Quantitative Recurrence Properties for Systems with Non-uniform Structure

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Abstract. Let X be a subshift with non-uniform structure, and $\sigma\colon X \to X$ be a shift map. Further, define

$$R(\psi) := \{x \in X : d(\sigma^n x, x) < \psi(n) \text{ for infinitely many } n\}$$

and

$$R(f) := \left\{x \in X : d(\sigma^n x, x) < e^{-S_n f(x)} \text{ for infinitely many } n\right\},$$

where $\psi\colon \mathbb{N} \to \mathbb{R}^+$ is a nonincreasing and positive function and $f\colon X \to \mathbb{R}^+$ is a continuous positive function. In this paper, we give quantitative estimates of the above sets, that is, $\dim_H R(\psi)$ can be expressed by ψ and $\dim_H R(f)$ is the solution of the Bowen equation of topological pressure. These results can be applied to a large class of symbolic systems, including β-shifts, S-gap shifts, and their factors.

1. Introduction

Let (X, T, d) be a topological dynamical system, where (X, d) is a compact metric space and $T\colon X \to X$ is a continuous map. For the measure-preserving dynamical system (X, T, μ, d), the Poincaré Recurrence Theorem states that the orbit of almost every point in any positive measure set E returns to E an infinite number of times. These results are qualitative in nature. There are fruitful results about the descriptions of the recurrence. We refer the reader to [4] and the references therein. These results do not address either the rate at which the orbit will return to the initial point or in what manner the neighborhood of the initial point will shrink. In 1993, Boshernitzan [2] presented the following result for general systems.

Theorem 1.1. [2] *Let (X, T, μ, d) be a measure-preserving dynamical system. Assume that, for some $\alpha > 0$, the α-dimensional Hausdorff measure H^α of the space X is σ-finite. Then for μ-almost all $x \in X$,*

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n x, x) < \infty.$$

If, moreover, $H^\alpha(X) = 0$, then for μ-almost all $x \in X$,

$$\liminf_{n \to \infty} n^{1/\alpha} d(T^n x, x) = 0.$$

Later, Barreira and Saussol [1] studied the shrinking rate which is tightly related to the local pointwise dimension.

Theorem 1.2. [1] *Let $T \colon X \to X$ be a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^m$ for some $m \in \mathbb{N}$, and let μ be a T-invariant probability measure on X. Then μ-almost surely,*

$$\lim_{n \to \infty} n^{1/\alpha} d(T^n x, x) < \infty$$

for any $\alpha > \underline{d}_\mu(x)$, where $\underline{d}_\mu(x)$ is the lower pointwise dimension of x with respect to μ, given by

$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

Clearly, Boshernitzan showed almost all points have a low recurrence rate. And Barreira and Saussol showed that the shrinking rate for the recurrence may relate to some indicators of x. Another direction is the question of how large will the set of points be when the shrinking rate for recurrence is related to other functions? In [5–7], Hill and Velani introduced a shrinking target problem originating from number theory and presented quantitative studies of the recurrence. Let $T \colon \mathbf{J} \to \mathbf{J}$ be an expanding rational map of the Riemann sphere acting on its Julia set $\mathbf{J}$ and $f \colon \mathbf{J} \to \mathbb{R}$ denote a Hölder continuous function satisfying $f(x) \geq \log |T'(x)|$ for all $x \in \mathbf{J}$. For any $z_0 \in \mathbf{J}$, Hill and Velani [5] studied the set of 'well approximable' points

$$D_{z_0}(f) := \left\{ x \in \mathbf{J} : d(y, x) < e^{-S_n f(y)} \text{ for infinitely many pairs } (y, n) \in \mathbf{I} \right\},$$

where $\mathbf{I} = \mathbf{I}(z_0)$ denotes the set of pairs (y, n) $(n \in \mathbb{N})$ such that $T^n y = z_0$ and $S_n f(y) = \sum_{i=0}^{n-1} f(T^i y)$. In fact, they gave the following result.

Theorem 1.3. *The set $D_{z_0}(f)$ has Hausdorff dimension $s(f)$, where $s(f)$ is the unique solution to the pressure equation*

$$P(-sf) = 0.$$

In [10], Tan and Wang investigated the metric properties as well as estimates of the Hausdorff dimension of the recurrence set for β-transformation dynamical systems. More precisely, the β-transformation $T_\beta \colon [0, 1] \to [0, 1]$ is defined by $T_\beta x = \beta x - \lfloor \beta x \rfloor$ for all $x \in [0, 1]$. The spotlight is on the size of the set

$$\{ x \in [0, 1] : d(T_\beta^n x, x) < \psi(n) \text{ for infinitely many } n \},$$

where $\psi(n)$ is a positive function. In fact, this has evoked a rich subsequent literature on the so-called Diophantine approximation. We refer the reader to [8, 11] for the related work about this set. It is worth mentioning that the study of recurrence for Diophantine approximation is focused on the β-transformations. In other words, the question is whether we can give a quantitative estimate of recurrence for more general dynamical systems.

In this paper, we consider a class of symbolic systems which were studied in [3]. That is, (X, σ) is a symbolic system with non-uniform structure. The concept of a non-uniform structure is basically defined in the following way: let $\mathcal{L}(X)$ be the language of X, there exists a $\mathcal{G} \subset \mathcal{L}(X)$ with (W)-specification and $\mathcal{L}(X)$ is edit approachable by $\mathcal{G}$. The details of the definitions will be given in the next section.

Now we state our main results. Fix a symbolic system (X, σ, d), where $\sigma \colon X \to X$ is a shift map and d is the metric of X. Write $M(X)$, $M_\sigma(X)$ for the probability measure, respectively, invariant measure, with the weak* topology. For a map $\psi \colon \mathbb{N} \to \mathbb{R}^+$, we define

$$R(\psi) := \{x \in X : d(\sigma^n x, x) < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Theorem 1.4. *Let X be a shift space with $\mathcal{L} = \mathcal{L}(X)$. Suppose that $\mathcal{G} \subset \mathcal{L}$ has a (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$. For a positive function $\psi(n) \colon \mathbb{N} \to \mathbb{R}^+$,*

(C1) *if $\liminf_{n \to \infty} \psi(n) > 0$, then*

$$\dim_H R(\psi) = h.$$

(C2) *if ψ is nonincreasing, then*

$$\dim_H R(\psi) = \frac{h}{1+b} \quad \text{with } b = \liminf_{n \to \infty} \frac{-\log \psi(n)}{n},$$

where $\dim_H(\cdot)$ denotes the Hausdorff dimension of a set and $h := h_{\text{top}}(X)$ denotes the topological entropy of X.

