dots,p_L\}$ and is continuous on $\mathcal{M}$. Then construct a metric

$$g=-\frac{4}{3}(K-K_1)(K-K_2)(K+K_1+K_2)\omega\overline{\omega}.$$

One can prove g is a non-CSC HCMU metric, K is the Gauss curvature of g with K_1, K_2 being the maximum and the minimum of K and ω is the character 1-form of g. Therefore g has the conical singularities at the zeros and the poles of ω. At the zeros of ω the singular angles of g are of the form $2\pi(\mathrm{ord}_p(\omega)+1)$, and at the poles of ω the singular angles of g are of the form $2\pi\frac{\mathrm{Res}_p(\omega)}{\sigma}$ or $2\pi\frac{\mathrm{Res}_p(\omega)}{\lambda\sigma}$ depending on the sign of $\mathrm{Res}_p(\omega)$. $\frac{\mathrm{Res}_p(\omega)}{\sigma}=1$ or $\frac{\mathrm{Res}_p(\omega)}{\lambda\sigma}=1$ means that p is a smooth point of g. □

By Theorem 3.2, to get a prescribed non-CSC HCMU metric we only need to get a suitable meromorphic 1-form which satisfies the conditions in Theorem 3.2. In general, the existence of this kind of meromorphic 1-form is complicated since in the poles of the meromorphic 1-form there are some unknown smooth points of the metric and one also need to determine which points in the given points are the zeros of the meromorphic 1-form.

3.2. Proof of Theorem 1.1

First it can be proved that a meromorphic 1-form on S^2 which satisfies the conditions (1) and (2) in Theorem 3.2 satisfies the condition (3) in Theorem 3.2 automatically (cf. [6]). Regard S^2 as $\mathbb{C}\cup\{\infty\}$. Without loss of generality, we assume $p_1=b_1,\dots,p_{N-2}=b_{N-2}, p_{N-1}=0, p_N=\infty$, where $b_1,\dots,b_{N-2}\in\mathbb{C}$, $\alpha_1,\dots,\alpha_{N-2}$ are integers and $\sum_{i=1}^{N-2}(\alpha_i-1)+\alpha_{N-1}>\alpha_N$. Let $S=\sum_{i=1}^{N-2}(\alpha_i-1)$. Then we have the following propositions.

Proposition 3.3. *There are S numbers $c_1,\dots,c_S\in\mathbb{C}$, not necessarily different from each other, such that the following equation holds:*

$$\frac{\alpha_{N-1}}{z}+\sum_{j=1}^{S}\frac{1}{z-c_j}=\frac{(\alpha_{N-1}+S)\prod_{i=1}^{N-2}(z-b_i)^{\alpha_i-1}}{z\prod_{j=1}^{S}(z-c_j)} \tag{3.2}$$

on $\mathbb{C}\setminus\{0,b_1,\dots,b_{N-2},c_1,\dots,c_S\}$.

Proof. Suppose

$$\prod_{i=1}^{N-2}(z-b_i)^{\alpha_i-1}=z^S+\lambda_1 z^{S-1}+\cdots+\lambda_{S-1}z+\lambda_S.$$

Let

$$\sigma_j=\frac{(\alpha_{N-1}+S)\lambda_j}{\alpha_{N-1}+S-j},\quad j=1,2,\dots,S$$

and

$$P(z)=z^S+\sigma_1 z^{S-1}+\cdots+\sigma_S.$$

Then one can prove that

$$\alpha_{N-1}P(z) + zP'(z) = (\alpha_{N-1} + S)\prod_{i=1}^{N-2}(z - b_i)^{\alpha_i - 1}. \tag{3.3}$$

Let $c_1, \ldots, c_S$ be the roots of $P(z) = 0$, that is, $P(z) = \prod_{j=1}^{S}(z - c_j)$. Divide both sides of the equation (3.3) by $zP(z)$. Then we get the equation (3.2). □

Proposition 3.4. *Suppose $c_1, \ldots, c_S \in \mathbb{C}$ satisfy the equation (3.2). Then there are two cases:*

(1) *If $\forall\, b_i$, $i = 1, 2, \ldots, N-2$ and $\forall\, c_j$, $j = 1, 2, \ldots, S$, $b_i \neq c_j$ then $c_j \neq 0$, $j = 1, 2, \ldots, S$ and $c_1, \ldots, c_S$ are different from each other.*

(2) *If $\exists\, b_i$, $i = 1, 2, \ldots, N-2$ and $\exists\, c_j$, $j = 1, 2, \ldots, S$ such that $b_i = c_j$ then there are just α_i numbers in $c_1, \ldots, c_S$ taking the value b_i.*

Proof. (1) If $\exists\, j$, $j = 1, 2, \ldots, S$, $c_j = 0$ or $\exists\, j, j' \in \{1, 2, \ldots, S\}$, $j \neq j'$, $c_j = c_{j'}$ then we multiply both sides of the equation (3.2) by $z\prod_{j=1}^{S}(z - c_j)$ and take limits as $z \to c_j$ on both sides of the equation multiplied by $z\prod_{j=1}^{S}(z - c_j)$. The limit of the left side is zero but the limit of the right side is nonzero, a contradiction.