Let f be a positive continuous function defined on X. In this paper, we set $S_n f(x) = \sum_{i=0}^{n-1} f(\sigma^i x)$. Define

$$R(f) = \left\{x \in X : d(\sigma^n x, x) < e^{-S_n f(x)} \text{ for infinitely many } n \in \mathbb{N}\right\}.$$

Theorem 1.5. *Let X be a shift space with $\mathcal{L} = \mathcal{L}(X)$. Let f be a positive continuous function defined on X. Suppose that $\mathcal{G} \subset \mathcal{L}$ has a (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$. The Hasudorff dimension of $R(f)$ is the unique solution s of the following pressure equation*

$$P(-s(f+1)) = 0,$$

where $P(\cdot)$ denotes the topological pressure.

2. Preliminaries

2.1. Non-uniform structure

In this paper, we consider symbolic spaces. Let $p \geq 2$ be an integer and $\mathcal{A} = \{1, 2, \ldots, p\}$. Let

$$\mathcal{A}^{\mathbb{N}} = \{(w_i)_{i=1}^{\infty} : w_i \in \mathcal{A} \text{ for } i \geq 1\}.$$

Then $\mathcal{A}^{\mathbb{N}}$ is compact in the product discrete topology. We can define a metric for $\mathcal{A}^{\mathbb{N}}$ as follows. For any $u, v \in \mathcal{A}^{\mathbb{N}}$, define

$$d(u, v) := e^{-|u \wedge v|},$$

where $|u \wedge v|$ is the maximal length n such that $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$. We say that (X, σ) is a subshift over $\mathcal{A}$ if X is a compact subset of $\mathcal{A}^{\mathbb{N}}$ and $\sigma(X) \subset X$, where σ is the left shift map on $\mathcal{A}^{\mathbb{N}}$

$$\sigma((w_i)_{i=1}^{\infty}) = (w_{i+1})_{i=1}^{\infty}, \quad \forall (w_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}.$$

In particular, (X, σ) is called a full shift over $\mathcal{A}$ if $X = \mathcal{A}^{\mathbb{N}}$. For $n \in \mathbb{N}$ and $w \in \mathcal{A}^n$, we write

$$[w] = \left\{ (w_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : w_1 \cdots w_n = w \right\},$$

and call it an nth word in $\mathcal{A}^{\mathbb{N}}$. The language of X, denoted by $\mathcal{L} = \mathcal{L}(X)$, is the set of finite words that appear in some $x \in X$. More precisely,

$$\mathcal{L}(X) = \{w \in \mathcal{A}^* : [w] \cap X \neq \emptyset\},$$

where $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$. Given $w \in \mathcal{L}$, let $|w|$ denote the length of w. For any collection $\mathcal{D} \subset \mathcal{L}$, let $\mathcal{D}_n$ denote $\{w \in \mathcal{D} : |w| = n\}$. Thus, $\mathcal{L}_n$ is the set of all words of length n that appear in sequences belonging to X. Given the words u, v, we use juxtaposition uv to denote the word obtained by concatenation.

Definition 2.1. [3] Given a shift space X and its language $\mathcal{L}$, consider a subset $\mathcal{G} \subset \mathcal{L}$. Given $\tau \in \mathbb{N}$, we say that $\mathcal{G}$ has a (W)-specification with gap length τ if for every $v, w \in \mathcal{G}$ there exists $u \in \mathcal{L}$ such that $vuw \in \mathcal{G}$ and $|u| \leq \tau$.

Definition 2.2. [3] Define an edit of a word $w = w_1 \cdots w_n \in \mathcal{L}$ to be a transformation of w by one of the following actions, where $u^j \in \mathcal{L}$ are arbitrary words and $a, a' \in \mathcal{A}$ are arbitrary symbols.

(1) Substitution: $w = u^1 a u^2 \mapsto w' = u^1 a' u^2$.

(2) Insertion: $w = u^1 u^2 \mapsto w' = u^1 a' u^2$.

(3) Deletion: $w = u^1 a u^2 \mapsto w' = u^1 u^2$.

Given $v, w \in \mathcal{L}$, define the edit distance between v and w to be the minimum number of edits required to transform the word v into the word w. We will denote this by $\widehat{d}(v, w)$. Now we introduce the key definition, which requires that any word in $\mathcal{L}$ can be transformed into a word in $\mathcal{G}$ with a relatively small number of edits.

Definition 2.3. [3] We say that a non-decreasing function $g \colon \mathbb{N} \to \mathbb{N}$ is a mistake function if $g(n)/n$ converges to 0. We say that $\mathcal{L}$ is edit approachable by $\mathcal{G}$, where $\mathcal{G} \subset \mathcal{L}$, if there is a mistake function g such that for every $w \in \mathcal{L}$, there exists $v \in \mathcal{G}$ with $\widehat{d}(v, w) \leq g(|w|)$.

We can get the following proposition by applying Proposition 4.2 and Lemma 4.3 in [3].

Proposition 2.4. *If $\mathcal{L}$ is edit approachable by $\mathcal{G}$ and $\mathcal{G}$ has a (W)-specification, then there is an $\mathcal{F} \subset \mathcal{L}$ that has the free concatenation property (i.e., for all $u, w \in \mathcal{F}$, we have $uw \in \mathcal{F}$) and $\mathcal{L}$ is edit approachable by $\mathcal{F}$.*

Remark 2.5. We do not require $\mathcal{F}_n \neq \emptyset$, for each $n \in \mathbb{N}$.

To estimate the lower bound of the *Hausdorff dimension* of a set, we need the following mass distribution principle.

Theorem 2.6. [9] *Let E be a Borel measurable set in X and μ be a Borel measure with $\mu(E) > 0$. Assume that there exist two positive constants c, η such that, for any set U with diameter $\operatorname{diam} U < \eta$, $\mu(U) \leq c \operatorname{diam}(U)^s$, then*

$$\dim_H E \geq s.$$

2.2. Topological pressure

Given a collection $\mathcal{D} \subset \mathcal{L}$, the entropy of $\mathcal{D}$ is

$$h(\mathcal{D}) := \limsup_{n \to \infty} \frac{1}{n} \log \sharp \mathcal{D}_n,$$

where $\mathcal{D}_n = \{w \in \mathcal{D} : |w| = n\}$. We write $h_{\mathrm{top}}(X) := h(\mathcal{L})$. Let $C(X)$ denote all the continuous functions from X to $\mathbb{R}$. For a fixed potential function $\varphi \in C(X)$, the pressure of $\mathcal{D} \subset \mathcal{L}$ is

$$P(\mathcal{D}, \varphi) := \limsup_{n \to \infty} \frac{1}{n} \log \Lambda_n(\mathcal{D}, \varphi),$$

where $\Lambda_n(\mathcal{D}, \varphi) = \sum_{w \in \mathcal{D}_n} e^{\sup_{x \in [w]} S_n \varphi(x)}$ and $S_n \varphi(x) = \sum_{k=0}^{n-1} \varphi(\sigma^k x)$. We write $P(\varphi) := P(\mathcal{L}, \varphi)$.

Proposition 2.7. [3] *If $\mathcal{L}$ is edit approachable by $\mathcal{G}$, then $P(\mathcal{G}, \varphi) = P(\varphi)$ for every $\varphi \in C(X)$.*

In the following, we set $\mathcal{N}(\mathcal{F}) := \{n \in \mathbb{N} : \mathcal{F}_n \neq \emptyset\}$.

Definition 2.8. Let $g = -(f+1) \in C(X)$ with $f > 0$ and suppose $\mathcal{L}$ is edit approachable by $\mathcal{F}$.

(1) For any $n \geq 1$ and $n \in \mathcal{N}(\mathcal{F})$, define $s_n(\mathcal{F})$ to be the unique solution of the equation

$$\sum_{w \in \mathcal{F}_n} \left(e^{\sup_{x \in [w]} S_n g(x)} \right)^s = 1.$$

(2) For any $n \geq 1$ and $n \in \mathcal{N}(\mathcal{F})$, define $\widehat{s}_n(\mathcal{F})$ to be the unique solution of the equation

$$\sum_{w \in \mathcal{F}_n} \left(e^{\inf_{x \in [w]} S_n g(x)} \right)^s = 1.$$

Remark 2.9. Since $f + 1 > 1$ is a continuous function on X, the above definitions are well defined.

Proposition 2.10. *Assume $s(\mathcal{F})$ to be the solution of the pressure equation*

$$P(\mathcal{F}, -s(f+1)) = 0.$$

For the increasing sequence $\{n_j\}_{j \geq 1} = \mathcal{N}(\mathcal{F})$, we have

$$\lim_{j \to \infty} s_{n_j}(\mathcal{F}) = s(\mathcal{F}) \quad and \quad \lim_{j \to \infty} \widehat{s}_{n_j}(\mathcal{F}) = s(\mathcal{F}).$$

Proof. By virtue of the definition of topological pressure, it is easy to see that the solution of $P(\mathcal{F}, -s(f+1)) = 0$ is unique and the pressure function $f \mapsto P(\mathcal{F}, f)$ is continuous. We claim that $s_n(\mathcal{F})$ is bounded for each $n \in \mathcal{N}(\mathcal{F})$. Since

$$\sum_{\mathcal{F}_n} e^{-n s_n(\mathcal{F})\|f+1\|_{\max}} \leq 1 \leq \sum_{\mathcal{F}_n} e^{-n s_n(\mathcal{F})\|f+1\|_{\min}},$$

we have

$$0 < \frac{1}{\|f+1\|_{\max}} \frac{\log \#\mathcal{F}_n}{n} \leq s_n(\mathcal{F}) \leq \frac{1}{\|f+1\|_{\min}} \frac{\log \#\mathcal{F}_n}{n},$$

where $\#E$ denotes the cardinality of the set E.

Using the fact that $\limsup_{n \to \infty} (\log \#\mathcal{F}_n)/n \leq \lim_{n \to \infty} (\log \#\mathcal{L}_n)/n = h_{\mathrm{top}}(X)$, we see that $s_n(\mathcal{F})$ is bounded. Moreover, from the continuity of the pressure function $f \mapsto P(\mathcal{F}, f)$ and $\mathcal{L}$ is edit approachable by $\mathcal{F}$, one can readily verify that $\liminf_{j \to \infty} s_{n_j}(\mathcal{F})$ and $\limsup_{j \to \infty} s_{n_j}(\mathcal{F})$ are the solutions of $P(\mathcal{F}, -s(f+1)) = 0$. Hence,

$$\lim_{j \to \infty} s_{n_j}(\mathcal{F}) = s(\mathcal{F}).$$

On the other hand, it is obvious that we can give another equivalent definition of topological pressure for any $\mathcal{D} \subset \mathcal{L}$ as follows. For a fixed potential function $\varphi \in C(X)$ and $\mathcal{D} \subset \mathcal{L}$, define

$$\widehat{P}(\mathcal{D}, \varphi) := \limsup_{n \to \infty} \frac{1}{n} \log \widehat{\Lambda}_n(\mathcal{D}, \varphi),$$

where $\widehat{\Lambda}_n(\mathcal{D}, \varphi) = \sum_{w \in \mathcal{D}_n} e^{\inf_{x \in [w]} S_n \varphi(x)}$. Note that

$$\left| \sup_{x \in [w]} S_n g(x) - \inf_{x \in [w]} S_n g(x) \right| = o(n),$$

by the continuity of g which leads to that $\widehat{P}(\mathcal{F}, -s(f+1)) = P(\mathcal{F}, -s(f+1))$. Similar to the above proof, we can see that $\lim_{j \to \infty} \widehat{s}_{n_j}(\mathcal{F})$ is the solution of $\widehat{P}(\mathcal{F}, -s(f+1)) = 0$, which is the same as the solution of $P(\mathcal{F}, -s(f+1)) = 0$. Hence, $\lim_{j \to \infty} \widehat{s}_{n_j}(\mathcal{F}) = s(\mathcal{F})$. $\qquad \square$

From Proposition 2.7, we have the following corollary.

Corollary 2.11. *Suppose $\mathcal{L}$ is edit approachable by $\mathcal{F}$. Assume $s(X)$ and $s(\mathcal{F})$ to be, respectively, the solution of the pressure equations $P(-s(f+1)) = 0$, $P(\mathcal{F}, -s(f+1)) = 0$. Then*

$$s(X) = s(\mathcal{F}).$$

3. Proof of Theorem 1.4

First, we consider (C1). By $\liminf_{n \to \infty} \psi(n) > 0$. Namely, there are $\epsilon_0 > 0$ and $N > 0$ such that for any $n \geq N$, $\psi(n) \geq \epsilon_0$. Clearly, we have $b = \liminf_{n \to \infty} (-\log \psi(n))/n = 0$. It suffices to show that $\dim_H R(\psi) \geq h$, module the upper bound given in (C2). By the upper semi-continuity of the entropy map $\mu \mapsto h_\mu(\sigma)$ and the variational principle for (X, σ), we can choose an ergodic measure μ such that $h_\mu(\sigma) = h$. By the Poincaré recurrence theorem, we have $\mu(R(\psi)) = 1$. Hence,

$$\dim_H R(\psi) \geq \dim_H \mu.$$

It follows from the Shannon-McMillan-Breiman Theorem that for μ a.e. $x \in X$,

$$h_\mu(\sigma) = \lim_{n \to \infty} \frac{-\log \mu([x_1 x_2 \cdots x_n])}{n} = \dim_H \mu.$$

So $\dim_H R(\psi) \geq h$.

Secondly, the proof of (C2) is divided into two parts.