(2) Fix b_i. If the number of the numbers in $c_1, \ldots, c_S$ taking b_i is less than α_i then we take limits as $z \to b_i$ on both sides of the equation (3.2) and get that the limit of the left side is ∞ but the limit of the right side is finite, a contradiction. If the number of the numbers in $c_1, \ldots, c_S$ taking b_i is more than α_i then we first reduce the right side of the equation (3.2) and then multiply both sides of the reduced equation by the denominator of the reduced right side. Take limits as $z \to b_i$ on both sides of the equation and get that the limit of the left side is zero but the limit of the right side is nonzero. This leads to a contradiction. □

We now construct ω using Propositions 3.3 and 3.4.

In case 1, let $\lambda = -(\alpha_{N-1} + S)/\alpha_N$ and

$$\omega = -\frac{(\alpha_{N-1} + S)\prod_{i=1}^{N-2}(z - b_i)^{\alpha_i - 1}}{z\prod_{j=1}^{S}(z - c_j)}\, dz.$$

By (3.2) ω is a meromorphic 1-form on S^2 satisfying the conditions (1) and (2) in Theorem 3.2 (∞ is also a simple pole of ω). Then consider the equations

$$(K_1 - K_2)(2K_1 + K_2) = 3 \quad \text{and} \quad \frac{2K_1 + K_2}{2K_2 + K_1} = -\lambda. \tag{3.4}$$

We get one solution of (3.4)

$$K_1 = -\frac{2\lambda+1}{\sqrt{3}\sqrt{\lambda(\lambda+1)}} \quad \text{and} \quad K_2 = \frac{2+\lambda}{\sqrt{3}\sqrt{\lambda(\lambda+1)}}.$$

Note $\lambda < -1$ so we have $K_1 > 0$ and $K_1 > K_2 > -(K_1+K_2)$. By (3.4), $\sigma \triangleq \frac{-3}{(K_1-K_2)(2K_1+K_2)} = -1$. Then by the proof of Theorem 3.2 there exists a non-CSC HCMU metric g with conical singularities $b_1,\dots,b_{N-2},0,\infty$. Moreover $b_1,\dots,b_{N-2}$ are the saddle points of the Gauss curvature K with the singular angles $2\pi\alpha_1,\dots,2\pi\alpha_{N-2}$ respectively, 0 is the maximum point of K with the singular angle $2\pi\alpha_{N-1}$ and ∞ is the minimum point of K with the singular angle $2\pi\alpha_N$.

In case 2, without loss of generality, we assume $b_1,\dots,b_T$ $(T<N-2)$ satisfy the hypothesis in Proposition 3.4(2) and $b_1 = c_1 = \cdots = c_{\alpha_1}, b_2 = c_{\alpha_1+1} = \cdots = c_{\alpha_1+\alpha_2},\dots, b_T = c_{\alpha_1+\cdots+\alpha_{T-1}+1} = \cdots = c_{\alpha_1+\cdots+\alpha_T}$. Then the equation (3.2) can be reduced to be

$$\text{(3.5)} \quad \frac{\alpha_{N-1}}{z} + \sum_{t=1}^{T}\frac{\alpha_t}{z-b_t} + \sum_{k=\left(\sum_{t=1}^{T}\alpha_t\right)+1}^{S}\frac{1}{z-c_k} = \frac{(\alpha_{N-1}+S)\prod_{h=T+1}^{N-2}(z-b_h)^{\alpha_h-1}}{z\prod_{t=1}^{T}(z-b_t)\prod_{k=\left(\sum_{t=1}^{T}\alpha_t\right)+1}^{S}(z-c_k)},$$

where c_k, $k = \left(\sum_{t=1}^{T}\alpha_t\right)+1,\dots,S$, are different from each other. Let $\lambda = -(\alpha_{N-1}+S)/\alpha_N$ and

$$\omega = -\frac{(\alpha_{N-1}+S)\prod_{h=T+1}^{N-2}(z-b_h)^{\alpha_h-1}}{z\prod_{t=1}^{T}(z-b_t)\prod_{k=\left(\sum_{t=1}^{T}\alpha_t\right)+1}^{S}(z-c_k)}\,dz.$$

By the equation (3.5) ω is a meromorphic 1-form on S^2 satisfying the conditions (1) and (2) in Theorem 3.2. Also let

$$K_1 = -\frac{2\lambda+1}{\sqrt{3}\sqrt{\lambda(\lambda+1)}} \quad \text{and} \quad K_2 = \frac{2+\lambda}{\sqrt{3}\sqrt{\lambda(\lambda+1)}}.$$

Then $\sigma \triangleq \frac{-3}{(K_1-K_2)(2K_1+K_2)} = -1$ and $K_1 > 0$, $K_1 > K_2 > -(K_1+K_2)$. Also by the proof of Theorem 3.2 there exists a non-CSC HCMU metric g with conical singularities $b_1,\dots,b_{N-2},0,\infty$. Moreover $b_1,\dots,b_T$ are the maximum points of the Gauss curvature K with the singular angles $2\pi\alpha_1,\dots,2\pi\alpha_T$ respectively, $b_{T+1},\dots,b_{N-2}$ are the saddle points of K with the singular angles $2\pi\alpha_{T+1},\dots,2\pi\alpha_{N-2}$ respectively, 0 is the maximum point of K with the singular angle $2\pi\alpha_{N-1}$ and ∞ is the minimum point of K with the singular angle $2\pi\alpha_N$. Therefore we finish the proof of Theorem 1.1.

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