3.1. Upper bound

The upper bound can be obtained by considering the natural covering system. It is obvious that

$$R(\psi) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(w_1, w_2, \ldots, w_n) \in \mathcal{L}_n} J(w_1, w_2, \ldots, w_n),$$

where

$$J(w_1, w_2, \ldots, w_n) := \{x \in X : x \in [w_1 w_2 \cdots w_n], d(\sigma^n x, x) < \psi(n)\}.$$

Obviously, we can estimate the diameter of $J(w_1, w_2, \ldots, w_n)$ by

$$\mathrm{diam}(J(w_1, w_2, \ldots, w_n)) \leq e^{-n} \psi(n).$$

For any $s > h/(1+b)$, and without loss of generality we can assume that $s = h(1+\delta)/(1+b)$ for some $\delta > 0$. For any $\epsilon > 0$ satisfying $(h(1+\delta)/(1+b) + 1)\epsilon < h\delta/2$, we have

$$\#\mathcal{L}_n(X) \leq e^{n(h+\epsilon)} \quad \text{and} \quad \psi(n) \leq e^{-n(b-\epsilon)}$$

for n large enough by the definition of the topological entropy and the definition of b. Hence,

$$H^s(R(\psi)) \leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} \sum_{(w_1 w_2 \cdots w_n) \in \mathcal{L}_n} \mathrm{diam}(J(w_1, w_2, \ldots, w_n))^s$$

$$\leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} e^{n(h+\epsilon)} (e^{-n} \psi(n))^s$$

$$\leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} e^{-nh\delta/2}.$$

Furthermore,

$$H^s(R(\psi)) < \infty.$$

This implies

$$\dim_H(R(\psi)) \leq \frac{h}{1+b}.$$

We remark that this argument on upper bound is valid for any positive function ψ instead of merely monotonic ones.

3.2. Lower bound

Construction of the Moran set. Fix $\eta > 0$. By Propositions 2.4 and 2.7, we can choose M large enough so that

$$\log \#\mathcal{F}_M \geq (1 - \eta) M h.$$

Choose a largely sparse subsequence $\{n_k\}_{k \geq 1}$ of $\mathbb{N}$ such that

$$(3.1) \qquad \liminf_{n \to \infty} \frac{-\log \psi(n)}{n} = \lim_{k \to \infty} \frac{-\log \psi(n_k)}{n_k}, \quad \frac{n_k}{k} \geq \max \left\{ \sum_{j=1}^{k-1} n_j, -\log \psi(n_{k-1}) \right\}$$

for any $k \geq 2$.

For $k = 1$, define l_1, i_1 such that $n_1 = l_1 M + i_1$, $0 \leq i_1 < M$. We define $\widehat{n}_1 = l_1 M$, and an integer $\widehat{t}_1$ satisfying

$$e^{-\widehat{t}_1} < \psi(\widehat{n}_1) \leq e^{-\widehat{t}_1 + 1}.$$

Then we choose t_1 by modifying $\widehat{t}_1$ such that $\widehat{t}_1 + M \geq t_1 \geq \widehat{t}_1$ and $M \mid t_1$. As a consequence, we obtain

$$e^{-t_1} < \psi(\widehat{n}_1) \leq e^{-t_1 + M + 1}.$$

Since ψ is nonincreasing, we have

$$e^{-t_1 + M + 1} \geq \psi(\widehat{n}_1) \geq \psi(n_1).$$

Define the rational number r_1 by

$$\widehat{n}_1 r_1 = \widehat{n}_1 + t_1.$$

For $k \geq 2$, define l_k, i_k such that $n_k - (\widehat{n}_{k-1} + t_{k-1}) = l_k M + i_k$, $0 \leq i_k < M$ and then we define $\widehat{n}_k := \widehat{n}_{k-1} + t_{k-1} + l_k M$. Define the integer $\widehat{t}_k$ by

$$e^{-\widehat{t}_k} < \psi(\widehat{n}_k) \leq e^{-\widehat{t}_k + 1},$$

and choose t_k satisfying $M \mid t_k$ and $\widehat{t}_k + M \geq t_k \geq \widehat{t}_k$. As a consequence, we have

$$e^{-t_k} < \psi(\widehat{n}_k) \leq e^{-t_k + M + 1}.$$

Since $\psi(n)$ is nonincreasing, we have

$$(3.2) \qquad e^{-t_k + M + 1} \geq \psi(\widehat{n}_k) \geq \psi(n_k).$$

Define the rational number r_k by

$$\widehat{n}_k r_k = \widehat{n}_k + t_k.$$

From these definitions, we can see that

$$(3.3) \qquad n_k - M \leq \widehat{n}_k \leq n_k.$$

We are now in a position to construct a Moran subset of $R(\psi)$ as follows. We realize the events $d(\sigma^n x, x) < \psi(n)$ infinitely many times along the subsequence $\{\widehat{n}_k\}_{k \geq 1}$.

Level 1 of the Moran set.   Employing the definitions of t_1 and r_1,

$$\mathbb{F}(1) = \bigcup [(\underbrace{w_1 \cdots w_M}_{M} \cdots \underbrace{w_{(l_1-1)M+1} \cdots w_{l_1 M}}_{M})^{r_1}],$$

where the union is taken over all blocks $(w_{lM+1}, \ldots, w_{(l+1)M}) \in \mathcal{F}_M$ for each $0 \le l \le l_1 - 1$. Since $\mathcal{F}$ has the free concatenation property, the concatenation is admissible. By construction, we have that for any word $I \in \mathbb{F}(1)$ and $x \in I$, the prefix of $\sigma^{\widehat{n}_1} x$ and x coincide for the first t_1 digits. So, $d(\sigma^{\widehat{n}_1} x, x) < e^{-t_1} \le \psi(\widehat{n}_1)$. This realizes that event $d(\sigma^n x, x) < \psi(n)$ for one time.

Level 2 of the Moran set.   For each word $J_1 \in \mathbb{F}(1)$, let

$$\mathbb{F}(2) = \bigcup_{J_1 \in \mathbb{F}(1)} \mathbb{F}(2, J_1),$$

where for a fixed $J_1 \in \mathbb{F}(1)$, writing $J_1 = [(w_1 \cdots w_{\widehat{n}_1 + t_1})]$ and

$$
\begin{aligned}
&\mathbb{F}(2, J_1) \\
&= \bigcup [(w_1 \cdots w_{\widehat{n}_1 + t_1} \underbrace{w_{\widehat{n}_1 + t_1 + 1} \cdots w_{\widehat{n}_1 + t_1 + M}}_{M} \cdots \underbrace{w_{\widehat{n}_1 + t_1 + (l_2-1)M+1} \cdots w_{\widehat{n}_1 + t_1 + l_2 M}}_{M})^{r_2}],
\end{aligned}
$$

where the union is taken over all blocks $(w_{\widehat{n}_1 + t_1 + lM+1}, \ldots, w_{\widehat{n}_1 + t_1 + (l+1)M}) \in \mathcal{F}_M$ for each $0 \le l \le l_2 - 1$. Since $\mathcal{F}$ has the free concatenation property, the concatenation is admissible. By construction, we obtain that for any word $I \in \mathbb{F}(2)$ and $x \in I$, the prefix of $\sigma^{\widehat{n}_2} x$ and x coincide for the first t_2 digits.

From level k to level $k + 1$.   Provided that $\mathbb{F}(k)$ has been defined, we define $\mathbb{F}(k+1)$ as follows:

$$\mathbb{F}(k+1) = \bigcup_{J_k \in \mathbb{F}(k)} \mathbb{F}(k+1, J_k),$$

where for any $J_k = [(w_1 \cdots w_{\widehat{n}_k + t_k})] \in \mathbb{F}(k)$,

$$
\begin{aligned}
&\mathbb{F}(k+1, J_k) \\
&= \bigcup [(w_1 \cdots w_{\widehat{n}_k + t_k} \underbrace{w_{\widehat{n}_k + t_k + 1} \cdots w_{\widehat{n}_k + t_k + M}}_{M} \cdots \underbrace{w_{\widehat{n}_k + t_k + (l_{k+1}-1)M+1} \cdots w_{\widehat{n}_k + t_k + l_{k+1} M}}_{M})^{r_{k+1}}],
\end{aligned}
$$

where the union is taken over all blocks $(w_{\widehat{n}_k + t_k + lM+1}, \ldots, w_{\widehat{n}_k + t_k + (l+1)M}) \in \mathcal{F}_M$ for each $0 \le l \le l_{k+1} - 1$. Since $\mathcal{F}$ has the free concatenation property, the concatenation is admissible. By construction, we obtain that for any word $I \in \mathbb{F}(k+1)$ and $x \in I$, the prefix of $\sigma^{\widehat{n}_{k+1}} x$ and x coincide for the first t_{k+1} digits.

The Moran set. We have constructed a nested sequence $\{\mathbb{F}(k)\}_{k \geq 1}$ composed of words. The Moran set is obtained by

$$\mathbb{F}_\infty = \bigcap_{k=1}^\infty \mathbb{F}(k).$$

From the above constructions, we get

$$\mathbb{F}_\infty \subset R(\psi).$$

Supporting measure. Now we construct a probability measure μ on $\mathbb{F}_\infty$. For any $J_k \in \mathbb{F}(k)$, let $J_{k-1} \in \mathbb{F}(k-1)$ be its mother word, i.e., $J_k \in \mathbb{F}(k, J_{k-1})$. The measure of J_k is defined as

$$\mu(J_k) := \frac{1}{\#\mathbb{F}(k, J_{k-1})} \mu(J_{k-1}) = \prod_{j=1}^k \frac{1}{(\#\mathcal{F}_M)^{l_j}}.$$

This means that the measure of any mother word is evenly distributed among her offspring. For any $n \geq 1$, and n long word $I_n = [w_1 \cdots w_n]$ with $I_n \cap \mathbb{F}_\infty \neq \emptyset$, let $k \geq 2$ be an integer satisfying $\widehat{n}_{k-1} + t_{k-1} < n \leq \widehat{n}_k + t_k$. We just set

$$\mu([w_1 \cdots w_n]) = \sum_{J_k \subset I_n} \mu(J_k),$$

where the summation is taken over all the words $J_k \in \mathbb{F}(k)$ contained in I_n. In fact, we have the following expression for the measure of a word.

(1) If $\widehat{n}_{k-1} \leq n \leq \widehat{n}_{k-1} + t_{k-1}$,

$$\mu(I_n) = \mu(I_{\widehat{n}_{k-1}+t_{k-1}}).$$

(2) If $\widehat{n}_{k-1} + t_{k-1} < n < \widehat{n}_k$, assume $n = \widehat{n}_{k-1} + t_{k-1} + lM + i$. For $i = 0$ and $0 \leq l \leq l_k - 1$,

$$\mu(I_n) = \mu(I_{\widehat{n}_{k-1}+t_{k-1}}) \frac{1}{(\#\mathcal{F}_M)^l}.$$

For $i \neq 0$ and $0 \leq l \leq l_k - 1$,

$$\mu(I_n) \leq \mu(I_{\widehat{n}_{k-1}+t_{k-1}+lM}).$$

The Hölder exponent of the measure. By (3.2) and the fact that $b = \lim_{k \to \infty} \frac{-\log \psi(n_k)}{n_k}$, we obtain

$$\lim_{k \to \infty} \frac{t_k}{n_k} \leq b.$$

Accordingly, from (3.3), we have

$$\lim_{k \to \infty} \frac{t_k}{\widehat{n}_k} \leq b.$$

Furthermore, by virtue of (3.1), (3.2) and (3.3), there exists k_0 such that $k \geq k_0$ satisfying

$$\frac{\widehat{n}_k - \widehat{n}_{k-1} - t_{k-1}}{\widehat{n}_k + t_k} \geq \frac{1-\eta}{1+b}.$$

Since

$$\begin{aligned}
\frac{-\log \mu(J_k)}{\widehat{n}_k + t_k} &= \frac{\sum_{j=1}^{k} l_j \log \#\mathcal{F}_M}{\widehat{n}_k + t_k} \\
&\geq \frac{\widehat{n}_k - \widehat{n}_{k-1} - t_{k-1}}{\widehat{n}_k + t_k} \times \frac{\log \#\mathcal{F}_M}{M} \\
&\geq \frac{h(1-\eta)^2}{1+b} := s,
\end{aligned}$$

we have

$$\mu(J_k) \leq e^{-(\widehat{n}_k + t_k)s}.$$

(1) If $\widehat{n}_{k-1} \leq n \leq \widehat{n}_{k-1} + t_{k-1}$,

$$\mu(I_n) = \mu(I_{\widehat{n}_{k-1} + t_{k-1}}).$$

Thus

$$\frac{-\log \mu(I_n)}{n} \geq \frac{-\log \mu(I_{\widehat{n}_{k-1} + t_{k-1}})}{\widehat{n}_{k-1} + t_{k-1}} \geq s.$$

This implies that

$$\mu(I_n) \leq \text{diam}(I_n)^s$$

for n large enough.

(2) If $\widehat{n}_{k-1} + t_{k-1} < n < \widehat{n}_k$, set $n = \widehat{n}_{k-1} + t_{k-1} + lM + i$. For $i = 0$ and $0 \leq l \leq l_k - 1$,

$$\mu(I_n) = \mu(I_{\widehat{n}_{k-1} + t_{k-1}}) \frac{1}{(\#\mathcal{F}_M)^l} \leq \mu(I_{\widehat{n}_{k-1} + t_{k-1}}) e^{-l(1-\eta)Mh}.$$

Then

$$\begin{aligned}
\frac{-\log \mu(I_n)}{n} &\geq \frac{-\log \mu(I_{\widehat{n}_{k-1} + t_{k-1}}) + l(1-\eta)Mh}{\widehat{n}_{k-1} + t_{k-1} + lM} \\
&= \min\left\{ \frac{-\log \mu(I_{\widehat{n}_{k-1} + t_{k-1}})}{\widehat{n}_{k-1} + t_{k-1}}, (1-\eta)h \right\} \\
&\geq s.
\end{aligned}$$

This implies that

$$\mu(I_n) \leq \text{diam}(I_n)^s$$

for n large enough.

For $i \neq 0$ and $0 \leq l \leq l_k - 1$,

$$\mu(I_n) \leq \mu(I_{\widehat{n}_{k-1} + t_{k-1} + lM}).$$

Since
$$\operatorname{diam}(I_n) \geq e^{-M} \operatorname{diam}(I_{\widehat{n}_{k-1}+t_{k-1}+lM}),$$

we have
$$\mu(I_n) \leq e^{Ms} \operatorname{diam}(I_n)^s$$

for n large enough.

Finally, by Theorem 2.6 and letting $\eta \to 0$, we complete the proof of (C2).

4. Proof of Theorem 1.5

Naturally, the proof is divided into two parts.

4.1. Upper bound

The proof is similar to the proof of the upper bound of Theorem 1.4. Clearly,

$$R(f) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(w_1 w_2 \cdots w_n) \in \mathcal{L}_n} J(w_1, w_2, \ldots, w_n),$$

where

$$J(w_1, w_2, \ldots, w_n) := \left\{ x \in X : x \in [w_1 w_2 \cdots w_n], d(\sigma^n x, x) < e^{-S_n f(x)} \right\}.$$

For each $(w_1 w_2 \cdots w_n)$, we can choose y such that

$$S_n f(y) = \inf_{x \in [w_1 w_2 \cdots w_n]} S_n f(x).$$

By the continuity of f, for each $\delta > 0$ and n large enough, we have

$$J(w_1, w_2, \ldots, w_n) \subset \left\{ x \in X : x \in [w_1 w_2 \cdots w_n], d(\sigma^n x, x) < e^{-S_n f(y)} e^{n\delta} \right\},$$

where $S_n f(y) = \inf_{x \in [x_1 \cdots x_n]} S_n f(x)$. Thus,

(4.1) $$\operatorname{diam}(J(w_1, w_2, \ldots, w_n)) \leq e^{-S_n f(y) + n\delta - n}.$$

We define $s(\delta)$ to be the solution of $P((s(-1 - f + \delta)) = 0$. By the continuity of the pressure function $f \mapsto P(f)$ and the boundedness of $s(\delta)$, we obtain $\lim_{\delta \to 0^+} s(\delta) = s(X)$. At the same time, we put $P := P((s(\delta) + \delta)(-1 - f + \delta)) < 0$. There exists $\epsilon(\delta) > 0$ such that

$$\sum_{w \in \mathcal{L}_n(X)} \left(e^{-\sup_{x \in [w]} S_n(f(x) + 1 - \delta)} \right)^{s(\delta) + \delta} \leq e^{-n\epsilon(\delta)},$$

for n large enough. Moreover, from (4.1), we have

$$H^{s(\delta)+\delta}(R(f)) \leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} \sum_{(w_1 w_2 \cdots w_n) \in \mathcal{L}_n} \operatorname{diam}(J(w_1, w_2, \ldots, w_n))^{s(\delta)+\delta}$$

$$\leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} e^{-n\epsilon(\delta)} < \infty.$$

This implies that

$$\dim R(f) \leq s(\delta) + \delta.$$

Letting $\delta \to 0$, we have finished the proof.

4.2. Lower bound

By the continuity of f, we can choose a $y \in [x_1 \cdots x_n]$ such that $S_n f(y) = \sup_{x \in [x_1 \cdots x_n]} S_n f(x)$, i.e., y depends on n and $[x_1 \cdots x_n]$. It suffices to show that the result holds for the set

$$\left\{ x \in X : d(x, \sigma^n x) < e^{-S_n f(y)} \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

Fix $\eta > 0$. By Proposition 2.10, we can choose $M_0 \in \mathbb{N}$ and $M \in \mathcal{N}(\mathcal{F})$, $M > M_0$ such that

(4.2)
$$\sup\{|f(x) - f(y)| : x, y \in X, d(x,y) \leq e^{-M_0}\} \leq \frac{\eta}{4},$$
$$\frac{M_0 \|f\|}{M} < \frac{\eta}{4} \quad \text{and} \quad |\widehat{s}_M(\mathcal{F}) - s(X)| < \eta.$$

Construction of the Moran set.   In the following, for any $[w_1 \cdots w_n]$, we set $y \in [w_1 \cdots w_n]$ satisfying

$$S_n f(y) := \sup_{x \in [w_1 \cdots w_n]} S_n f(x).$$

For $k = 1$, choose $m_1 = 1$ and define $n_1 := M$. For any $(w_1 \cdots w_{n_1}) \in \mathcal{F}$, define $\widehat{t}_1 = \widehat{t}(w_1 \cdots w_{n_1})$ to be the integer such that

$$e^{-\widehat{t}_1} < e^{-S_{n_1} f(y)} \leq e^{-\widehat{t}_1 + 1}.$$

Moreover, we choose t_1 by modifying $\widehat{t}_1$ in such a way that $\widehat{t}_1 + M \geq t_1 \geq \widehat{t}_1$ and $M \mid t_1$. So

$$e^{-t_1} < e^{-S_{n_1} f(y)} \leq e^{-t_1 + M + 1}.$$

Define r_1 by

$$n_1 r_1 = n_1 + t_1.$$

For $k \geq 2$, we choose m_k large enough such that

$$(4.3) \qquad (n_{k-1} + t_{k-1})\|f\| \leq \frac{m_k M \eta}{2} \quad \text{and} \quad \frac{m_k}{k} \geq m_1 + \cdots + m_{k-2} + m_{k-1},$$

and then define $n_k = m_k M + n_{k-1} + t_{k-1}$. Define $\widehat{t}_k$ to be the integer such that

$$e^{-\widehat{t}_k} < e^{-S_{n_k} f(y)} \leq e^{-\widehat{t}_k + 1},$$

and then we choose t_k satisfying $M \mid t_k$ and $\widehat{t}_k + M \geq t_k \geq \widehat{t}_k$. As a consequence, we have

$$(4.4) \qquad\qquad e^{-t_k} < e^{-S_{n_k} f(y)} \leq e^{-t_k + M + 1}.$$

Define r_k by

$$n_k r_k = n_k + t_k.$$

We are now in a position to construct the Moran subset of $R(f)$ as follows.

Level 1 of the Moran set.   Employing the definitions of t_1 and r_1, let

$$\mathbb{F}(1) = \bigcup [(w_1^1)^{r_1}],$$

where the union is taken over all blocks $(w_1^1) \in \mathcal{F}_M$ with $r_1 M = m_1 M + t_1$. Since $\mathcal{F}$ has the free concatenation property, the concatenation is admissible. By construction, we have that for any word $I \in \mathbb{F}(1)$ and $x \in I$, the prefix of $\sigma^{n_1} x$ and x coincide for the first t_1 digits.

Level 2 of the Moran set.   The second level sets are composed of the collection of the words of each word $J_1 \in \mathbb{F}(1)$:

$$\mathbb{F}(2) = \bigcup_{J_1 \in \mathbb{F}(1)} \mathbb{F}(2, J_1),$$

where for a fixed $J_1 \in \mathbb{F}(1)$, we write $J_1 = [(w_1 \cdots w_{m_1 M + t_1})]$, and put

$$\mathbb{F}(2, J_1) = \bigcup [(w_1 \cdots w_{m_1 M + t_1} w_1^2 \cdots w_{m_2}^2)^{r_2}]$$

where the union is taken over all blocks $w_j^2 \in \mathcal{F}_M$ for all $1 \leq j \leq m_2$, with $r_2 n_2 = n_1 + t_1 + m_2 M + t_2$. Since $\mathcal{F}$ has the free concatenation property, the concatenation is admissible.

From level k to level $k+1$. Provided that $\mathbb{F}(k)$ has been defined, we define $\mathbb{F}(k+1)$ as follows:

$$\mathbb{F}(k+1) = \bigcup_{J_k \in \mathbb{F}(k)} \mathbb{F}(k+1, J_k),$$

where for any $J_k = [(w_1 \cdots w_{t_k+n_k})] \in \mathbb{F}(k)$,

$$\mathbb{F}(k+1, J_k) = \bigcup[(w_1 \cdots w_{t_k+n_k} w_1^{k+1} \cdots w_{m_{k+1}}^{k+1})^{r_{k+1}}],$$

where the union is taken over all blocks $w_j^{k+1} \in \mathcal{F}_M$ for each $1 \leq j \leq m_{k+1}$, with $r_2 n_{k+1} = n_k + t_k + m_{k+1}M + t_{k+1}$. Since $\mathcal{F}$ has the free concatenation property, the concatenation is admissible.

The Moran set. We have obtained a nested sequence $\{\mathbb{F}(k)\}_{k \geq 1}$ composed of words. The Moran set is obtained as

$$\mathbb{F}_\infty = \bigcap_{k=1}^{\infty} \mathbb{F}(k).$$

From the above constructions, we get

$$\mathbb{F}_\infty \subset \left\{ x \in X : d(x, \sigma^n x) < e^{-S_n f(y)} \text{ for infinitely many } n \in \mathbb{N} \right\} \subset R(f).$$

4.2.1. Supporting measure

Now we construct a probability measure μ on $\mathbb{F}_\infty$. For any $J_k \in \mathbb{F}(k)$, let $J_{k-1} \in \mathbb{F}(k-1)$ be its mother word, i.e., $J_k \in \mathbb{F}(k, J_{k-1})$. The measure of J_k is defined by

$$\mu(J_k) := \prod_{i=1}^{m_k} e^{-s_M M - s_M S_M f(y_i^k)} \mu(J_{k-1})$$

$$= \prod_{j=1}^{k} \prod_{i=1}^{m_j} e^{-s_M M - s_M S_M f(y_i^j)}$$

where $s_M := \widehat{s}_M(\mathcal{F})$ and $S_M f(y_i^j) = \sup_{x \in [w_i^j]} S_M f(x)$ with $y_i^j \in [w_i^j]$, $1 \leq i \leq m_k$. This means that the measure of any mother word is evenly distributed among her offspring. For any $n \geq 1$, and n long word $I_n = [w_1 \cdots w_n]$ with $I_n \cap \mathbb{F}_\infty \neq \emptyset$, n_k and t_k depend on the digits. More precisely, given a block $[w_1 \cdots w_n]$ of length n, it determines t_1 if $n \geq n_1$. If t_{k-1} can be determined, we then compare n with $n_k = n_{k-1} + t_{k-1} + m_k$. If $n \geq n_k$, it determines t_k; otherwise, we have $n_{k-1} \leq n < n_k + t_{k-1} + m_k = n_k$, and the block $[w_1 \cdots w_n]$ determines n_1 up to n_{k-1}.

Now we consider any word with length $n_{k-1} \leq n < n_k$. We just set

$$\mu([w_1 \cdots w_n]) = \sum_{J_k \subset I_n} \mu(J_k),$$

where the summation is taken over all words $J_k \subset \mathbb{F}(k)$ contained in I_n. In fact, we have the following expression for the measure of a word.

(1) When $n_{k-1} \le n \le n_{k-1} + t_{k-1}$,

$$\mu(I_n) = \mu(I_{n_{k-1}+t_{k-1}}) = \prod_{j=1}^{k-1} \prod_{i=1}^{m_j} e^{-s_M M - s_M S_M f(y_i^j)}.$$

(2) When $n_{k-1} + t_{k-1} < n < n_k$,

$$\mu(I_n) = \sum_{J_k \subset I_n} \mu(J_k) = \sum_{(w_{n+1}\cdots w_{n_k}) \in \Xi} \mu(I_{n_k + t_k}(w_1 \cdots w_n w_{n+1} \cdots w_{n_k})),$$

where Ξ denotes the sets of $(w_{n+1} \cdots w_{n_k})$ such that $(w_{n_k - m_k + 1} \cdots w_n w_{n+1} \cdots w_{n_k}) \in (\mathcal{F}_M)^{m_k}$.

Hölder exponent of the measure. First, we consider the J_k. By (4.2), for k large enough,

$$\mu(J_k) \le \prod_{j=1}^{k} \left(\prod_{i=1}^{m_j} e^{-M - S_M f(y_i^j)} \right)^{s(X) - \eta}.$$

From (4.2) and (4.3), for any $J_k = [w_1 \cdots w_{n_k + t_k}]$,

$$\sum_{j=1}^{k} \left| \sum_{i=1}^{m_j} S_M f(y_i^j) - S_{n_j} f(y) \right|$$

$$(4.5) \quad \le \sum_{j=1}^{k} \left(\left| \sum_{i=1}^{m_j} S_M f(y_i^j) - S_{Mm_j} f(\sigma^{n_{j-1}+t_{j-1}} y_0^j) \right| + \left| S_{Mm_j} f(\sigma^{n_{j-1}+t_{j-1}} y_0^j) - S_{n_j} f(y) \right| \right)$$

$$\le \sum_{j=1}^{k} \left(\frac{m_j M \eta}{4} + \frac{m_j M \eta}{4} + \frac{m_j M \eta}{2} \right) \le 2 m_k M \eta,$$

where $y \in J_k$ and $y_0^j \in [w_1 \cdots w_{n_j}]$ with $1 \le j \le k$. Furthermore,

$$\mu(J_k) \le \prod_{j=1}^{k} \left(\prod_{i=1}^{m_j} e^{-M - S_M f(y_i^j)} \right)^{s(X) - \eta}$$

$$\le \prod_{j=1}^{k} \left(e^{-Mm_j - S_{n_j} f(y)} \right)^{s(X) - \eta} e^{2 m_k M \eta (s(X) - \eta)}$$

$$\le \prod_{j=1}^{k} \left(e^{-Mm_j - t_j + M + 1} \right)^{s(X) - \eta} e^{2 m_k M \eta (s(X) - \eta)}$$

$$\le e^{-(n_k + t_k)(s(X) - \eta) + 2 m_k M \eta (s(X) - \eta) + 2M(s(X) - \eta)}$$

$$\le e^{-(n_k + t_k)(s(X) - \eta - 2\eta(s(X) - \eta)) + 2M(s(X) - \eta)}$$

$$= C(\eta) e^{-(n_k + t_k)(s(X) - \Delta(\eta))},$$

where $C(\eta) := e^{2M(s(X)-\eta)}$ is bounded, and $\Delta(\eta) := \eta + 2\eta(s(X) - \eta)$ satisfying $\Delta(\eta) \to 0$ as $\eta \to 0$. The second inequality follows from (4.5) and the third inequality follows from (4.4). Now, we can make the following estimates.

(1) If $n_{k-1} \le n \le n_{k-1} + t_{k-1}$, then

$$\begin{aligned}
\mu(I_n) = \mu(I_{n_{k-1}+t_{k-1}}) &\le C(\eta) \operatorname{diam}(J_{k-1})^{(s(X)-\Delta(\eta))} \\
&= C(\eta) e^{-n(s(X)-\Delta(\eta))} e^{-(n_{k-1}+t_{k-1}-n)(s(X)-\Delta(\eta))} \\
&\le C(\eta) \operatorname{diam}(I_n)^{(s(X)-\Delta(\eta))}.
\end{aligned}$$

(2) If $n_{k-1} + t_{k-1} < n < n_k$, let $n = n_{k-1} + t_{k-1} + l$. Then

$$(4.6) \qquad \mu(I_n) = \mu(J_{k-1}) \sum_{w_{n+1}\cdots w_{n_k} \in \Xi} \prod_{i=1}^{m_k} e^{-s_M M - s_M S_M f(y_i^k)},$$

where Ξ denotes the set of $(w_{n+1} \cdots w_{n_k})$ such that $(w_{n_k-Mm_k+1} \cdots w_n w_{n+1} \cdots w_{n_k}) \in (\mathcal{F}_M)^{m_k}$. Next we assume $l = M(q-1) + p$, with $1 \le q \le m_k$, $1 < p < M$. We estimate the following summation,

$$\sum_{w_{n+1}\cdots w_{n_k} \in \Xi} \prod_{i=1}^{m_k} e^{-s_M M - s_M S_M f(y_i^k)}$$
$$\le \prod_{i=1}^{q-1} e^{-s_M M - s_M S_M f(y_q^k)} \prod_{i=q}^{m_k} \sum_{y \in [w], w \in \mathcal{F}_M} e^{-s_M M - s_M S_M f(y)}.$$

By the definition of s_M, we have

$$\sum_{y \in [w], w \in \mathcal{F}_M} e^{-s_M M - s_M S_M f(y)} = 1.$$

Hence, we have

$$(4.7) \qquad \begin{aligned}
\sum_{w_{n+1}\cdots w_{n_k} \in \Xi} \prod_{i=1}^{m_k} e^{-s_M M - s_M S_M f(y_i^k)} &\le \prod_{i=1}^{q-1} e^{-s_M M - s_M S_M f(y_q^k)} \\
&\le \prod_{i=1}^{q-1} e^{-s_M M} \\
&\le e^{-M(q-1)(s(X)-\eta)}.
\end{aligned}$$

From (4.6) and (4.7), we have

$$\begin{aligned}
\mu(I_n) &\le C(\eta) e^{-(n_{k-1}+t_{k-1})(s(X)-\Delta(\eta)) - M(q-1)(s(X)-\eta)} \\
&= C(\eta) e^{-(n_{k-1}+t_{k-1}+M(q-1))(s(X)-\eta)} \\
&\le C(\eta) e^{M(s(X)-\Delta(\eta))} \operatorname{diam}(I_n)^{s(X)-\Delta(\eta)}.
\end{aligned}$$

By Theorem 2.6 and letting $\eta \to 0$, we finish the proof of Theorem 1.5.

Remark 4.1. At the end, we pose a question about Theorem 1.4. Does this result remain valid for $\psi(n)$ without monotonicity?

5. Applications

S-gap shifts. An S-gap shift Σ_S is a subshift of $\{0,1\}^{\mathbb{Z}}$ defined by the rule that for a fixed $S \subset \{0,1,2,\ldots\}$, the number of 0s between consecutive 1s is an integer in S. That is, the language is

$$\{0^n 10^{n_1} 10^{n_2} 10^{n_3} 1 \cdots 10^{n_k} 10^m : 1 \leq i \leq k \text{ and } n_i \in S, n, m \in \mathbb{N}\},$$

together with $\{0^n : n \in \mathbb{N}\}$, where we assume that S is infinite.

β-shifts. Fix $\beta > 1$, write $b = \lceil \beta \rceil$ and let $w^{\beta} \in \{0,1,\ldots,b-1\}^{\mathbb{N}}$ be the greedy β-expansion of 1. Then w^{β} satisfies $\sum_{j=1}^{\infty} w_j^{\beta} \beta^{-j} = 1$, and has the property that $\sigma^j(w^{\beta}) \prec w^{\beta}$ for all $j \geq 1$, where $\prec$ denotes the lexicographic ordering. The β-shift is defined by

$$\Sigma_{\beta} = \left\{ x \in \{0,1,\ldots,b-1\}^{\mathbb{N}} : \sigma^j(x) \prec w^{\beta} \text{ for all } j \geq 1 \right\}.$$

In [3], the authors showed that S-gap shifts, β shifts, and their factors have a non-uniform structure, i.e., for $X := \Sigma_S$ or Σ_{β}, there exists $\mathcal{G} \subset \mathcal{L}(X)$ that has a (W)-specification and $\mathcal{L}(X)$ is edit approachable by $\mathcal{G}$. Now suppose given a positive function $\psi(n) \colon \mathbb{N} \to \mathbb{R}$ satisfying (C1) $\liminf_{n\to\infty} \psi(n) > 0$, or (C2) ψ is nonincreasing. Set

$$R(\psi) := \{x \in X : d(\sigma^n x, x) < \psi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Then we have

$$\dim_H R(\psi) = \frac{h}{1+b}, \quad \text{with } b = \liminf_{n\to\infty} \frac{-\log \psi(n)}{n}.$$

Let f be a positive continuous function defined on X. Set

$$R(f) = \left\{ x \in X : d(\sigma^n x, x) \leq e^{-S_n f(x)} \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

The Hasudorff dimension of the set $R(f)$ is the unique solution s of the following pressure equation

$$P(-s(f+1)) = 0.$$

Acknowledgments

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